Geometric transitions: from hyperbolic to AdS geometry

Jeffrey Danciger

October 31, 2010

Abstract

This article is meant to be a summary of some of the highlights of my thesis work on transitional geometry. I’ll stick mainly to intuitive descriptions and sketches, rather than detailed arguments.

1 Introduction and summary of results

1.1 the basic problem

Let $M$ be a manifold and let $X_1$ represent a homogeneous geometry, e.g. hyperbolic geometry. Given a path of degenerating $X_1$ structures on $M$, the basic goal is to construct a geometric transition: a natural continuation of the path into $X_2$ structures on $M$, where $X_2$ is some other geometry. Both Hodgson [Hod86] and Porti [Por98] have studied transition problems, including the transition between the constant curvature three dimensional geometries. In this setting, a path of hyperbolic structures collapsing to a point can be rescaled to converge to a Euclidean structure and then continued into spherical structures.

The degeneration behavior of interest in this article is that of three dimensional structures that collapse down to a hyperbolic plane.

To begin, we focus our attention on singular hyperbolic structures on a compact three-manifold $M$. Let $D_t : \tilde{M} \to H^3$ be a family of developing maps defined for $t > 0$ and suppose $D_t$ converges to $D_0$ a local submersion onto a two dimensional hyperbolic plane. Then $D_0$ defines a co-dimension one transverse hyperbolic foliation. The problem of regenerating hyperbolic structures from this data was first examined by Hodgson [Hod86], and later in a specific case by Porti [Por10]. However, it has not yet been established how to construct a geometric transition in this context. Our point of view, based on projective geometry, is that such a degeneration naturally suggests a transition to anti de Sitter or AdS geometry, a constant curvature 2+1 Lorentzian geometry. Further, if the hyperbolic structures have cone singularities, then the AdS structures generated on the “other side” of the transition have tachyon singularities, a natural analogue to cone singularities studied by Barbot-Bonsante-Schlenker in [BBS09].

Many parallels exist between the study of hyperbolic structures and AdS structures in three dimensions. One prominent example is the remarkable similarity between Mess’s classification of maximal compact globally hyperbolic AdS space-times in [Mes07] and the classical simultaneous uniformization theorem of Bers [Ber60] for quasifuchsian hyperbolic structures. The analogy here extends in the case of manifolds with particles (cone singularities along an infinite line), studied by Lecuire-Schlenker [LS09] and Moroianu-Schlenker [MS09] in the hyperbolic case, and by Bonsante-Schlenker [BS09] in the AdS case. We also
mention the Wick rotation-rescaling theory of Benedetti-Bonsante [BB09], which gives a correlation between the boundaries of convex cores of constant curvature space-times and those of geometrically finite hyperbolic three-manifolds. The author’s work in [Dan10a] and [Dan10b] on the description and theory of the hyperbolic-to-AdS transition establishes an explicit connection between the two geometries where previously there was only an analogy.

1.2 Results: transition theory

Both hyperbolic and anti de Sitter geometry can be modeled as real projective geometries. So it is natural to try to realize a geometric transition in the context of projective geometry. Here, we focus on hyperbolic cone structures (see for example [CHK00]) and their AdS analogues, tachyon structures (see [BBS09]). A tachyon is a singularity along a space-like axis such that the holonomy of a meridian encircling the axis is a Lorentz boost perpendicular to the axis. The author proves the following in [Dan10a]:

**Theorem 1.** Let \( h_t \) be a path of hyperbolic cone structures on a manifold \( M \) defined for \( t > 0 \). Suppose that as \( t \to 0 \), \( h_t \) limits to a transverse hyperbolic foliation with holonomy \( \rho : \pi_1 M \to SO(2,1) \). Then if \( H^1(\pi_1 M, so(2,1)_{Ad}) = \mathbb{R} \), we can construct a geometric transition: a path of projective structures \( p_t \) defined in a neighborhood of \( t = 0 \) so that

- \( p_t \) is equivalent to \( h_t \) for \( t > 0 \) and
- \( p_t \) is equivalent to an AdS tachyon structure for \( t < 0 \).

The same result holds when the roles of hyperbolic and AdS structures are interchanged.

**Remark.** The cohomology condition in Theorem 1 is satisfied by a variety of examples, including the class of examples described in Section 4.2. This condition, reminiscent of a similar condition appearing in Porti’s regeneration theorem for Euclidean cone structures [Por98], is simply a way to guarantee smoothness of the representation variety. Our construction of a geometric transition really only requires that a transition exists on the level of representations. Deciding such questions can be difficult because the character variety may in general have singularities at the locus of degenerated structures. The description and geometric implications of these singularities is the subject of an ongoing investigation. See Section 5 for an interesting example.

Central to our transition construction is the introduction of a new transitional geometry that we call half-pipe geometry or HP geometry (see Section 2). Just as euclidean geometry serves as the transitional geometry between hyperbolic and spherical geometry, HP geometry bridges the gap between hyperbolic and AdS geometry in the sense that the projective structure at time \( t = 0 \) in Theorem 1 is an HP structure.

1.3 Results: triangulated structures

The transition problem is also quite interesting when the broader class of Dehn surgery singularities is allowed. We introduce tools for studying transitions in this more complicated setting in the case when an ideal triangulation is available. Ideal triangulations are useful in a variety of contexts including the volume maximization program of Casson-Rivin [Riv94] and the related variational formulation of the Poincaré conjecture by Luo [Luo10].
Let $M^3$ have a union of tori as boundary. A particularly fruitful method to study deformations of singular hyperbolic structures is to build $M$ as a union of hyperbolic ideal tetrahedra. We generalize this method to study deformation spaces of triangulated AdS and HP structures. Just as a hyperbolic ideal tetrahedron is determined by a complex shape parameter, AdS and HP ideal tetrahedra are determined by parameters in other two-dimensional real algebras. Using this novel construction, we frame the *triangulated transition problem* in terms of solving Thurston’s edge consistency equations [Thu80] over a transitioning family of algebras. The study of real solutions to Thurston’s equations, which give triangulated co-dimension one transverse hyperbolic foliations, turns out to be crucial to solving the transition problem in this context. We show in particular that if the real deformation variety is smooth at a given transverse hyperbolic foliation, then any tangent vector with positive coordinates determines regenerations to robust hyperbolic and anti de Sitter structures.

The author proves the following in [Dan10b]:

**Theorem 2.** Let $M^3$ be a punctured torus bundle with anosov monodromy and let $\mathcal{T}$ be the monodromy ideal triangulation of $M$. The deformation variety of transverse hyperbolic foliations on $(M, \mathcal{T})$ is a smooth one dimensional variety. Further, the tangent direction to this variety has all positive components.

**Corollary.** Any transverse hyperbolic foliation on $(M, \mathcal{T})$ regenerates to both hyperbolic and anti de Sitter structures.

### 2 Transition theory: half-pipe structures

Our description of the transition between hyperbolic and AdS geometry hinges on the understanding of an interesting new transitional geometry, which we call *half-pipe* or HP geometry, that bridges the gap between hyperbolic and AdS geometry. In order to define $\text{HP}^n$ we revisit the hyperboloid models of $\mathbb{H}^n$ and $\text{AdS}^n$. Though the main focus of this article is the case $n = 3$, we develop this part of the theory in all dimensions $n \geq 2$.

#### 2.1 $\mathbb{H}^n$ and $\text{AdS}^n$ as hyperboloids in $\mathbb{R}^{n+1}$

Consider the family $\eta_s$ of diagonal forms on $\mathbb{R}^{n+1}$ given by

$$\eta_s = \begin{pmatrix}
-1 & 0 & 0 \\
0 & I_{n-1} & 0 \\
0 & 0 & s
\end{pmatrix},$$

where $s$ is a real parameter and $I_{n-1}$ represents the identity matrix. Each form $\eta_s$ defines a hyperboloid $X_s$ in $\mathbb{R}^{n+1}$ by the equation

$$x^T \eta_s x = -x_1^2 + x_2^2 + \ldots + x_n^2 + sx_{n+1}^2 = -1.$$

With metric induced by $\eta_s$, $X_s$ is a homogeneous geometric space with isometry group $G_s = \text{Isom}^+(\eta_s) \subset \text{GL}_{n+1}(\mathbb{R})$. The usual hyperboloid model for hyperbolic geometry is given by $X_{+1}$. In fact, for all $s > 0$ the map

$$c_s = \begin{pmatrix}
I_n & 0 \\
0 & \sqrt{|s|}
\end{pmatrix}$$
gives an isometry \( X_s \to X_{s+1} \). Similarly, \( X_{-1} \) is the common hyperboloid model for anti de Sitter geometry and for all \( s < 0 \), the map \( c_s \) gives an isometry \( X_s \to X_{-1} \).

**Remark.** By considering the intersection of \( X_s \) with lines through the origin in \( \mathbb{R}^{n+1} \), \( X_s \) describes a convex set in \( \mathbb{R}P^n \) and \( G_s \) is the group of projective transformations preserving this convex set. We interpret the map \( c_s \) as a simple change of coordinates in projective space.

There is a distinguished codimension one hyperbolic space \( P \) defined by

\[
x_{n+1} = 0 \quad \text{and} \quad -x_1^2 + x_2^2 + \ldots + x_n^2 = -1.
\]

Note that \( P \) is contained in \( X_s \) for all \( s \).

### 2.2 Rescaling the degeneration - definition of \( \text{HP}^n \)

![Figure 1: The family of hyperboloids \( X_s \) as \( s \to 0 \) from above. For \( s > 0 \), \( X_s \) (left four figures) give models for \( \mathbb{H}^2 \), while \( X_0 \) (shown right) is a model for \( \text{HP}^2 \). The distinguished codimension one hyperbolic space \( P \cong \mathbb{H}^1 \) is shown in red.](image)

The space \( X_0 \) is a natural intermediary space between \( \mathbb{H}^n \) and \( \text{AdS}^n \). However, as the metric \( \eta_0 \) is degenerate, the full group of isometries of \( X_0 \) makes the structure too flimsy to be of much use in our transition context. In order to determine a useful structure group for \( X_0 \) we examine the degeneration context in which we hope to construct a transition.

Consider a family of developing maps

\[
D_t : \tilde{M} \to X_{s+1} \quad \text{with holonomy} \quad \rho_t : \pi_1 M \to G_{s+1} = \text{SO}(n,1),
\]

defined for \( t \geq 0 \). Suppose that at time \( t = 0 \), our developing maps degenerate to \( D_0 \), a local submersion onto the codimension one hyperbolic space \( P \). In particular the last coordinate \( x_{n+1} \) converges to the zero function. The holonomies \( \rho_t \) then converge to a representation \( \rho_0 \) with image in the subgroup \( H_0 = \text{O}(n-1,1) \) that preserves \( P \). As \( D_0 \) is a local submersion, it defines a *codimension one transverse hyperbolic foliation*. We assume for simplicity that the the fibers of the foliation are orientable so that in particular the holonomy \( \rho_0 \) of the transverse structure has image in the subgroup

\[
H_0^+ = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in \text{SO}(n-1,1) \right\}.
\]

Next, apply the projective change of coordinates \( c_{t_2}^{-1} \) to get the family \( c_{t_2}^{-1}D_t : \tilde{M} \to X_t \). This effectively rescales the \( x_{n+1} \) coordinate by \( 1/t \). Suppose that \( c_{t_2}^{-1}D_t \) converges as \( t \to 0 \) to a local diffeomorphism \( D : \tilde{M} \to X_0 \) (see Proposition 1 below). The holonomy \( \rho_D \) is the
limit of the holonomies of $c_{t^2}^{-1}D_{t}$, which are given by $c_{t^2}^{-1}\rho_{t}c_{t^2}$. For a particular $\gamma \in \pi_1 M$, we write

$$\rho_{t}(\gamma) = \begin{pmatrix} A(t) & w(t) \\ v(t) & a(t) \end{pmatrix}$$

where $A$ is $n \times n$, $w,v^T \in \mathbb{R}^n$, and $a \in \mathbb{R}$. Then

$$c_{t^2}^{-1}\rho_{t}(\gamma)c_{t^2} \xrightarrow{t \to 0} \begin{pmatrix} A(0) & 0 \\ v'(0) & 1 \end{pmatrix} = \rho_{D}(\gamma). \quad (1)$$

We say that any path of representations $\rho_{t}$ satisfying the above limit is compatible to first order at $t = 0$ with $\rho_{D}$. The special form of $\rho_{D}$ motivates the following definition.

**Definition 1.** Define $\text{HP}^n = X_0$ and $G_{\text{HP}}$ to be the subgroup of $\text{GL}(n + 1, \mathbb{R})$ having the form $\begin{pmatrix} A & 0 \\ v & 1 \end{pmatrix}$ where $A \in \text{SO}(n - 1, 1)$ and $v^T \in \mathbb{R}^n$. We refer to $G_{\text{HP}}$ as the group of half-pipe isometries. A structure modeled on $(\text{HP}^n, G_{\text{HP}})$ is called a half-pipe structure.

**Remark.** Both Lie algebras $\mathfrak{so}(n, 1)$ and $\mathfrak{so}(n - 1, 2)$ split with respect to the adjoint action of $\text{SO}(n - 1, 1)$ as the direct sum $\mathfrak{so}(n - 1, 1) \oplus \mathbb{R}^{n-1,1}$. In both cases, the $\mathbb{R}^{n-1,1}$ factor describes the tangent directions normal to $\text{SO}(n - 1, 1)$. The group $G_{\text{HP}}$ is really a semi-direct product

$$G_{\text{HP}} \cong \mathbb{R}^{n-1,1} \rtimes \text{SO}(n - 1, 1)$$

where an element $(v', A)$ is thought of as an infinitesimal deformation $v'$ of the element $A$ normal to $\text{SO}(n - 1, 1)$ (into either $\text{SO}(n, 1)$ or $\text{SO}(n - 1, 2)$).

### 2.3 The general transition theorem

The rescaling process described above can be made to converge with reasonable assumptions in place.

**Proposition 1 (Rescaling).** Let $M$ be a compact $n$-manifold with boundary and let $X = \mathbb{H}^n$ or $X = \text{AdS}^n$. Suppose $D_t : \tilde{M} \to X$ are developing maps such that the limit $D_0$ is a local submersion onto a codimension one hyperbolic space. Then after appropriate coordinate changes, the structures converge projectively to a half-pipe structure.

**Proof sketch.** The main concern is that some part of the structure could collapse faster than the rest so that even after rescaling, the limiting structure is not robust everywhere. However, since $M$ is compact we can control the rate of collapse uniformly over $M$ by (a path of) coordinate changes in the domain.

**Remark (Uniqueness).** The HP structure in the above Proposition is unique up to scale in the $x_{n+1}$ direction.

**Definition 2.** We say a half-pipe structure $H$ is transitional if there are degenerating families of both hyperbolic and anti de Sitter structures that projectively limit to $H$. All of this data together describes a geometric transition.
The next step of the transition process is to regenerate robust structures from an HP structure:

**Proposition 2 (Regeneration).** Let $M$ be a compact manifold (with boundary) endowed with an HP structure $H$. Let $X$ be either $\mathbb{H}^n$ or $\text{AdS}^n$. Let $\rho_t : \pi_1 M \to \text{Isom}(X)$ be a family of representations compatible to first order at time $t = 0$ with the holonomy of $H$ (in the sense of Equation 1). Then we can construct a family of $X$ structures with holonomy $\rho_t$ on a small time interval $t \in (0, \delta)$.

**Proof sketch.** The proof parallels the proof of Proposition 2.3 in Porti [Por98]. Think of $\text{HP}^n = X_0$ as the normal bundle of $P$ inside $X_t$ but with normal distances scaled by $t$. For each $t$, we exponentiate HP charts to build finitely many $X_t$ charts. In order to fit together properly, the charts need to be adjusted. We show that the holonomy compatibility condition guarantees that for small $t$ these adjustments are small enough and the resulting structure is robust.

Together, Propositions 1 and 2 prove:

**Theorem 3.** Let $M$ be a compact manifold with boundary and let a path of hyperbolic (resp. anti de Sitter) structures on $M$ that degenerate to a transverse hyperbolic foliation be given. Then, after appropriate re-parameterization, a transition to anti de Sitter (resp. hyperbolic) structures can be constructed if and only if the transition can be constructed on the level of representations.

Note that while this theorem applies in broader generality than Theorem 1 from the Introduction, it does not guarantee any control of the geometry at the boundary. We study behavior near the boundary in Section 3.

### 2.4 Example: singular torus

We give an illustrative example in dimension $n = 2$ of transitioning singular structures on a torus. Let $F_2 = \langle a, b \rangle$ be the free group on two generators. For $t > 0$ define the following representations into $\text{SO}(\eta+1) \cong \text{SO}(2,1)$:

$$
\rho_t(a) = \begin{pmatrix} 3 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho_t(b) = \begin{pmatrix} \sqrt{1+t^2} & 0 & t \\ 0 & 1 & 0 \\ t & 0 & \sqrt{1+t^2} \end{pmatrix}.
$$

For small $t$, the commutator $\rho_t[a, b]$ is elliptic, rotating by an amount $\theta(t) = 2\pi - 2t + O(t^2)$. These representations describe a family of hyperbolic cone tori with cone angle $\theta_t$. As $t \to 0$ these tori collapse onto a circle (the geodesic representing $a$). Next, we rescale this family using the projective change of coordinates $c_{t^2}$ to produce a limiting half-pipe representation:

$$
c_{t^2}^{-1} \rho_t(a) c_{t^2} = \rho_t(a) \quad \text{constant}.
$$

$$
c_{t^2}^{-1} \rho_t(b) c_{t^2} = \begin{pmatrix} \sqrt{1+t^2} & 0 & t^2 \\ 0 & 1 & 0 \\ 1 & 0 & \sqrt{1+t^2} \end{pmatrix}.
$$
After applying $c_{t^2}^{-1}$, the fundamental domains for the hyperbolic cone tori limit to a fundamental domain for a singular HP structure on the torus (see figure 2). The commutator
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
2 & -2\sqrt{2} & 1
\end{pmatrix}
\]
fixes the singular point and shears in the degenerate direction. This half-pipe isometry can be thought of as an infinitesimal rotation in $\mathbb{H}^2$.

Figure 2: Fundamental domains for hyperbolic cone tori collapsing to a circle (shown in red). The collapsing structures are rescaled to converge to a robust HP structure (right).

Next we consider a family of singular AdS$^2$ structures on the torus given by the $\text{SO}(\eta_{-1}) \cong \text{SO}(1,2)$ representations defined for $t < 0$:

\[
\sigma_t(a) = \begin{pmatrix}
3 & 2\sqrt{2} & 0 \\
2\sqrt{2} & 3 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad \sigma_t(b) = \begin{pmatrix}
\sqrt{1-t^2} & 0 & t \\
0 & 1 & 0 \\
t & 0 & \sqrt{1-t^2}
\end{pmatrix}.
\]

Here the commutator $\sigma_t[a, b]$ acts as a Lorentz boost by hyperbolic angle $\varphi(t) = 2t + O(t^2)$ about a fixed point in AdS$^2$. These representations describe a family of singular AdS tori with hyperbolic cone angle $\varphi(t)$. The singular point is the Lorentzian analogue of a cone point in Riemannian geometry. We describe the three-dimensional version of this singularity in more detail in Section 3. Again, as $t \to 0$ these tori collapse onto a circle (the geodesic representing $a$). Similar to the above, we have that

\[
c_{t^2}^{-1} \sigma_t(b) c_{t^2} \xrightarrow{t \to 0} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix},
\]

showing that the limiting HP representation for these AdS structures is the same as for the above hyperbolic structures. So we have described a transition on the level of representations. Indeed, applying $c_{t^2}^{-1}$ to fundamental domains for the collapsing AdS structures gives the same limiting HP structure as in the hyperbolic case above.

3 Singular three dimensional structures

One main focus of the author’s thesis work is to study geometric transitions in the context of singular compact three dimensional structures. In this section, we restrict our attention to hyperbolic cone structures and their AdS analogues, tachyon structures. We assume the reader is familiar with hyperbolic cone structures, which are widely studied and play
crucial roles in the proofs of the Orbifold Theorem [BLP05, CHK00] and the Bers Density Conjecture [Bro07]. Some basics of AdS and HP geometry are given in Sections 3.1 and 3.2 respectively.

### 3.1 AdS geometry and singularities

As in hyperbolic geometry, the orientation-preserving isometries $H_L$ of $\text{AdS}^3 = \mathbb{X}^3 - 1$ that preserve a geodesic axis $L$ form a 2-dimensional subgroup. In the case that the axis $L$ is space-like, $H_L \cong \mathbb{R} + \mathbb{R} + \mathbb{Z}/2$, generated by translating along the axis, shearing perpendicular to the axis, and a half rotation around the axis. The shearing isometries, which fix $L$ pointwise, act locally as Lorentz boosts perpendicular to $L$ (see figure 3). The lifted group $\tilde{H}_L$, of isometries of $\tilde{\mathbb{X}}^3 - 1 - L$, is isomorphic to $\mathbb{R} + \mathbb{R} + \pi \mathbb{Z}$ where the discrete factor consists of half rotations around $L$. We note that the rotational part of an element of $H_L$ can not be deformed.

![Figure 3: Action of a Lorentz boost of hyperbolic angle $\varphi = \frac{1}{2}$ on a neighborhood of the origin in $\mathbb{R}^{1,1}$. Contour lines of constant square-norm, which are preserved by the transformation, are drawn for reference. This picture describes a cross section of the action of an AdS$^3$ boost isometry perpendicular to its fixed space-like axis.](image)

AdS structures on a 3-manifold with torus boundary can occasionally be Dehn filled (compare hyperbolic structures). In this case the filled manifold will be Seifert fibered. However, fillable structures are less frequent than in hyperbolic geometry. The (space-like) analogue of a cone singularity is called a tachyon:

**Definition 3.** A tachyon is a singularity along a space-like geodesic axis characterized by the property that the meridional holonomy around the singular axis is a full $2\pi$ rotation plus a boost. The magnitude of the boost is called the tachyon mass.

**Remark.** The tachyon mass is also referred to as the hyperbolic angle. Tachyon structures and other singular AdS structures are studied in [BBS09].

### 3.2 Half-pipe geometry and singularities

As half-pipe geometry is a projective geometry, lines and planes are naturally defined. Specifically, a line in $\text{HP}^3 = \mathbb{X}^3_0$ comes simply from intersection with any two-dimensional linear subspace of $\mathbb{R}^4$. Similarly, planes come from intersecting with three-dimensional subspaces. Recall that we choose $P$, defined by the equation $x_4 = 0$, as the distinguished codimension one hyperbolic plane in $\text{HP}^3$. In fact, any plane coming from intersection with a non-degenerate three-space inherits a hyperbolic metric. Note that any non-degenerate plane...
(resp. line) is the image of $P$ (resp. a geodesic in $P$) under a shear isometry $S_v = \begin{pmatrix} I_3 & 0 \\ v & 1 \end{pmatrix}$ with $v \in \mathbb{R}^3$. The action of $S_v$ is entirely in the $x_4$ direction (see Figure 4) and either fixes a geodesic $L$ in $P$ or moves $P$ off of itself entirely. In the first case, we may think of $S_v$ as an infinitesimal rotation or as an infinitesimal boost about the axis $L$. The magnitude of $S_v$, also referred to as the infinitesimal angle, is measured by $\|v\|_2$. The subgroup $H_L$ of HP isometries preserving $L$ is isomorphic to $\mathbb{R}+\mathbb{R}+\mathbb{Z}/2$ generated by translations along $L$, shear isometries fixing $L$, and a half rotation. The lifted group $\widetilde{H}_L$, of isometries of $\widetilde{X}_0 - L$, is isomorphic to $\mathbb{R}+\mathbb{R}+\pi\mathbb{Z}$. As in the AdS case, the rotational part can not be deformed.

Figure 4: Action of a shear isometry of infinitesimal angle $\varphi = \frac{1}{2}$ on a neighborhood of the fixed point in $\mathbb{H}P^2$. Contour lines of constant distance from the fixed point, which are preserved by the transformation, are drawn for reference. This picture describes a cross section of the action of an $\mathbb{H}P^3$ shear isometry perpendicular to its fixed (non-degenerate) axis.

The analogue of a cone singularity in HP geometry is called an infinitesimal cone singularity:

**Definition 4.** An infinitesimal cone singularity is a singularity along a non-degenerate line $L$ characterized by the property that the meridional holonomy around the axis is a full $2\pi$ rotation plus a shear about $L$. The shearing magnitude is called the infinitesimal cone angle.

### 3.3 Transition theory

We first consider hyperbolic cone structures which degenerate to a transverse hyperbolic foliation. In this case each cone angle must approach a multiple of $\pi$. We assume for simplicity that the limiting cone angles are $2\pi$. The limiting singular transverse hyperbolic foliation will have fillable singularities (see Section 4 for more on transverse hyperbolic foliations with Dehn surgery singularities).

**Proposition 3** (Rescaling - with cone singularities). Consider a family of compact hyperbolic cone manifolds that degenerates to a transverse hyperbolic foliation such that the cone angles limit to $2\pi$. Then, after appropriate coordinate changes, the structures converge projectively to a half-pipe structure with infinitesimal cone singularities.

**Proof sketch.** We use the same techniques as for Proposition 1, but we must pay special attention to the degeneration of the geometry near the boundary. We construct special collapsing cylindrical coordinates near the singular axis. Using these coordinates, we describe a neighborhood of the singular axis as a family of elliptical tubular neighborhoods that flatten out as the structures collapse. After appropriate rescaling, these flattening tubes
converge to robust tubular neighborhoods of an HP infinitesimal cone singularity. As a nice consequence, the construction leads to a definition of “cylindrical” coordinates around an axis in HP$^3$.

A similar statement holds for AdS structures with tachyon singularities. In this case, the tachyon mass goes to zero as the structures collapse.

**Proposition 4** (Rescaling - with tachyon singularities). Consider a family of compact AdS tachyon structures that degenerates to a transverse hyperbolic foliation. Then, after appropriate coordinate changes, the structures converge projectively to a half-pipe structure with infinitesimal cone singularities.

**Proof sketch.** Similar to the above, but without a good analogue of cylindrical coordinates about a space-like axis in AdS, the computations are a bit more messy.

The previous two propositions generalize Proposition 1 to establish that HP structures with infinitesimal cone singularities are the natural intermediary structures between hyperbolic cone structures and AdS tachyon structures. Next we generalize Proposition 2 in this context:

**Proposition 5** (Regeneration). Let $M$ be a compact manifold with torus boundary endowed with an HP structure $H$ having infinitesimal cone singularities at the boundary. Let $X = \mathbb{H}^3$ (resp. $X = \text{AdS}^3$) and let $\rho_t : \pi_1 M \to \text{Isom}(X)$ be a family of representations such that

- $\rho_t$ is compatible to first order at time $t = 0$ with the holonomy of $H$ (in the sense of Equation 1).
- $\rho_t(\mu)$ is elliptic (resp. a boost) for all meridians $\mu$ of the singular curves.

Then we can construct a family of hyperbolic cone structures (resp. AdS tachyon structures) with holonomy $\rho_t$ on a small time interval $t \in (0, \delta)$.

**Proof sketch.** Proposition 2 builds developing maps for the desired structures outside of a neighborhood of the boundary. Using “flattening” tubular neighborhoods, as described in Proposition 3, we build model developing maps in the neighborhood of the boundary having the prescribed holonomy. The structure around these model singularities “flattens out” and collapses as $t \to 0$. After appropriate rescaling, these model neighborhoods of the singular curves converge and match up with the given HP structure. It then follows that the model tubular neighborhoods (after some adjustment) can be fit together with the rest of the structure.

With these components in place, we prove Theorem 1, restated here for convenience:

**Theorem 1.** Let $h_t$ be a path of hyperbolic cone structures on a compact manifold $M$ defined for $t > 0$. Suppose that as $t \to 0$, $h_t$ limits to a transverse hyperbolic foliation with holonomy $\rho : \pi_1 M \to SO(2,1)$. Suppose further that $H^1(\pi_1 M, S\mathfrak{o}(2,1)) = \mathbb{R}$. Then we can construct a geometric transition: a path of projective structures $p_t$ defined in a neighborhood of $t = 0$ so that

- $p_t$ is equivalent to $h_t$ for $t > 0$ and
- $p_t$ is equivalent to an AdS tachyon structure for $t < 0$. 

\[ \text{\blacksquare} \]
The same result holds when the roles of hyperbolic and AdS structures are interchanged.

Proof sketch. By Proposition 3, the degenerating path $h_t$ can be re-parameterized to converge to an HP structure with infinitesimal cone singularities. Proposition 5 then produces the desired path of AdS tachyon structures provided we can construct a path of representations satisfying the hypotheses of the proposition. The cohomology condition guarantees that the SO(2, 1) representation variety is smooth at $\rho$. Since SO(3, 1) is (up to finite index) the complexification of SO(2, 1), it follows that the SO(3, 1) representation variety is smooth. Similarly, since SO(2, 2) is (up to finite index) a product of two copies of SO(2, 1), the SO(2, 2) variety is also smooth. So the desired path of representations exists.

Remark. We note that the cohomology condition in Theorem 1 holds for a class of examples coming from punctured torus bundles (see Theorem 2 in Section 4.2). Other examples come from small Seifert fiber spaces with a curve transverse to the fiber removed.

3.4 Dehn surgery singularities

The cone and tachyon singularities studied so far in this section are special cases of the more general class of Dehn surgery singularities. Consider a general $(X, G)$ geometry (in the sense of Thurston [Thu80]), with $L$ a line in $X$. Let $H_L \subset G$ be the subgroup that preserves $L$ and let $\tilde{H}_L$ be the group of diffeomorphisms of $\tilde{X} - L$ that are lifts of elements of $H_L$.

Definition 5. A Dehn surgery singularity for an $(X, G)$ structure is modeled locally on $\tilde{X} - L/\Gamma$ where $\Gamma$ is a discrete subgroup of $\tilde{H}_L$.

The deformation theory of hyperbolic three manifolds with Dehn surgery singularities has been studied by many authors going back to Thurston [Thu80]. For a given link complement $M^3$, the boundary of hyperbolic Dehn surgery space provides many examples of co-dimension one transverse hyperbolic foliations that are limits of degenerating singular hyperbolic structures. The above definition extends the notion of Dehn surgery singularities to the context of Lorentzian structures, in particular AdS structures. In many cases, the boundaries of hyperbolic Dehn surgery space and AdS Dehn surgery space actually meet at the locus of transverse hyperbolic foliations. The singular transition theory described here and in Section 4 allows for the description of paths of structures passing freely from one Dehn surgery space to the other.

4 Ideal triangulations

In this section, we construct transverse hyperbolic foliations, anti de Sitter structures, and half-pipe structures out of ideal tetrahedra. Assume that $M^3$ has a fixed topological ideal triangulation $T = \{T_1, \ldots, T_n\}$ and that $\partial M$ is a union of tori.

4.1 Shape Parameters

Recall that a hyperbolic ideal tetrahedron is described up to isometry by a complex shape parameter. Defining a hyperbolic structure on $(M, \mathcal{T})$ amounts to assigning complex shape parameters $z_i$ to each tetrahedron $T_i$ in $\mathcal{T}$ such that the $z_i$ satisfy the edge consistency equations. The solutions to these algebraic equations make up the deformation variety. A point
(z_j) for which all tetrahedra are positively oriented (i.e. Im(z_j) > 0) determines a robust hyperbolic structure. Y.E. Choi shows in [Cho04] that the deformation variety is smooth near such a point. This method for producing \( \mathbb{H}^3 \) structures has been studied by many going back to Thurston [Thu80]. A degenerate \( \mathbb{H}^2 \) ideal tetrahedron is a hyperbolic tetrahedron with real shape parameter. Such a tetrahedron is collapsed onto a plane. The real deformation variety of real solutions to the edge consistency equations describes transverse hyperbolic foliations built out of these degenerate tetrahedra (under some assumptions). These transverse foliations have Dehn surgery singularities as defined by Hodgson in [Hod86].

In [Dan10b], the author constructs deformation spaces of AdS structures built of anti de Sitter ideal tetrahedra. All faces of these tetrahedra are space-like hyperbolic ideal triangles. It turns out that the shape of a tetrahedron is determined by a parameter \( z \) lying in the algebra \( \mathbb{R} \times \tau \) where \( \tau \) commutes with \( \mathbb{R} \) and satisfies \( \tau^2 = 1 \). Given an element \( a + b\tau \), its square-norm is defined by \( |a + b\tau|^2 = a^2 - b^2 \). We call \( a \) the real part, and \( b \) the imaginary part of \( a + b\tau \). The parameter \( z \) determines an ideal tetrahedron if and only if \( z, \frac{1}{1-z}, \) and \( \frac{z-1}{z} \) (are defined and) have positive square-norm (such elements are called space-like). The AdS deformation variety consists of solutions to the edge consistency equations over the algebra \( \mathbb{R} + \mathbb{R}\tau \). A point \((z_j)\) for which all tetrahedra are positively oriented (\( \text{Im}(z_j) > 0 \)) determines a robust AdS structure with Dehn surgery singularities (see Definition 3). However, the deformation variety may not be smooth at such a point in contrast to the hyperbolic case.

Though it is not a field (the elements with square-norm zero are not invertible), the algebra \( \mathbb{R} + \mathbb{R}\tau \) plays a similar role for AdS\(^3\) geometry as \( \mathbb{C} \) plays for hyperbolic geometry. We discuss a few important properties. First, the square-norm \( | \cdot |^2 \) makes \( \mathbb{R} + \mathbb{R}\tau \) into the Lorentz plane \( \mathbb{R}^{1,1} \). The space-like elements of \( \mathbb{R} + \mathbb{R}\tau \) acting by multiplication form the group of similarities of \( \mathbb{R}^{1,1} \). Second, the two light-like idempotents \( \frac{1+\tau}{2}, \frac{1-\tau}{2} \) decompose the algebra as a direct sum of two copies of \( \mathbb{R} \). This implies that a triangulated AdS structure is the same information as two triangulated transverse hyperbolic foliations (satisfying some conditions). So, the deformation variety is a product.

**Remark.** In his thesis, the author constructs a half-space model of AdS\(^3\) for which the isometries are given by \( \text{PSL}_2(\mathbb{R} + \mathbb{R}\tau) \) acting by Mobius transformations. The ideal boundary in this model is a projectivization of \( (\mathbb{R} + \mathbb{R}\tau)^2 \).

### 4.2 Regeneration results

In the context of ideal triangulations, the regeneration problem becomes more straightforward, especially in the presence of smoothness assumptions.

**Proposition 6.** If the real deformation variety is smooth at a point \((z_j) \in \mathbb{R}^N\), then any positive tangent vector \((v_j) \in \mathbb{R}^N\) determines regeneration to robust hyperbolic and anti de Sitter structures.

**Proof.** The imaginary tangent vector \((iv_j)\) can be integrated to give a path of complex solutions to the edge consistency equations. Similarly, the imaginary tangent vector \((\tau v_j)\) can be integrated to give a path of \( \mathbb{R} + \mathbb{R}\tau \) solutions. In both cases the solutions have positive imaginary part, so they determine robust structures. \( \square \)

In light of this proposition, we ask the following question:

**Question.** For which triangulated three-manifolds \((M, \mathcal{T})\) is the real deformation variety smooth with positive tangent vectors?
The author answers this question in a special case by proving the following theorem in [Dan10b]:

**Theorem 2.** Let $M^3$ be a punctured torus bundle with anosov monodromy and let $T$ be the monodromy ideal triangulation on $M$. The deformation variety of transverse hyperbolic foliations on $(M, T)$ is a smooth one dimensional variety, parametrized by the trace of the puncture curve. Further, the tangent direction to this variety has all positive components.

*Proof remark.* To prove Theorem 2, the author makes use of Gueritaud’s description of Thurston’s equations, given in [Ga06].

*Corollary.* Any transverse hyperbolic foliation on $(M, T)$ regenerates to both hyperbolic and anti de Sitter structures.

*Corollary.* The deformation variety of anti de Sitter structures on $(M, T)$ is a smooth one $\mathbb{R} + \mathbb{R} \tau$ dimensional variety parametrized by the $\mathbb{R} + \mathbb{R} \tau$ length of the puncture curve. In particular, tachyon structures are parametrized by the tachyon mass and the mass can be increased without bound.

*Remark.* The $\mathbb{R} + \mathbb{R} \tau$ length, referred to in the above corollary, of an AdS$^3$ isometry is defined analogously to the complex length of a hyperbolic isometry.

### 4.3 Triangulated transitions and HP ideal tetrahedra

The shape parameter algebra of hyperbolic tetrahedra intersects that of AdS tetrahedra exactly in the shape parameter algebra of degenerate $\mathbb{H}^2$ tetrahedra: $\mathbb{C} \cap (\mathbb{R} + \mathbb{R} \tau) = \mathbb{R}$. In some sense, this intersection is transverse. In order to construct smooth transitions on $(M, T)$, we enlarge the shape parameter coefficient algebra to the *generalized Clifford algebra* $\mathcal{C}$ generated by $i$ and $\tau$:

$$\mathcal{C} = \langle 1, i, \tau : i^2 = -1, \tau^2 = +1, i\tau = -\tau i \rangle.$$

Consider the following path in $\mathcal{C}$ (defined for $t \neq 0$):

$$\mathcal{J}(t) = \frac{(1 + t)i + (1 - t)\tau}{2|t|^\frac{1}{2}}.$$

Note that for $t > 0$, $\mathcal{J}^2 = -1$, while for $t < 0$, $\mathcal{J}^2 = +1$. We define the following smooth path of two dimensional sub-algebras:

$$\mathcal{B}_t = \mathbb{R} + \mathbb{R} \cdot |t|^\frac{1}{2} \mathcal{J}(t).$$

The path $\mathcal{B}_t$ satisfies the following properties:

- If $t > 0$ then $\mathcal{B}_t \cong \mathbb{C}$ via the isomorphism $\mathcal{J}(t) \mapsto i$.
- If $t < 0$ then $\mathcal{B}_t \cong \mathbb{R} + \mathbb{R} \tau$ via the isomorphism $\mathcal{J}(t) \mapsto \tau$.
- $\mathcal{B}_0 = \mathbb{R} + \mathbb{R} \sigma$, where $\sigma = \frac{i + \tau}{2}$. Note that $\sigma^2 = 0$.

To describe geometric transitions of triangulated structures, one constructs a smooth path of solutions to the edge consistency equations over the varying algebra $\mathcal{B}_t$. A solution for some $t \neq 0$ is interpreted as an assignment of shape parameters for either hyperbolic (if...
t > 0) or anti de Sitter (if t < 0) ideal tetrahedra via the isomorphisms given above. The transitional shape parameter algebra \( B_0 = \mathbb{R}^+\mathbb{R}\sigma \) can be thought of as the cotangent bundle of \( \mathbb{R} \). In fact, this algebra describes shape parameters for half-pipe ideal tetrahedra or HP tetrahedra.

**Remark.** In his thesis, the author builds a half-space model of HP\(^3\) for which the isometries \( G_{HP} \) are given by \( \text{PSL}_2(\mathbb{R} + \mathbb{R}\sigma) \) acting by Mobius transformations. The ideal boundary in this model is a projectivization of \((\mathbb{R} + \mathbb{R}\sigma)^2\).

We formalize the discussion of transitioning shape parameters with the following definition:

**Definition 6.** A geometric transition on \((M, T)\) is described by a smooth path of shape parameters \((z_j(t))\) that solve the edge consistency equations over the varying algebra \( B_t \) such that for \( t > 0 \), \((z_j(t))\) determines a hyperbolic structure, for \( t < 0 \), \((z_j(t))\) determines an AdS structure, and \((z_j(0))\) determines an HP structure.

**Remark.** A geometric transition on \((M, T)\) determines a transition on \( M \) in the sense of Theorem 1, except that the singularities at the boundary may lie in the more general class of Dehn surgery singularities (see Definition 3).

Similar to the hyperbolic and AdS case, the shape parameters at \( t = 0 \) determine a robust HP structure on \( M \) if each \( z_j(0) = a_j + b_j \) has imaginary part \( b_j > 0 \). The real parts \( a_j \) are shape parameters for the limiting transverse hyperbolic foliation on \((M, T)\) when the hyperbolic and AdS structures are allowed to collapse. The imaginary parts \( b_j \) describe the derivatives of the imaginary parts for the complex (resp. \( \mathbb{R} + \mathbb{R}\tau \)) shape parameters at \( t = 0^+ \) (resp. \( t = 0^- \)).

**Proposition 7.** The data of a geometric transition on \((M, T)\) is equivalent to the following:

1. A path of hyperbolic structures on \((M, T)\), defined for \( s > 0 \), determined by shape parameters \( z_j(s) = c_j(s) + d_j(s)i \).

2. A path of AdS structures on \((M, T)\), defined for \( s < 0 \), determined by shape parameters \( z_j(s) = c_j(s) + d_j(s)\tau \).

3. An HP structure determined by shape parameters \( z_j(0) = c_j(0) + d_j(0)\sigma \)

where, for all \( j \), \( c_j \) and \( d_j \) are smooth functions on a neighborhood of \( s = 0 \) with \( d_j(0) = 0 \).

**Proof.** To convert from the data given in the proposition to a smooth geometric transition, use the re-parameterization \( t = \text{sign}(s)s^2 \). When \( s > 0 \), replace \( i \) with \( \Im(t) \). When \( s < 0 \), replace \( \tau \) with \( \Im(t) \).

The above proposition and proposition 6 give the following corollary.

**Corollary.** If the real deformation variety is smooth at a point \((z_j) \in \mathbb{R}^N\), then any positive tangent vector \((v_j) \in \mathbb{R}^N\) determines a geometric transition on \((M, T)\).
4.4 Example: figure eight knot complement

Let \( M \) be the figure eight knot complement. Let \( T \) be the decomposition of \( M \) into two ideal tetrahedra (four faces, two edges, and one ideal vertex) well-known from [Thu80]. The edge consistency equations reduce to the following:

\[
z_1(1 - z_1)z_2(1 - z_2) = 1. \tag{2}
\]

We are interested in solutions to (2) over the reals, which determine transverse hyperbolic foliations. Suppose \( z_1 \in \mathbb{R} \). Then \( z_2 \) is real if and only if \( z_1 < 0 \) or \( z_1 > 1 \). In the first case, we must have \( z_2 > 1 \) and in the latter we must have \( z_2 < 0 \).

**Remark.** There are solutions to (2) with both \( z_1, z_2 > 1 \) or both \( z_1, z_2 < 0 \). However, the angular part of the holonomy around the edges is not \( 2\pi \) (one edge has \( 0\pi \) and the other has \( 4\pi \)). These are not considered legitimate solutions to the edge consistency equations and we ignore them.

It is straightforward to show that the variety of real solutions to (2) (with angular holonomy \( 2\pi \) around each edge) is a smooth one-dimensional variety with positive tangent vectors. Thus, any transverse hyperbolic foliation on \((M, T)\) regenerates to robust hyperbolic and AdS structures by Proposition 6. As \( M \) is a punctured torus bundle, this is a special case of Theorem 2 above.

Next, we consider hyperbolic cone structures on \( M \), with singular meridian being the curve around the puncture (this is also the longitude of the knot). Such a structure, with cone angle \( \theta < 2\pi \) is constructed by solving the equations

\[
z_1(1 - z_1) = z_2^{-1}(1 - z_2)^{-1} = e^{i\frac{\theta}{2}}
\]

over \( \mathbb{C} \). Similarly, AdS tachyon structures with hyperbolic angle \( \varphi \) are constructed by solving the equations

\[
z_1(1 - z_1) = z_2^{-1}(1 - z_2)^{-1} = -e^{\frac{\varphi}{2}}
\]

over \( \mathbb{R} \). In order to construct a smooth transition between these two types of structures, we consider a generalized version of these equations defined over the transitioning family \( \mathcal{B}_t \) of sub-algebras of \( \mathcal{C} \) (see Section 4.3). The idea is to replace \( i \) in (3) (resp. \( \tau \) in (4)) by the algebraically equivalent elements \( \mathfrak{I}(t) \). We recall the definition

\[
\mathfrak{I}(t) = \frac{(1 + t)i + (1 - t)\tau}{2\sqrt{|t|}} \in \mathcal{C}
\]

and that for \( t > 0 \), \( \mathfrak{I}^2 = -1 \), while for \( t < 0 \), \( \mathfrak{I}^2 = +1 \). The generalized version of (3) and (4) that we wish to solve is

\[
z_1(1 - z_1) = z_2^{-1}(1 - z_2)^{-1} = -e^{\sqrt{|t|}\mathfrak{I}(t)}. \tag{5}
\]

Note that the right hand side (which can be defined in terms of Taylor series) is a smooth function of \( t \). In fact, solving (5) over the varying algebra \( \mathcal{B}_t = \mathbb{R} + \mathbb{R}\sqrt{|t|}\mathfrak{I}(t) \) for small \( t \), gives a smooth path \((z_1(t), z_2(t))\) of shape parameters for transitioning structures. For
$t > 0$, $(z_1, z_2)$ determines a hyperbolic cone structure with cone angle $\theta = 2\pi - \sqrt{|t|}$. For $t < 0$, $(z_1, z_2)$ determines an AdS tachyon structure with hyperbolic angle $\phi = \sqrt{|t|}$. At $t = 0$, interpreting $\sqrt{|t|}I(t)$ as $\sigma = i + \tau$, we get shape parameters for a robust half-pipe structure: $z_1(0) = \frac{1 + \sqrt{5}}{2} + \frac{1}{\sqrt{5}}\sigma$, $z_2(0) = \frac{1 - \sqrt{5}}{2} + \frac{1}{\sqrt{5}}\sigma$.

Figure 5: The C-length of the singular curve is plotted as hyperbolic cone structures (red) transition to AdS tachyon structures (blue). After rescaling (solid lines), the transition is realized as a smooth path passing through a half-pipe structure.

5 Borromean Rings Example

Here we will construct examples of transitioning structures on the Borromean rings complement $M$ (with one boundary component required to be a parabolic cusp). In this case, the $\text{SO}(2,1)$ character variety is singular at the locus of degenerated structures, so this example does not fit into the theory we have developed so far in Sections 3 and 4. We will see that a transitional HP structure on $M$ can be deformed to nearby HP structures that do not regenerate to hyperbolic structures. However, in this case these nearby HP structures do regenerate to AdS structures. Such examples can be constructed using ideal tetrahedra and the methods of Section 4 (in fact, $M$ is the union of eight tetrahedra). However, for brevity, we observe this phenomenon only at the level of representations.

5.1 The PSL description of isometry groups

It will be convenient to use a different model of $\text{AdS}^3$ in this section, namely the half-space model (see [Dan10c] for a construction). The isometry group in this model is given by $\text{PSL}_2(\mathbb{R} + \mathbb{R} \tau)$ acting by Mobius transformations. Note that the idempotents $\frac{1 + \tau}{2}$ and $\frac{1 - \tau}{2}$ decompose $\mathbb{R} + \mathbb{R} \tau$ as a direct sum of two copies of $\mathbb{R}$. Thus $\text{PSL}_2(\mathbb{R} + \mathbb{R} \tau) \cong \text{PSL}_2\mathbb{R} \times \text{PSL}_2\mathbb{R}$.

It is convenient to think of $\text{PSL}_2\mathbb{C}$ and $\text{PSL}_2(\mathbb{R} + \mathbb{R} \tau)$ as intersecting in $\text{PSL}_2\mathbb{R}$, the subgroup preserving a distinguished hyperbolic plane $P$ (and its orientation) in both $\mathbb{H}^3$ and $\text{AdS}^3$.

Next, the isometry group of $\text{HP}^3$ can be described as $\text{PSL}_2(\mathbb{R} + \mathbb{R} \sigma)$. Recall that $\sigma^2 = 0$ and that we think of $\mathbb{R} + \mathbb{R} \sigma$ as the cotangent bundle of $\mathbb{R}$. Similarly, $\text{PSL}_2(\mathbb{R} + \mathbb{R} \tau)$ can be thought of as the cotangent bundle of $\text{PSL}_2\mathbb{R}$. We interpret an element $A + B\sigma \in \text{PSL}_2(\mathbb{R} + \mathbb{R} \sigma)$...
as an infinitesimal deformation of the $\text{PSL}_2\mathbb{R}$ element $A$ in the pure imaginary direction into either $\text{PSL}_2\mathbb{C}$ or $\text{PSL}_2(\mathbb{R} + \mathbb{R} \tau)$ (compare to the Remark at the end of Section 2.2).

We rephrase Proposition 2 in this framework:

**Proposition 8 (Regeneration).** Let $M^3$ be a compact manifold whose interior has an HP structure with holonomy $\Phi : \pi_1 M \to \text{PSL}_2(\mathbb{R} + \mathbb{R} \sigma)$. Let $\kappa = i$ or $\kappa = \tau$ and suppose that for $t \geq 0$, $\rho_t : \pi_1 M \to \text{PSL}_2(\mathbb{R} + \mathbb{R} \kappa)$ is a path of representations satisfying the following:

(i) $\rho_0 = \text{Re}(\Phi)$

(ii) $\text{Im}(\rho'_0) = \text{Im}(\Phi)$.

Then, for short time, $\rho_t$ is the holonomy of a family of geometric structures, hyperbolic if $\kappa = i$, or anti de Sitter if $\kappa = \tau$.

We say that a path $\rho_t$ as in the above proposition is compatible to first order with the HP representation $\Phi$. This is exactly the condition defined by Equation 1 of Section 2 translated into this framework.

### 5.2 Borromean Rings Example: $\text{PSL}_2\mathbb{R}$ character variety

Consider the three-torus $T^3$ defined by identifying opposite faces of a cube. Now, define $M^3 = T^3 - \{\alpha, \beta, \gamma\}$, where $\alpha, \beta, \gamma$ are disjoint curves freely homotopic to the generators $a, b, c$ of $\pi_1 T^3$ as shown in Figure 6. Then $M$ is homeomorphic to the complement of the Borromean rings in $S^3$.

Figure 6: We remove the three curves $\alpha, \beta, \gamma$ shown in the diagram from the three-torus $T^3$ (opposite sides of the cube are identified). The resulting manifold $M$ is homeomorphic to the complement of the Borromean rings in $S^3$.

Borromean rings in $S^3$ (this is stated in [Hod86]). A presentation for $\pi_1 M$ is given by:

$$\pi_1 M = \langle a, b, c : [a, b], c = [c, b^{-1}], a = 1 \rangle.$$  

We study the character variety $X_{\text{par}}(M)$ of representations $\rho : \pi_1 M \to \text{PSL}_2\mathbb{R}$ up to conjugacy such that $\rho[a, b]$ is parabolic (and so $\rho(c)$ is parabolic with the same fixed point). These representations correspond to transversely hyperbolic foliations which are “cursed” at one boundary component and have Dehn surgery singularities at the other two boundary components.
Let $T$ denote the punctured torus, with $\pi_1 T = \langle a, b \rangle$. Then $\pi_1 T \to \pi_1 M$, so that $X_{\text{par}}(M) \to X_{\text{par}}(T)$. The elements of $X_{\text{par}}(T)$ determine hyperbolic punctured tori (with a cusp at the puncture). A representation $\rho : \pi_1 T \to \text{PSL}_2 \mathbb{R}$ satisfies the parabolic condition if and only if $\rho(a), \rho(b)$ are parabolic elements with

$$\sinh \frac{l(a)}{2} \sinh \frac{l(b)}{2} \sin \varphi = 1$$

where $l(a), l(b)$ are the translation lengths of $\rho(a), \rho(b)$ respectively and $\varphi$ is the angle between the axes. To lift such a representation to a representation of $\pi_1 M$, we must assign $\rho(c)$ so that the relations of $\pi_1 M$ are satisfied. Since $\rho(c)$ must commute with the parabolic $\rho[a, b]$, $\rho(c)$ is parabolic with the same fixed point. Let $x$ denote the amount of parabolic translation of $\rho(c)$ relative to $\rho[a, b]$, so if $\rho[a, b] = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, then $\rho(c) = \begin{pmatrix} -1 & x \\ 0 & -1 \end{pmatrix}$. It turns out (by a nice geometric argument) that there are exactly two solutions for $x$:

Either $x = 0$ or $x = \frac{1}{2} \text{sech} \frac{l(a)}{2} \text{sech} \frac{l(b)}{2} \cot \varphi$.

This describes the character variety $X_{\text{par}}$ rather explicitly as the union of two irreducible two-dimensional components $X_T$ and $X_R$. The first component $X_T$ (‘T’ for Teichmüller) consists of the obvious representations with $\rho(c) = 1$ and $\rho(a), \rho(b)$ generating a hyperbolic punctured torus group. The associated transversely hyperbolic foliations are products (with two fillable singularities at $\alpha$ and $\beta$). The second component $X_R$ (‘R’ for regenerate) describes transversely hyperbolic foliations with more interesting structure. This component, in fact its complexification, is the relevant one for regenerating hyperbolic structures. Note that $X_T$ and $X_R$ meet (transversely) exactly at the locus of “rectangular” punctured tori ($\cot \varphi = 0$).

**Remark.** If we identify $X_T$ with the Teichmüller space $T_{1,1}$ of the punctured torus, then the singular locus for $X_{\text{par}}$, given by $X_T \cap X_R$, is exactly the *line of minima* for the curves $a$ and $b$. In other words, $X_T \cap X_R$ consists of the representations in $X_T$ where there is a relation between the differentials $dl(a)$ and $dl(b)$. The relevance of such a relation in the context of regeneration questions is discussed in Section 3.17 of [Hod86].

### 5.3 Regenerating 3D structures

Fix a particular rectangular punctured torus $\rho_0 : \pi_1 T \to \text{PSL}_2 \mathbb{R}$, and lift $\rho_0$ to $\pi_1 M$ by setting $\rho_0(c) = 1$ (this is the only possible lift). Let $v$ be a tangent vector at $\rho_0$, tangent to the component $X_R$ but transverse to $X_T$. For suitably chosen $v$, the representation $\rho_0 + \sigma v : \pi_1 M \to \text{PSL}_2(\mathbb{R} + \mathbb{R} \sigma)$ is the holonomy of a robust HP structure (which can be constructed from eight tetrahedra). Now, as the variety $X_R$ is smooth, the complexified variety $X_R^c$ is smooth at $\rho_0$. Thus the Zariski tangent vector $iv$ is tangent to a path $\rho_t : \pi_1 M \to \text{PSL}_2 \mathbb{C}$ which is compatible to first order with $\rho_0 + \sigma v$. By Proposition 8, the HP structure can be exponentiated to produce a path of hyperbolic structures with holonomy $\rho_t$ (or alternatively, this path of hyperbolic structures can be constructed directly using tetrahedra). Similarly, the variety $X_R^{\mathbb{R}+\mathbb{T}}$ is smooth at $\rho_0$ yielding a path of holonomies $\rho_t : \pi_1 M \to \text{PSL}_2(\mathbb{R} + \mathbb{T} \tau)$ with $\rho_0 = \tau v$ so that Proposition 8 then produces a regeneration to AdS structures with holonomy $\rho_t$. Thus, our HP structure is transitional. Actually, in the AdS case, the representations can be constructed a bit more directly. Let $\sigma_t : \pi_1 M \to \text{PSL}_2 \mathbb{R}$
be a path with $\sigma_0' = v$. Then, a path $\rho_t$ of $\text{PSL}_2(\mathbb{R} + \mathbb{R}\tau)$ representations with $\rho_0' = \tau v$ is defined by

$$\rho_t = \frac{1 + \tau}{2} \sigma_t + \frac{1 - \tau}{2} \mu_t.$$

### 5.4 An interesting flexibility phenomenon

The transitional HP structure from the previous sub-section, with holonomy $\rho_0 + \sigma v$, can be deformed in an interesting way. Just as in the case of homogeneous Riemannian structures, nearby HP structures are determined by nearby holonomy representations. In this case, we consider a deformation of the form

$$\rho_0 + \sigma (v + \epsilon u)$$

where $\epsilon u$ is a small tangent vector at $\rho_0$, tangent to the component $X_T$ and transverse to $X_R$ (see Figure 7). Now, $X^C_{\text{par}}(M)$ is the union of its irreducible components $X^C_T$ and $X^C_R$. 3

![Figure 7: A schematic picture of the PSL$_2\mathbb{R}$ character variety $X_{\text{par}}(M)$. The variety is the union of two irreducible two-dimensional components which meet at the locus of rectangular punctured torus representations (with $c = 1$). We let $\rho_0$ be one such representation, with $v$ tangent to one component, and $u$ tangent to the other.](image)

(locally at $\rho_0$). So, as $u$ and $v$ are tangent to different components of $X_{\text{par}}(M)$, any Zariski tangent vector of the form $w + i(v + \epsilon u)$, for $w$ real, is not integrable. Thus, the deformed HP structure does not regenerate to hyperbolic structures. However, it does regenerate to AdS structures. To see this, consider paths $\sigma_t$ and $\mu_t$ with derivatives $2v$ and $2\epsilon u$ respectively at $t = 0$. Then,

$$\rho_t = \frac{1 + \tau}{2} \sigma_t + \frac{1 - \tau}{2} \mu_t$$

gives a family of $\text{PSL}_2(\mathbb{R} + \mathbb{R}\tau)$ representations with $\rho_0' = (v - \epsilon u) + \tau(v + \epsilon u)$. Proposition 8 now implies that the deformed HP structure regenerates to AdS structures.

**Remark.** The author thanks Joan Porti for suggesting the possibility of this phenomenon.
References


