Research Statement - Jeffrey Danciger

My research program studies *locally homogeneous geometric manifolds* and *discrete subgroups of Lie groups*. Locally homogeneous geometric manifolds are abstract mathematical objects designed to model the universe we live in. The term locally homogeneous refers to the presence of a high degree of local symmetry which is captured by a Lie group called the *symmetry group*. It is this symmetry group which governs the geometry. Indeed, the roots of this subject go back to Felix Klein's 1872 Erlangen program, which promoted the philosophy that geometry *is* symmetry: the meaningful quantities we can measure in a geometric manifold (e.g. lengths or angles) are exactly those which are invariant under the symmetry group. There are many possible symmetry groups, each leading to different types of geometric manifolds that are useful for many purposes across mathematics and physics. Indeed, there are geometric manifolds that describe space-time in the theory of relativity, as well as phase spaces, configuration spaces, and many other useful structures. There are also many different geometric manifolds with the same local symmetry group. These have the same local properties, but can look very different at large scales. This large scale behavior is reflected in their holonomy groups (also called monodromy groups), which are subgroups of the symmetry group that are often *discrete*: their elements are isolated from one another, arranged much like the atoms in a crystal lattice. Below I outline some of my recent accomplishments, focusing primarily on *convex real projective structures*, *affine structures*, *hyperbolic structures*, *anti de Sitter structures* and discrete subgroups of the associated symmetry groups for these geometries.

1. Discrete subgroups in the projective general linear group and convex real projective structures. From the decades of intense work surrounding Thurston's geometrization program, we now have a deep understanding of discrete groups in three-dimensional hyperbolic geometry, i.e. Kleinian groups. In other Lie groups, in particular higher rank semi-simple Lie groups, much less is known about the structure of the discrete subgroups, and there is a long way to go before any major classification results will be possible. Immediately problematic is the lack of good notions for large classes of well-behaved discrete subgroups. While lattices are well-studied, these are extremely special examples in the vast universe of all discrete subgroups. This component of my research program is dedicated to finding, deforming, and eventually classifying special classes of discrete subgroups of semi-simple Lie groups.

In the setting of Kleinian groups, convex cocompact subgroups are a well-studied middle-ground between the very special class of cocompact lattices and the general class of all finitely generated discrete subgroups. My joint work [DGK5, DGK4] with Guéritaud and Kassel introduced and developed a new notion of *convex cocompact subgroups in real projective space*, which generalizes the notion from Kleinian groups to the higher rank setting of discrete subgroups of the projective general linear group $\mathrm{PGL}(V)$ acting on the projective space $\mathbb{P}(V)$, where V is a finite dimensional vector space over the reals. The study of these groups has since blossomed into an active subject; many of the initial questions we posed in our paper have already received focused attention from a number of different authors (see e.g. Canary's notes [C], Kassel's ICM paper [Ka2], the papers [GlMo, IZ1, IZ2], and the paper [We] by my former Ph.D. student Weisman). Natural questions motivating further study include: *How common are convex cocompact subgroups amongst the discrete finitely generated subgroups of* $\mathrm{PGL}(V)$ *that preserve a properly convex open set? Can one parameterize their deformation spaces in terms of natural geometric data?* Though general answers to these questions are likely out of reach in the near future, some optimism is found from recent joint work [DGKLM], with with Guéritaud, Kassel, Lee, and Marquis, that gives complete answers in the special case that our discrete subgroup is a Coxeter group generated by reflections in the faces of a convex polytope in $\mathbb{P}(V)$. In the context of Vinberg's theory of linear reflection groups, we find precise conditions for a reflection group representation to be convex cocompact, allowing us to give a parameterization of the deformation space of convex cocompact reflection groups and the corresponding convex projective reflection orbifolds.

We emphasize that in contrast to the setting of convex cocompact Kleinian groups, many convex cocompact subgroups in $\mathbb{P}(V)$ fail to be word hyperbolic as abstract groups. Generalizing Benoist's dichotomy for groups dividing properly convex sets, we show that a group acting convex cocompactly on a properly convex open domain is word hyperbolic if and only if the domain is strictly convex and differentiable along the full orbital limit set (see also [Z]). Further, a word hyperbolic convex cocompact group is *Anosov*. Anosov subgroups, introduced by Labourie [L] and generalized by Guichard-Wienhard [GW3], occupy a central role in the study of discrete subgroups of Lie groups from a dynamical perspective. Our notion of convex cocompactness gives natural geometric structures associated to Anosov subgroups.

Another recent joint paper with Guéritaud and Kassel [DGK6] proves combination theorems for discrete subgroups of $\mathrm{PGL}(V)$ preserving properly convex open sets in $\mathbb{P}(V)$, in the spirit of Klein and Maskit [Kl, Ma2]. The approach generalizes the original idea of ping pong dynamics for Kleinian groups to the setting of real projective space, with the set containment arguments from Klein and Maskit replaced by a notion of convex set occultation: one properly convex set blocking others from viewing each other. We use this general framework to obtain a number of general results on discrete linear groups, including that the free product of two $\mathbb{Z}$-linear groups is again $\mathbb{Z}$-linear. We also produce examples of semi-simple Lie groups $H < G$ of higher rank, and a discrete, Zariski-dense subgroup $\Gamma < G$ of infinite co-volume, such that $H \cap \Gamma$ is a lattice, resolving a conjecture attributed to Nori in the negative, see [CV]. Our combination theorems behave well with respect to the notion of convex cocompact subgroup in $\mathbb{P}(V)$, and are used to establish, for example, that two convex cocompact groups may be conjugated into position so that they generate a convex cocompact subgroup isomorphic to the free product, and also that compact convex projective structures with planar boundary, as found in my prior work [BDL] with Ballas and Lee, may be glued together convexly.

The deformation space of convex real projective structures on a closed manifold N forms a union of connected components of $\mathrm{Hom}(\pi_1 N, \mathrm{PGL}(V))$. In the case that $\dim N = 2$, there is just one connected component. My joint work [BDLM] (in preparation) with Ballas, Lee, and Marquis shows that, suprisingly, when $\dim N = 3$, the deformation space can have multiple connected components, answering a question posed by Benoist during a 2012 mini-course. We find such examples via a new theory of *convex real projective Dehn filling*, generalizing the classical picture of Thurston's hyperbolic Dehn filling space. Thurston's famous hyperbolic Dehn filling theorem states that for M an orientable, finite volume, hyperbolic three-manifold with one torus cusp, the closed manifold $N = M_{p/q}$, obtained by Dehn filling M along the slope p/q, admits a hyperbolic structure for all but finitely many filling slopes $p/q \in \mathbb{Q} \cup \{\infty\}$. We prove a real projective analogue of Thurston's theorem, showing that for certain one-cusped, finite volume, hyperbolic 3-manifolds, there exists an open interval of filling slopes p/q for which the Dehn filling $M_{p/q}$ admits a convex projective structure which is *exotic*, i.e. is not a continuous deformation of the hyperbolic structure. The holonomy representations of the exotic convex real projective structures in the theorem give discrete, faithful, and Zariski dense representations of closed hyperbolic three-manifold groups $\pi_1 M_{p/q}$ into $\mathrm{SL}_4 \mathbb{R}$ which, by contrast to previous known examples, are not in the same connected component as the holonomy representation of the hyperbolic structure. One of the major steps in the proof of this theorem is a detailed picture of the local representation variety $\mathrm{Hom}(\pi_1 M, \mathrm{PGL}_4 \mathbb{R})$, building on the partial picture we obtained in [BDL].

2. Affine geometry, and the Auslander conjecture. Affine geometry remains one of the most mysterious of the classical geometries, and progress has been slow on the major conjectures in the subject. The *Auslander conjecture* (1962), is a generalization of Bieberbach's theory of Euclidean crystallographic groups to the affine geometry setting. It states that a compact, complete, flat affine manifold must have virtually solvable fundamental group, or equivalently a discrete group acting properly and cocompactly by affine automorphisms of $\mathbb{R}^n$ must be virtually solvable. This longstanding conjecture, which is now known up to dimension six [FrG, AMS3] is of fundamental importance in affine geometry and gives a large source of motivation for my work in this area. A major difficulty in the Auslander Conjecture is that the statement becomes false if the compactness assumption is removed. Indeed, Margulis [Ma1, Ma2] discovered properly discontinuous affine actions by (non-abelian) *free groups* in dimension three (and free groups are not virtually solvable). Margulis' discovery changed the landscape of affine geometry, destroying the expectation from the Euclidean setting that there should not be much difference between compact and non-compact quotients from a group theoretic perspective (see [Ab]). The quotients by proper affine actions of free groups in dimension three are complete flat affine 3-manifolds with free fundamental group, nowadays called *Margulis spacetimes.* The word 'spacetime' refers to the presence of an invariant flat Lorentzian structure, which models flat space-time in Einstein's theory of special relativity. Stemming from Margulis' discovery, there has been much work over the past several decades to understand the topology, geometry, and deformation theory of Margulis spacetimes.

My joint work with Guéritaud and Kassel developed powerful tools to study of Margulis spacetimes, leading to an essentially complete classification. Our prior works [DGK1, DGK2], from 2016, studied Margulis spacetimes with convex cocompact linear part. In particular, we found a new properness criterion for actions on Minkowski 3-space which implies that *a Margulis spacetime fibers in time-like lines over a hyperbolic surface.* As a consequence, we proved the *Tameness Conjecture* for Margulis Spacetimes. We gave a *parameterization of the moduli space of Margulis spacetimes* with fixed convex cocompact linear holonomy in terms of the complex of arc systems of an associated hyperbolic surface [DGK2]. This parametrization, which shares some similarities with Penner's famous cell decomposition of the decorated Teichmüller space [Pe], represents the type of result whose generalization to higher dimensions could lead to great progress on the Auslander Conjecture. Indeed, by the Tits Alternative, free groups are precisely the obstruction to virtual solvability, so a full understanding of their moduli space is crucial. Also in [DGK2], we proved Drumm–Goldman's *Crooked Plane Conjecture*, that every Margulis spacetime admits a fundamental domain in Minkowski space bounded by piecewise linear surfaces called crooked planes. In other words, by analogy to crystallography in Euclidean space, each proper affine action corresponds to a tiling of Minkowski space by fundamental cells which are polyhedra (although in contrast to Euclidean geometry, these fundamental cells are never convex!). We also prove a geometric transition statement: Every Margulis spacetime, with convex cocompact linear part, is a *rescaled limit* of a collapsing family of AdS spacetimes; this result makes rigorous the intuition underpinning our entire approach to the topic. Our work [DGK7] will prove the general case, allowing the linear part to have parabolic elements, for each of these four results. This work has been in progress for quite some time, but it is essentially complete now; we are working toward releasing a draft soon.

My recent work with Guéritaud and Kassel goes beyond free groups to study affine actions by more complicated non-solvable groups. As an indication of the wide open landscape in affine geometry, until our work there were essentially zero known examples of irreducible proper affine actions which are neither virtually solvable nor virtually free. In [DGK3], we give a large new class of examples, including the first examples for which the group acting is word hyperbolic but not virtually free. Specifically, we show that *any right-angled Coxeter group (RACG) admits a properly discontinuous action by affine transformations in some* $\mathbb{R}^n$*.* RACGs, famous for their role in the Virtual Haken Conjecture [AGM], are simple objects with tantalizingly flexible behavior. In particular, there exist word hyperbolic RACGs of arbitrarily large virtual cohomological dimension [JS]. Further, the subgroup structure of RACGs is extremely rich, hence our result leads to proper affine actions for many interesting groups, including *surface groups.* On the other hand, my work [DZ] with Zhang shows that proper affine actions by surface groups never have Hitchin linear part.

3. Hyperbolic and AdS geometry. Next, I will discuss a facet of my research program rooted in themes from my Ph.D. thesis. Anti de Sitter (AdS) space is the model for Lorentzian geometry of constant negative curvature, and is often called a Lorentzian analogue of hyperbolic geometry. Though historically the studies of hyperbolic and AdS geometry have been somewhat disjoint, many new parallels have appeared in the years following Mess's breakthrough result on globally hyperbolic maximal compact (GHMC) AdS space-times [Mes, ABB+, BB, BS] and its remarkable similarity to the Simultaneous Uniformization Theorem of Bers [Ber] for quasi-Fuchsian hyperbolic structures. Stemming from Mess's work, results and questions in hyperbolic and AdS geometry began to appear in tandem, suggesting the existence of a deeper link between the two geometries. My work on geometric transitions [Da1, Da2], which began with my 2011 Ph.D. thesis, established an explicit and natural connection between hyperbolic and AdS geometry. Progress toward better understanding the relationship between the two geometries continues to be a focus for me.

Celebrated work of Alexandrov and Pogorelov determines exactly which metrics on the sphere are induced on the boundary of a compact convex set in hyperbolic three-space. I have studied several different generalizations of this problem, in both hyperbolic and AdS geometry, to the setting of unbounded convex sets with special properties. My joint work [DMS] with Maloni and Schlenker studies the moduli space of *convex ideal polyhedra* in AdS^3, namely polyhedra with vertices at infinity; this paper won the Frontiers of Science Award in Mathematics at the ICBS 2024. By analogy to Rivin's results [Ri] from the hyperbolic geometry setting, we give two related parameterizations of the moduli space, the first in terms of extrinsic data, namely the collection of (Lorentzian) dihedral angles along the edges of the polyhedron, and the second in terms of intrinsic data, namely the hyperbolic metric induced on the boundary. As an application, we answer a long-standing open question, from Steiner's 1832 book [St], about inscribability of polyhedra in quadrics. Geometric transition ideas play an important role in our work. Indeed, some of the key arguments take place in the setting of *half-pipe geometry*, a homogeneous geometry introduced in my thesis [Da1] that naturally bridges the gap between hyperbolic geometry and AdS geometry.

My paper [BDMS], joint with Bonsante, Maloni, and Schlenker, focuses on convex sets in hyperbolic and AdS three-space, whose accumulation set in the boundary at infinity is a quasi-circle. Such a set arises, for example, as the universal cover of a convex neighborhood of the convex core in a quasi-Fuchsian hyperbolic manifold, or in a GHMC AdS spacetime; indeed, motivations for this work include the parallel conjectures of Thurston and Mess, which seek to parametrize the moduli spaces of quasi-Fuchsian manifolds in hyperbolic geometry, or of GHMC spacetimes in AdS geometry, via data on the boundary of the convex core. In our setting, it is natural to augment the notion of induced metric on the boundary to include a gluing map at infinity. We show that every quasi-symmetric map of the circle is realized as the gluing map at infinity for such a convex set bounded by two constant curvature K-surfaces. Our second paper [BDMS2] studies a notion of width of a Jordan curve in the ideal boundary of hyperbolic 3-space, demonstrating a surprising contrast in behavior between convex sets in hyperbolic space and those in anti de Sitter space.

4. Higher rank Teichmüller spaces and the limit cone. Next, I will discuss a new direction in my research program on higher rank Teichmüller spaces, with three papers in preparation—one with Stecker, and two with Guéritaud and Kassel. We are working toward releasing drafts soon. Fix a closed, oriented, surface S of genus at least two. The classical Teichmüller space $\mathcal{T}(S)$ parameterizes the hyperbolic structures on S. This moduli space of geometric structures has fascinated a broad community of mathematicians for more than a half century, and its study has uncovered deep structure hidden within the seemingly simple topology of a surface. A focal example is the notion of a *geodesic lamination*, which show up naturally in a number of different Teichmüller theory contexts, in particular as the locus of maximal stretching when two different hyperbolic structures are compared. In the past two decades, new moduli spaces called *higher rank Teichmüller spaces*, have emerged from the the study of *Anosov representations*, *maximal representations*, and *positive representations* of surface groups $\pi_1 S$ into higher rank semi-simple Lie groups G. These are connected components of $\mathrm{Hom}(\pi_1 S, G)$ consisting entirely of discrete, faithful representations. The most famous examples are the *Hitchin components*, which arise in the case that the semi-simple Lie group G is real split.

Given a representation in a Hitchin component, we study its *limit cone*, a convex cone in the positive Weyl chamber reflecting the spectral behavior of the representation. We aim to understand the shape of the limit cone and how it reflects geometric information. The crucial issue is to understand which elements of the group, or geodesic currents, find the boundary of the limit cone. We find that such extremizers are sometimes geodesic laminations, and in other cases exhibit interesting new lamination-like behavior, suggesting the possibility of a meaningful higher rank analogue of geodesic laminations.

In joint work [DGK8] with Guéritaud and Kassel, we study the *limit cones of multi-Fuchsian representations*. These are representations of a surface group $\pi_1 S$, where S is compact surface of negative Euler characteristic, possibly with boundary, into $G = (\mathrm{PSL}_2\,\mathbb{R})^d$ for which each factor is Fuchsian, corresponding to a (convex cocompact) hyperbolic structure on S. Associated to such a representation ρ is a $(\mathbb{R}^+)^d$-valued length function $\boldsymbol{\lambda}_\rho$ and the limit cone Λ_ρ is the closure of the positive span of the image of this map; it is (generically) a convex cone with non-empty interior. Alternatively, Λ_ρ is the image of the natural extension of $\boldsymbol{\lambda}_\rho$ to the space of geodesic currents on S, a natural completion of the space of weighted homotopy classes of curves. We give a criterion guaranteeing that a boundary face of $\partial\Lambda_{\boldsymbol{\rho}}$ be realized by a *simple* geodesic current, i.e. a measured geodesic lamination. The proof uses a strategy we call *slack calculus*, in which we carefully estimate the effect on $\boldsymbol{\lambda}_\rho$ of resolving a crossing of a long and complicated closed geodesic. We note that the special case $d = 2$ of this criterion is a well-known ingredient in the theory of Thurston's asymmetric metric on Teichmuller space: given a pair (ρ_1, ρ_2) of hyperbolic structures, the supremum of the geodesic length ratios is realized by a maximally stretched lamination. Using our criterion, we are able to give the first examples, going beyond $G = (\mathrm{PSL}_2\,\mathbb{R})^2$, of Hitchin/positive representations for which the limit cone may be explicitly computed. In particular, we show that for $G = (\mathrm{PSL}_2\,\mathbb{R})^3$, there exist multi-Fuchsian representations for which the limit cone is a finite-sided polyhedral cone of any number of sides ≥ 3, and other multi-Fuchsian representations for which the limit cone is strictly convex, i.e. every boundary ray is extreme. The surface S may be taken closed or with boundary in both cases.

Now consider the case of a general Hitchin representation ρ into a real split semi-simple Lie group G, such as $G = \mathrm{SL}_d\,\mathbb{R}$. Again, the limit cone Λ_ρ is the closed cone spanned by the image of the Jordan projection λ_ρ, an analogue of the vector valued length function above. In this general setting, we have no way to determine whether the curves and currents realizing the boundary of the limit cone are simple; indeed, my joint work in progress [DS] with Stecker explicitly computes the first non-trivial example of the limit cone of a Hitchin representation in $G = \mathrm{SL}_3\,\mathbb{R}$; we find that the extremizing curve/current is *not simple* (but nonetheless the self-crossings are rather tame). In recent seminar and conference talks, I have been promoting the following question: *Given S and G, describe the set $\mathscr{S}_{S,G}$ of curves, and the set $\mathscr{M}_{S,G}$ of geodesic currents, whose Jordan projection lies in the boundary of the limit cone Λ_ρ for some Zariski-dense Hitchin representation $\rho : \pi_1 S \to G$* These are mapping class group invariant collections of curves and currents which should have significant meaning linking the topology of S to fundamental properties of the Lie group G. For G any higher-rank semi-simple Lie group other than $(\mathrm{PSL}_2\,\mathbb{R})^2$, nothing is known about these sets. My ongoing work in progress [DGK9] with Guéritaud and Kassel starts to shed light on this question. Generalizing the slack calculus techniques from the multi-Fuchsian setting above, we study carefully what happens to the Jordan projection upon resolving a crossing with small angle, and find two new and intriguing types of lamination-like behavior on the boundaries of limit cones for Hitchin representations. Specifically, we show that a generic face of the limit cone is realized by either a *measured quasi-lamination* or a *measured anti-lamination*. Here, the notion of quasi-lamination is a modification of the usual notion of geodesic lamination: geodesic leaves are not allowed to cross at small angles, but are allowed to cross at large angles. On the other hand, anti-laminations display the opposite behavior: any two geodesic leaves that come close together are required to cross. Additionally, we show that the Jordan projections (spectra) of individual curves approximate such boundary faces exponentially well. Our future plans include further investigating the structure of the quasi-laminations and anti-laminations appearing in the theorem, and how this may be constrained by the Lie group G, toward the goal of characterizing the sets $\mathscr{S}_{S,G}$ and $\mathscr{M}_{S,G}$.

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