Research Statement - Jeffrey Danciger

My research program is dedicated to the study of *locally homogeneous geometric manifolds*. These are abstract mathematical objects designed to model the universe we live in. The term locally homogeneous refers to the presence of a high degree of local symmetry which is captured by a Lie group called the *(local) symmetry group*. It is this symmetry group which governs the geometry. Indeed, the roots of this subject go back to Felix Klein's 1872 Erlangen program, which promoted the philosophy that geometry *is* symmetry: the meaningful quantities we can measure in a geometric manifold (e.g. lengths or angles) are exactly those which are invariant under the symmetry group. A similar philosophy, known as Noether's theorem, permeates modern physics: conservation laws in a physical system are also governed by the symmetry group. There are many different possible symmetry groups and these each lead to different types of geometric manifolds which are useful for many purposes across mathematics and physics. Indeed, there are geometric manifolds that describe space-time in the theory of relativity, as well as phase spaces, configuration spaces, and many other useful structures.

There can also be many different geometric manifolds with the same local symmetry group. These all have the same local properties, but can look very different at large scales. The space of all such possibilities is called a *moduli space*; it is a topological space whose points are the possible geometric manifolds of a certain type and whose topology organizes those geometric manifolds into families whose features vary continuously. While the precise features (e.g. shape, size, etc.) of our universe is a question for empirical physics, a moduli space is the mathematical answer to the question of what possible features could the universe have.

My research addresses fundamental questions about locally homogeneous geometric manifolds and their moduli spaces, with focus on three different types of geometry: *flat affine geometry*, *real projective geometry*, and *constant curvature pseudo-Riemannian geometry*. These geometries have just the right amount of symmetry to allow for large interesting moduli spaces with mysterious but tractable behavior. These also have a close connection to the most famous and well-studied geometric manifolds in modern mathematics, namely the *hyperbolic manifolds*. These were featured prominently in Thurston's 1982 Geometrization Conjecture, which dictated the direction of research in low-dimensional topology for decades until it was proved by Pereleman in 2002. As a result, an immense trove of knowledge about hyperbolic manifolds now exists. Building on this foundation, my work is part of a growing movement to study new classes of geometric manifolds, with more complicated (higher rank, or even non-reductive) local symmetry groups, through techniques rooted in hyperbolic geometry. The extent to which this is possible has only become clear in the past decade or so. As an example, my 2011 Ph.D. thesis introduced a *geometric transition* between hyperbolic geometry and anti de Sitter geometry, a pseudo-Riemannian geometry which models curved spacetime in special relativity. What this showed, essentially, is that the moduli spaces of these two different types of geometric manifolds interact in a meaningful way. Since then, the understanding of similar interactions between many other moduli spaces has repeatedly proven fruitful and forms the backbone of my research program.

1. Affine geometry, Margulis spacetimes, and the Auslander conjecture. Affine geometry remains one of the most mysterious of the classical geometries, and progress has been slow on the major conjectures in the subject. One such conjecture, called the *Auslander conjecture* (1962), is a generalization of Bieberbach's theory of Euclidean crystallographic groups to the affine setting. It states that a compact, complete, flat affine manifold must have virtually solvable fundamental group, or equivalently a discrete group acting properly and cocompactly by affine automorphisms of $\mathbb{R}^n$ must be virtually solvable. This longstanding conjecture, which is now known up to dimension six [FrG, AMS3] is of fundamental importance in affine geometry and gives a large source of motivation for my work in this area.

The major difficulty in the Auslander Conjecture is that the statement becomes false if the compactness assumption is removed. Indeed, Margulis [Ma1, Ma2] discovered properly discontinuous affine actions by (non-abelian) *free groups* in dimension three (and free groups are not virtually solvable). Margulis' discovery changed the landscape of affine geometry, destroying the expectation from the Euclidean setting that there should not be much difference between compact and non-compact quotients from a group theoretic perspective (see [Ab]). The quotients by proper affine actions of free groups in dimension three are complete flat affine 3-manifolds with free fundamental group, nowadays called *Margulis spacetimes.* The word 'spacetime' refers to the presence of an invariant flat Lorentzian structure, making affine 3-space into a copy of the $(2+1)$-dimensional Minkowski space, which models flat space-time in Einstein's theory of special relativity. Stemming from Margulis' discovery, there has been much work over the past several decades to understand the topology, geometry, and deformation theory of Margulis spacetimes (see e.g. [Dr2, DrG1, GLM]) .

My joint work with Guéritaud and Kassel developed powerful tools for the study of Margulis spacetimes, leading to an essentially complete classification. In particular, we found a new properness criterion for actions on Minkowski 3-space which implies that *a Margulis spacetime fibers in time-like lines over a hyperbolic surface.* As a consequence, we proved the *Tameness Conjecture* [1] for Margulis Spacetimes [DGK1, DGK4]. In [DGK2, DGK4], we gave a *parameterization of the moduli space of Margulis spacetimes* with fixed linear holonomy in terms of the complex of arc systems of an associated hyperbolic surface. This parametrization, which shares some similarities with Penner's famous cell decomposition of the decorated Teichmüller space [Pe], represents the type of result whose generalization to higher dimensions could lead to great progress on the Auslander Conjecture. Indeed, by the Tits Alternative, free groups are precisely the obstruction to virtual solvability, so a full understanding of their moduli space is crucial. Also in [DGK2, DGK4], we proved Drumm–Goldman's *Crooked Plane Conjecture*, that every Margulis spacetime admits a fundamental domain in Minkowski space bounded by piecewise linear surfaces called crooked planes. In other words, by analogy to crystallography in Euclidean space, each proper affine action corresponds to a tiling of Minkowski space by fundamental cells which are polyhedra (although in contrast to Euclidean geometry, these fundamental cells are never convex!).

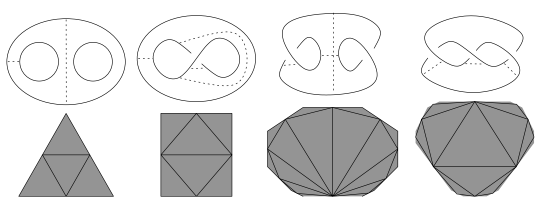

FIGURE 1. The moduli space of Margulis spacetimes associated to a fixed hyperbolic surface is parameterized in terms of the arc complex. Shown are the four surfaces of Euler characteristic -1 and their arc complexes.

The simple perspective that proved key to the development of this body of work is the following: Margulis spacetimes, which are flat, ought to behave like *microscopic analogues* of their negatively curved counterparts, called complete anti de Sitter (AdS) spacetimes. Indeed, AdS spacetimes were studied with great success by Guéritaud and Kassel [Ka1, GK] via a connection with the deformation theory of hyperbolic surfaces and Thurston's Lipschitz metric on Teichmüller space. It was the discovery of the correct microscopic analogues of some of the pieces in the AdS theory, including a theory of *contracting infinitesimal deformations*, that led the way forward for Margulis spacetimes. In [DGK1], we confirmed the simple perspective above by proving a geometric transition statement: Every Margulis spacetime is a *rescaled limit* of a collapsing family of AdS spacetimes. This and related geometric transition ideas (see e.g. [CDW]) suggest a promising path forward in higher-dimensional affine geometry. Margulis spacetimes are just the first stop on what we believe will be a deep and enriching journey making contact with many other types of geometry along the way.

[1] The conjecture was also proved independently by Choi, Drumm, and Goldman [ChG, ChDG].

Some of my recent work with Guéritaud and Kassel goes beyond free groups to study affine actions by more complicated non-solvable groups. As an indication of the wide open landscape in affine geometry, until our work there were essentially zero known examples of irreducible proper affine actions which are neither virtually solvable nor virtually free. In [DGK8], we give a large new class of examples, including the first examples for which the group acting is word hyperbolic but not virtually free. Specifically, we show that *any right-angled Coxeter group (RACG) admits a properly discontinuous action by affine transformations in some* $\mathbb{R}^n$*.* RACGs, recently in the news for their role in the Virtual Haken Conjecture [AGM], are simple objects with tantalizingly flexible behavior. In particular, there exist word hyperbolic RACGs of arbitrarily large virtual cohomological dimension [JS]. Further, the subgroup structure of RACGs is extremely rich, hence our result leads to proper affine actions for many interesting groups, including *surface groups*.

In the opposite direction, I have also developed tools for recognizing the failure of properness of an affine action. While my work just above finds examples of proper affine actions by surface groups, my joint work [DZ] with Zhang shows that an entire connected component of the space of affine surface group actions, coming from the well-studied Hitchin component from higher Teichmüller-Thurton theory, all fail to be properly discontinuous.

2. Discrete subgroups in the projective general linear group and convex real projective structures. After decades of progress, our understanding of finitely generated discrete groups in three-dimensional hyperbolic geometry, i.e. Kleinian groups, is now beginning to crystalize. In other Lie groups, in particular higher rank semi-simple Lie groups, much less is known about the structure of the discrete subgroups, and there is a long way to go before any major classification results will be possible. Immediately problematic is the lack of good notions for large classes of well-behaved discrete subgroups. Lattices are well-studied, but these are extremely special examples in the vast universe of all discrete subgroups. In the case of the projective general linear group $\mathrm{PGL}(V)$ of a finite dimensional real vector space V, one large and still mysterious source of discrete subgroups comes from automorphism groups of open subsets in the real projective space $\mathbb{P}(V)$ which are *properly convex*, meaning convex and bounded in some affine chart. The quotient $N = \Gamma\backslash\Omega$ by a discrete group $\Gamma < \mathrm{PGL}(V)$ acting on a properly convex open set $\Omega \subset \mathbb{P}(V)$ is called a (properly) *convex real projective manifold* (or orbifold, if Γ has torsion). Such manifolds, which inherit strong geometric properties from the convexity of Ω, are a major focus of my research program. My recent work on this topic naturally breaks up into two facets.

2.1 Convex real projective manifolds in dimension three. While every hyperbolic manifold is a convex projective manifold, (indecomposable) examples of convex projective three-manifolds whose underlying topology does not support a hyperbolic structure were found only about 15 years ago by Benoist [B6]. My recent joint work [BDL] with Sam Ballas and Gye-seon Lee uses deformation techniques to produce what we believe to be the largest known source of examples of convex projective three-manifolds which are not homeomorphic to hyperbolic manifolds. Essentially, we deform finite volume hyperbolic three-manifolds so that the cusps come in from infinity as totally geodesic (planar) torus boundary components. This of course is not possible within hyperbolic

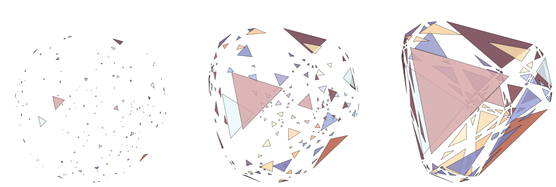

Figure 2. The properly convex sets associated to a convex projective deformation of a finite volume hyperbolic three-manifold under which the cusp becomes a totally geodesic torus coming in from infinity.

geometry, but it becomes possible in the larger context of convex real projective geometry, and we show that it frequently happens in examples. We then glue multiple such convex real projective manifolds together along the boundary tori to produce closed manifolds (the easiest way to do this is to double a single such manifold). As a corollary, the fundamental group of each such manifold admits a discrete faithful four-dimensional real linear representation. Hence, our work suggests new theoretical and experimental techniques in the search for low-dimensional matrix representations of three-manifold groups. Previous linearity results, obtained via cube complex techniques [PW], for groups of this type gave no control over the dimension of the representation.

In a different direction, but still in dimension three, new work in progress joint with Sam Ballas, Gye-Seon Lee, and Ludovic Marquis, will develop a theory of *convex real projective Dehn filling*, generalizing the picture of Thurston's hyperbolic Dehn filling space [T]. Thurston's famous hyperbolic Dehn filling theorem states that for M an orientable, finite volume, hyperbolic three-manifold with one cusp, most (all but finitely many) ways of topologically filling in the cusp with a solid torus give closed three-manifolds which support a hyperbolic structure. Our new work shows that in many cases, some of these so-called hyperbolic Dehn fillings also admit a second convex real projective structure. Further, that structure can not be deformed continuously to the hyperbolic structure. This answers a question of Benoist by giving the first known examples of a closed manifold whose moduli space of convex real projective structures is disconnected.

2.2 Convex cocompact groups in real projective geometry. In the setting of Kleinian groups and hyperbolic geometry, convex cocompact subgroups are a well-studied middle-ground between the very special class of cocompact lattices and the general class of all finitely generated discrete subgroups. In joint work [DGK5, DGK6] (see also [CM]) with Guéritaud and Kassel, we generalized the notion of convex cocompact subgroup to the higher rank setting of discrete subgroups of the projective general linear group $\mathrm{PGL}(V)$ acting on the projective space $\mathbb{P}(V)$. Much of my most recent and current work is dedicated to the thorough study of this new class of subgroups. We have high expectations that this new notion will emerge as a central tool for understanding the structure of the projective general linear group. Indeed, Kassel promoted the study of these groups in her address at the 2018 International Congress of Mathematicians in Rio de Janeiro, see [Ka2, §6].

Convex cocompact subgroups in $\mathbb{P}(V)$ have a close connection with another important class of discrete subgroups in higher rank Lie groups, namely the class of *Anosov subgroups*, which has earned a central role in Higher Teichmüller–Thurston theory. These are subgroups exhibiting special *dynamical* behavior, which leads to many desirable analogies with convex cocompact groups in hyperbolic geometry: see e.g. [L, GW3, KLPc]. There are interesting geometric structures, modeled on flag varieties, associated to Anosov representations: see e.g. [GW3, KLPb, GGKW]. However, natural *convex* (or metric) geometric structures associated to Anosov representations had been lacking in general. As a result, familiar geometric constructions from hyperbolic geometry, such as convex fundamental domains, had not played a role in the theory and examples going beyond surface groups and free groups had proven difficult to construct. Our introduction of the class of convex cocompact subgroups in projective space has begun to change this.

We proved, first in the context of pseudo-Riemannian hyperbolic geometry [DGK6] and then in the general projective geometry context [DGK5], that subgroups acting convex cocompactly on strictly convex domains Ω are Anosov and that conversely, essentially all Anosov subgroups admit such an action (in an appropriate projective space, see also independent work of Zimmer [Z]). As an application, we find a large new class of Anosov subgroups coming from Vinberg's classical study of discrete linear groups generated by reflections [V]. My joint work [DGKLM] with Guéritaud, Kassel, Lee, and Marquis analyzes deformation spaces of Anosov representations of Coxeter groups, showing in particular that any word hyperbolic Coxeter group admits Anosov representations (we had show this earlier in [DGK6] for the right-angled case). Considering the rich subgroup structure of Coxeter groups, this gives an enormous new source of examples with rich behavior.

We also emphasize that, while the Anosov subgroups are a vast and interesting class, the convex cocompact groups are a larger and richer class. By contrast to Anosov subgroups (and to convex cocompact subgroups in hyperbolic geometry), there exist discrete subgroups of $\mathrm{PGL}(V)$ which are convex cocompact in $\mathbb{P}(V)$, but *are not word hyperbolic* (including e.g. some reflection groups). For this reason, the study of convex cocompactness in real projective geometry promises access to new deformation spaces of well-behaved discrete subgroups beyond those typically studied in Higher Teichmüller–Thurston theory. My collaborators and I, and some of my Ph.D. students as well, are actively exploring the many open questions in this fertile new direction.

3. Hyperbolic and AdS geometry. Finally, I will discuss a facet of my research program rooted in themes from my Ph.D. thesis. Anti de Sitter (AdS) space is the model for Lorentzian geometry of constant negative curvature, and is often called a Lorentzian analogue of hyperbolic geometry. Though historically the studies of hyperbolic and AdS geometry have been somewhat disjoint, many new parallels between the two geometries have appeared in the years following Mess's breakthrough result on globally hyperbolic maximal compact (GHMC) AdS space-times [Mes, ABB+, BB, BS] and its remarkable similarity to the Simultaneous Uniformization Theorem of Bers [Ber] for quasi-Fuchsian hyperbolic structures. Stemming from Mess's work, results and questions in hyperbolic and AdS geometry have begun to appear in tandem, suggesting the existence of a deeper link between the two geometries. My work on geometric transitions [Da1, Da2], which began with my 2011 Ph.D. thesis, established an explicit and natural connection between hyperbolic and AdS geometry. Progress toward better understanding the relationship between the two geometries continues to be a focus for me.

My joint work [DMS] with Maloni and Schlenker studies the moduli space of *convex ideal polyhedra* in AdS^3, giving parallel results to those of Rivin [Ri] from the hyperbolic geometry setting. We give two related parameterizations of the moduli space, the first in terms of extrinsic data, namely the collection of (Lorentzian) dihedral angles along the edges of the polyhedron, and the second in terms of intrinsic data, namely the hyperbolic metric induced on the boundary . As an application, we answer a long open question, from Steiner's 1832 book [St], about inscribability of polyhedra in quadrics in three-space. One note of interest is that geometric transition ideas play a central role in our work. Indeed, some of the key arguments take place in the setting of *half-pipe geometry*, a homogeneous geometry introduced in my thesis [Da1] that naturally bridges the gap between hyperbolic geometry and AdS geometry. A recent preprint [BDMS], joint with Bonsante, Maloni, and Schlenker, generalizes some of the ideas of [DMS] to a more complicated setting, namely the space of convex sets bounded by constant curvature K-surfaces whose accumulation set at infinity is a quasicircle. Our eventual goal is to parametrize moduli spaces of compact AdS manifolds with convex boundary.

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