Thurston's asymmetric metric on Teichmüller space - revisited

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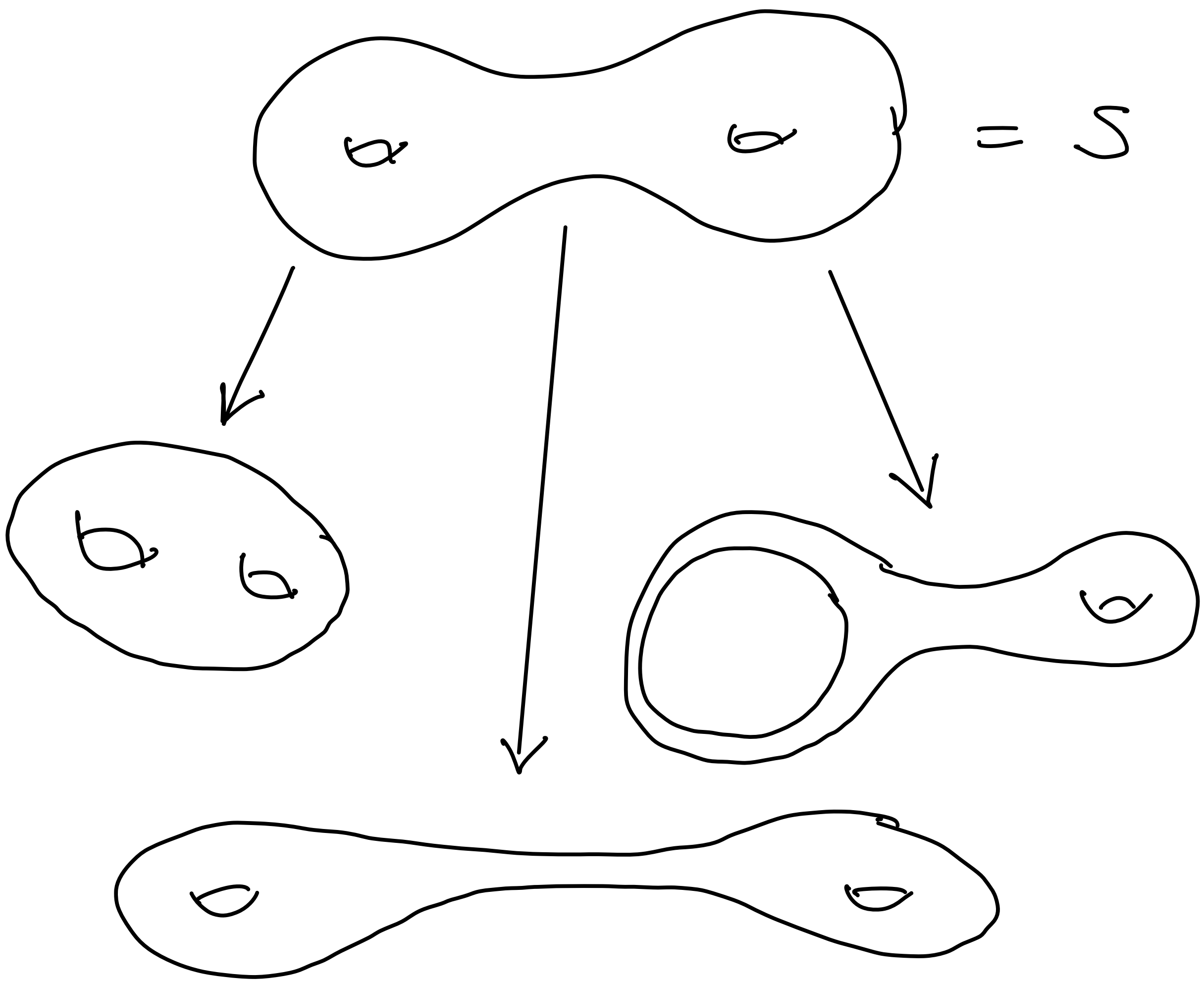
Todd Drumm Memorial Conference
Brinn Mathematics Research Center

Hyperbolic Structures on surfaces

S = closed, oriented surface,
genus $g \geq 2$



many hyperbolic structures

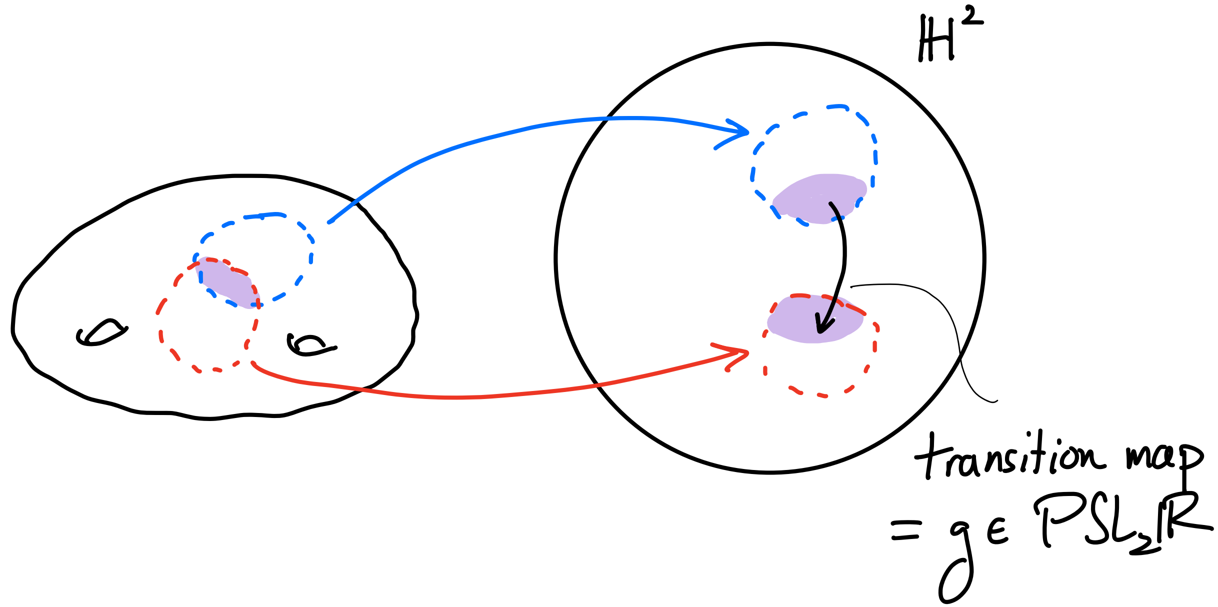
Teichmüller space : $\mathcal{T}(S) = \{\text{hyperbolic structures on } S\}/\sim$

What is a hyperbolic structure?

a Riemannian metric ρ on S
with curvature $= -1$.

OR

a structure ρ locally modeled on the hyperbolic plane
$\mathbb{H}^2 = {PSL_2\mathbb{R}}/{PSO_2}$



OR

a injective homomorphism with discrete image

$$\rho : \pi_1 S \longrightarrow PSL_2\mathbb{R}$$

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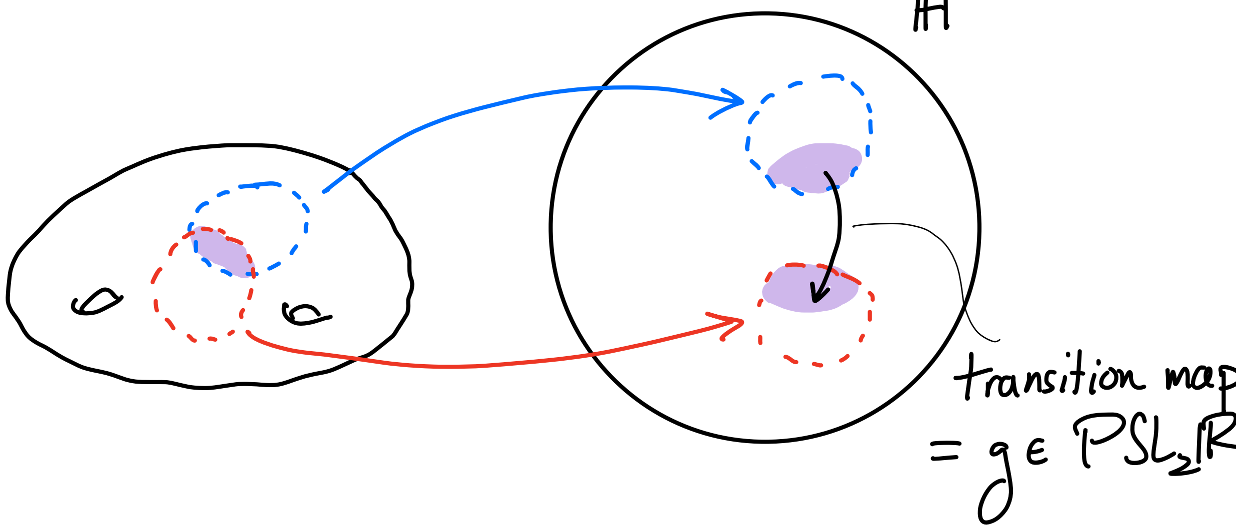


OR

a injective homomorphism $\rho : \pi_1 S \longrightarrow PSL_2\mathbb{R}$
with discrete image

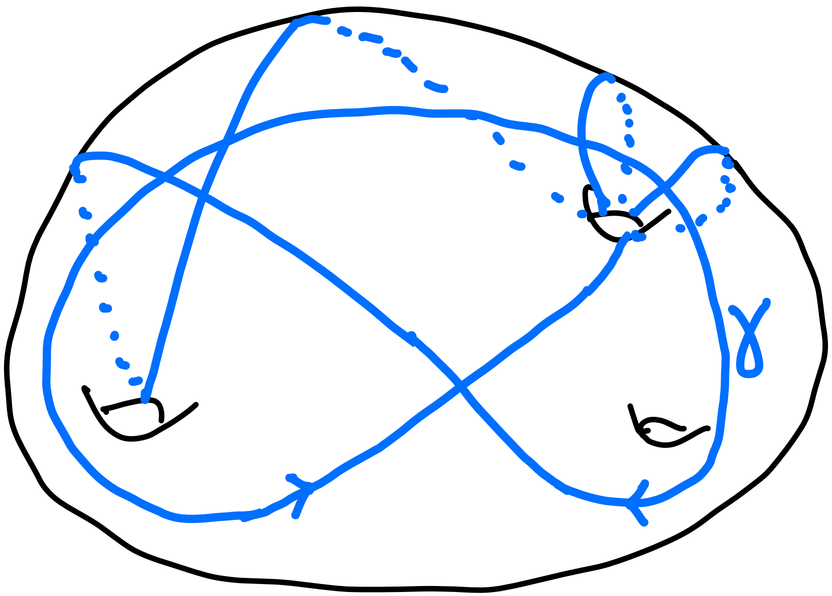
A higher Teichmüller space? : special representations $\pi_1 S \longrightarrow G$ semi-simple
e.g. $G = (PSL_2\mathbb{R})^d \rightsquigarrow \underbrace{\mathcal{T}(S) \times \dots \times \mathcal{T}(S)}_{d}$

ρ a hyperbolic structure on S

Length Spectrum

$$\lambda_\rho : \underbrace{\left\{ \begin{array}{c} \text{free homotopy classes} \\ \text{of closed curves} \\ \text{on } S \end{array} \right\}}_{\textstyle \cong\ [\pi_1 S]} \to \mathbb{R}^+$$

$$\gamma \longmapsto \begin{array}{c} \text{length of} \\ \text{geodesic} \\ \text{representative} \end{array}$$



Thurston (1986)

$\rho_1, \rho_2 \in \mathcal{T}(S)$

$$C_\lambda(\rho_1,\rho_2) := \sup_{\gamma \in [\pi_1 S]\setminus 1} \frac{\lambda_{\rho_2}(\gamma)}{\lambda_{\rho_1}(\gamma)}$$

"supremal length ratio"

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$$d_{Th}(\rho_1,\rho_2) := \log C_\lambda(\rho_1,\rho_2)$$

Thurston's asymmetric metric

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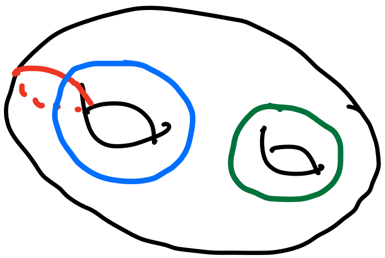
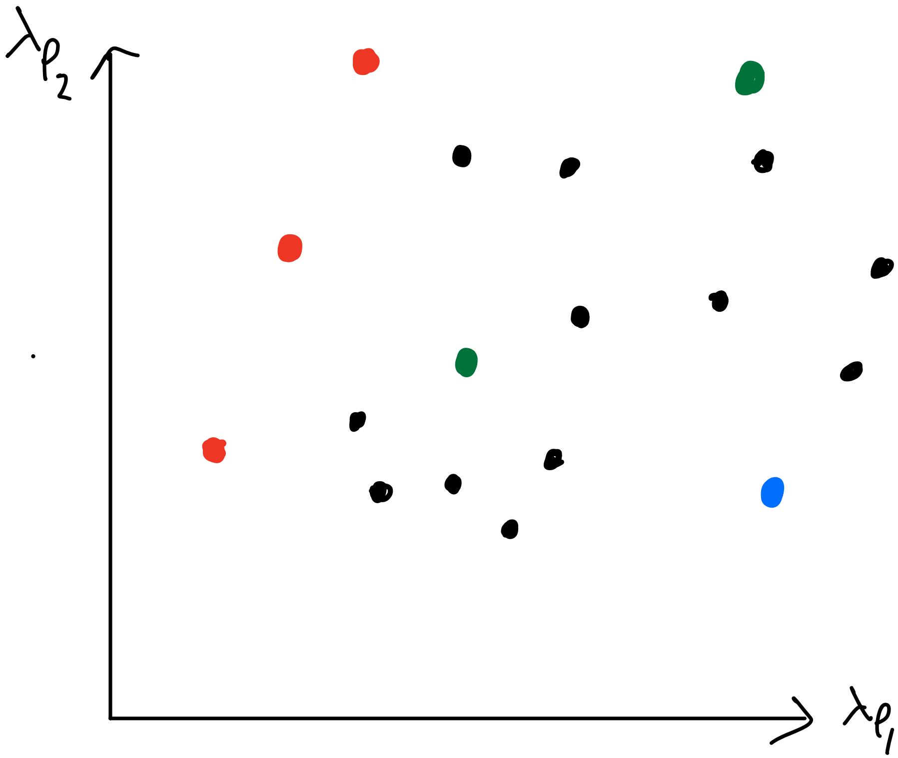
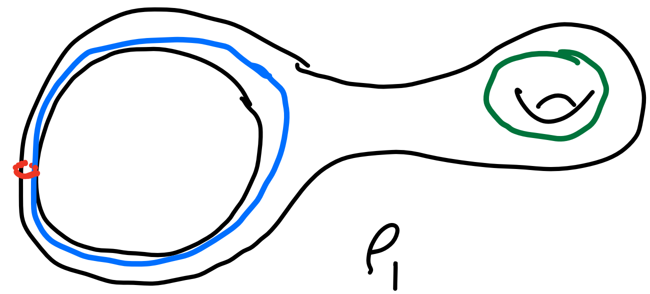


ρ_2





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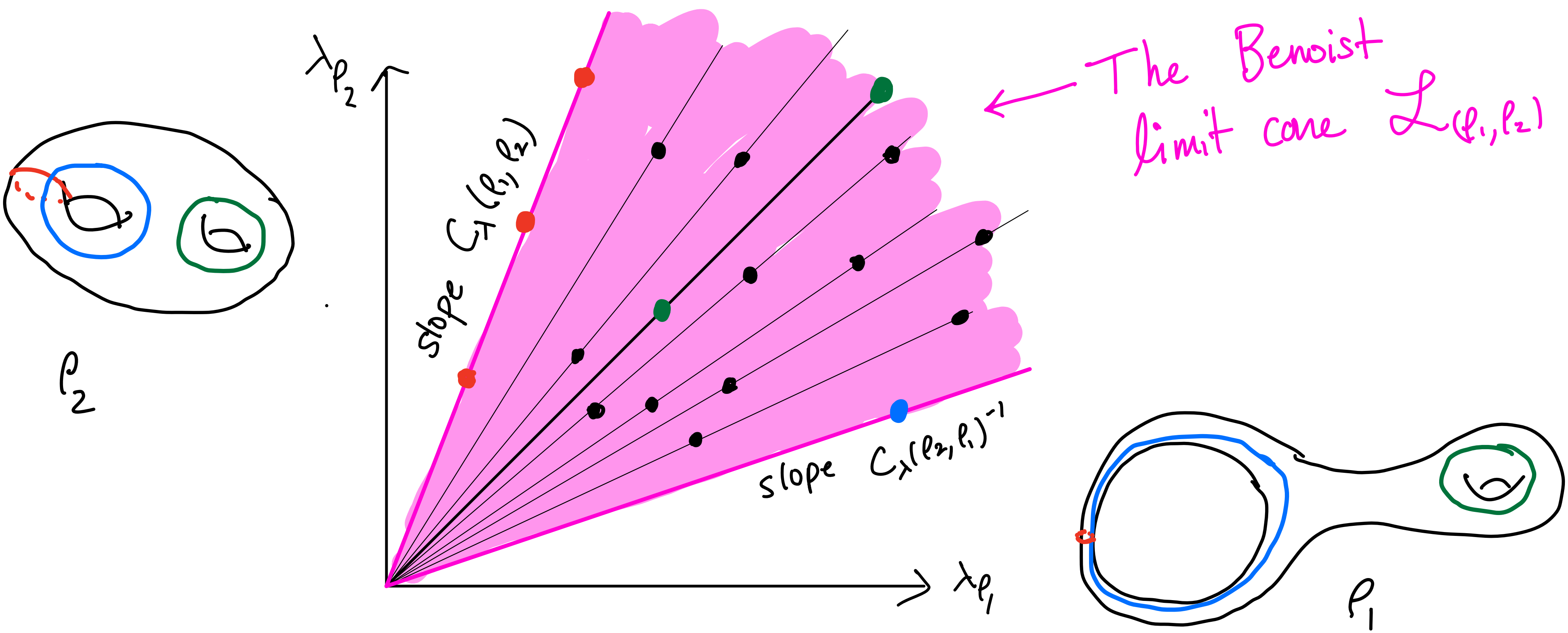

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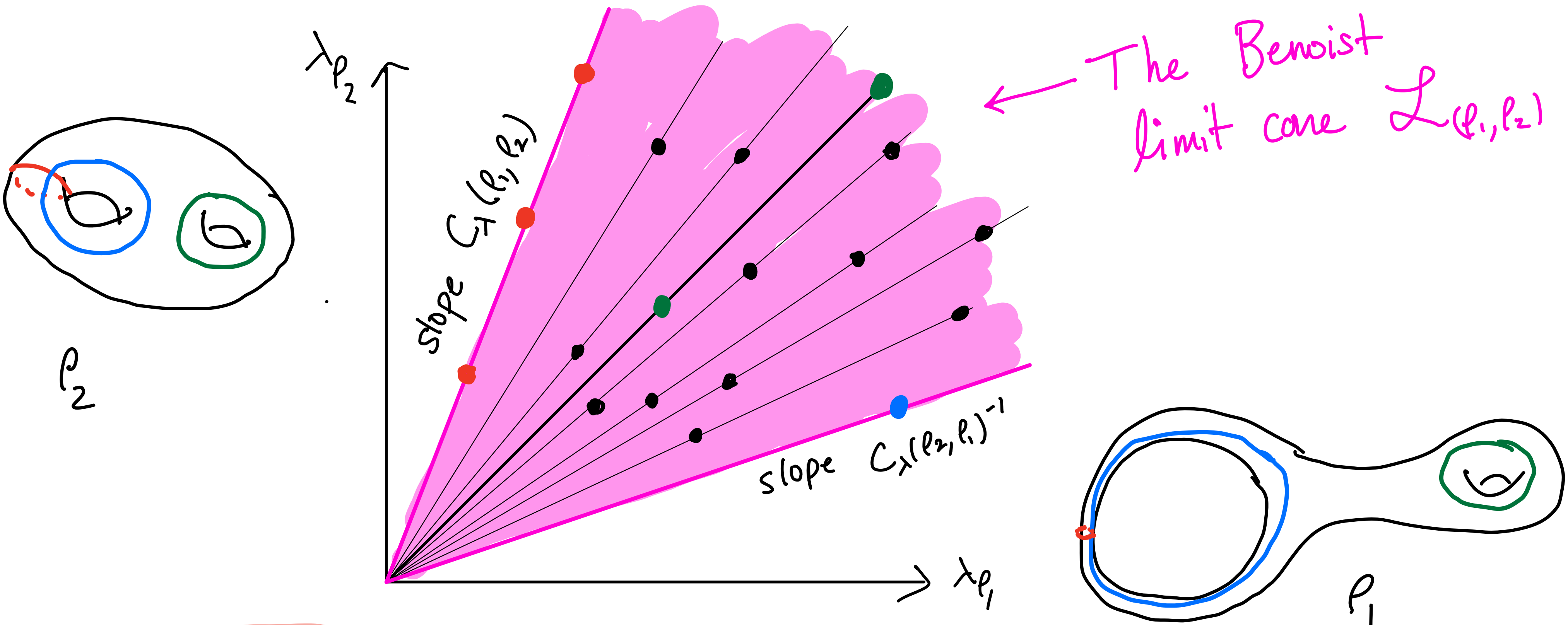

Thurston's "Simple" Theorem: The sup is achieved over $\mathcal{S} = \{$simple closed curves$\}$

Approach : optimization over a discrete set $\rightsquigarrow$ optimization over a compact set.

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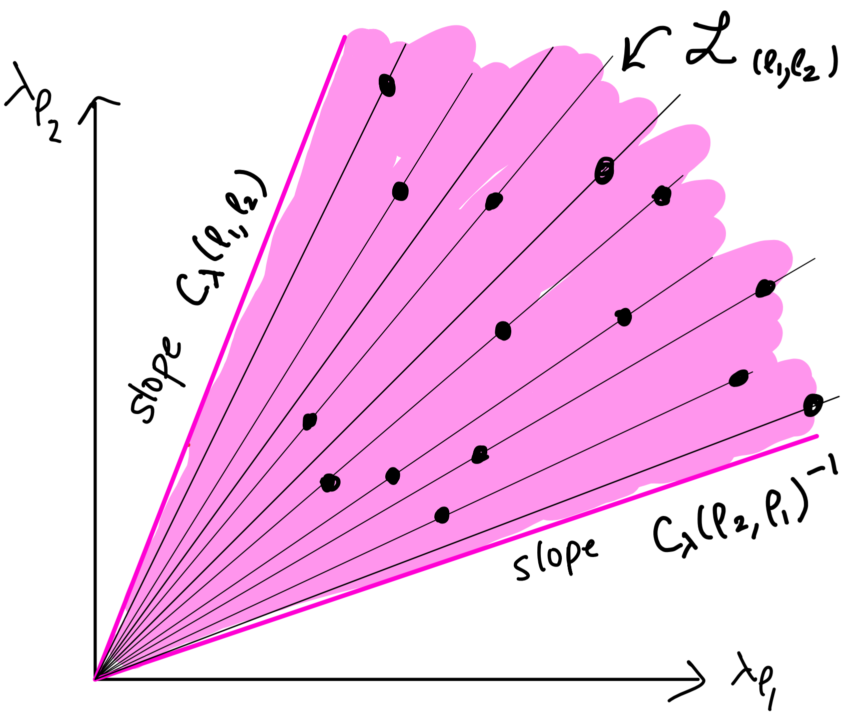
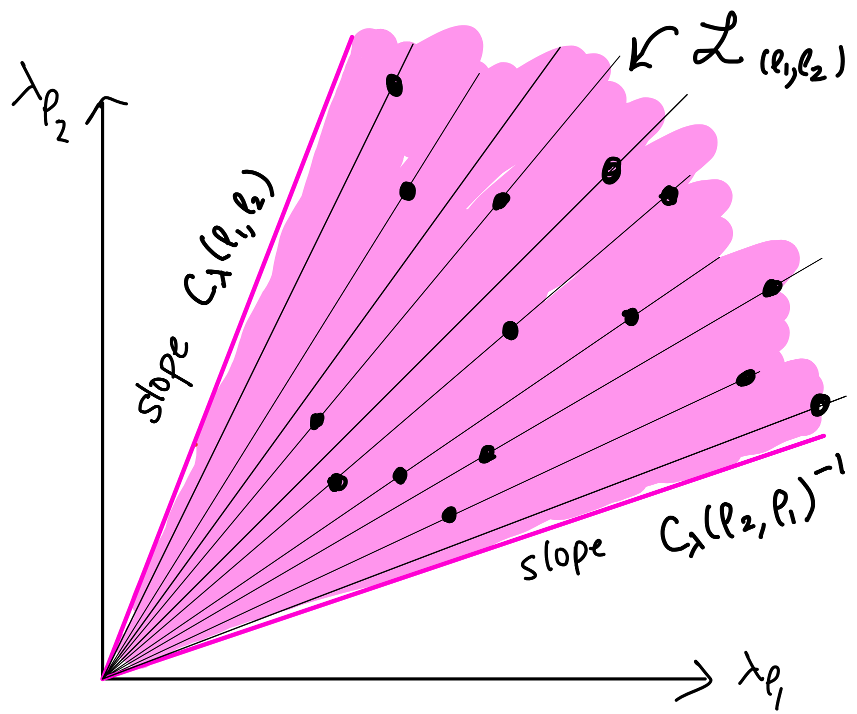
$$C_\lambda = \sup_{\gamma \in [\Pi, S]} \frac{\lambda_{\rho_2}(\gamma)}{\lambda_{\rho_1}(\gamma)}$$



Approach : optimization over a discrete set $\rightsquigarrow$ optimization over a compact set.

$$C_\lambda = \sup_{\gamma \in [\pi_1 S]} \frac{\lambda_{\rho_2}(\gamma)}{\lambda_{\rho_1}(\gamma)} \leqslant C_{Lip} = \inf \left\{ \mathrm{Lip}(f) : \begin{array}{c} f:(S,\rho_1) \to (S,\rho_2) \\ f \sim id \end{array} \right\}$$



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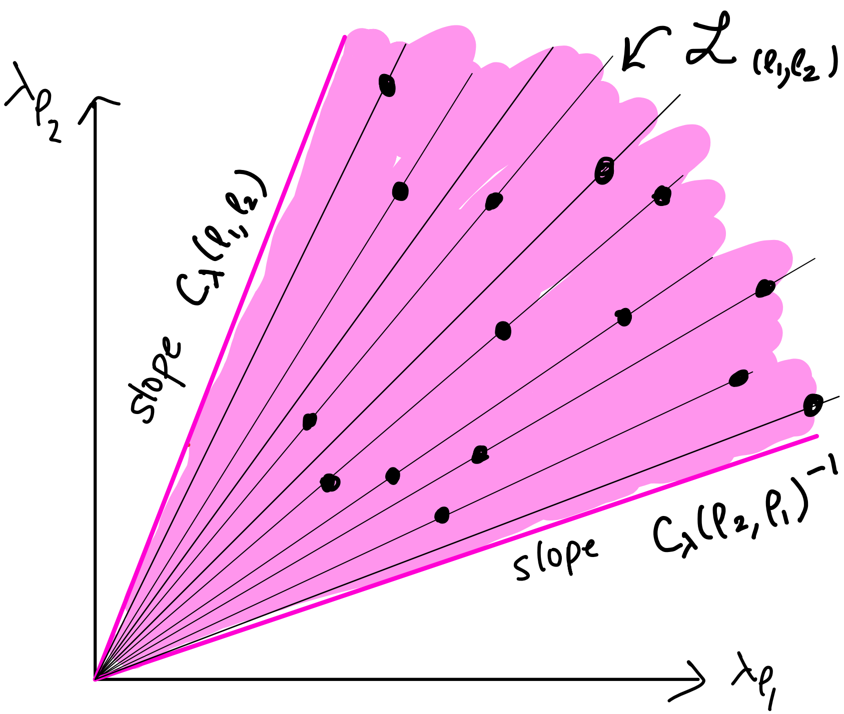

Thurston, Kassel, Guéritaud-Kassel:

if $C_\lambda > 1$, then $C_\lambda = C_{Lip}$

and $\exists$ optimal $f : (S, \rho_1) \to (S, \rho_2)$

that stretches by C_λ along

a geodesic lamination $\Lambda \subset (S, \rho_1)$

a closed subset foliated by geodesics.

$\exists$ C_λ-Lipschitz $f:(S,\rho_1)\to(S,\rho_2)$ that stretches by exactly C_λ along a geodesic lamination Λ.

a geodesic lamination!

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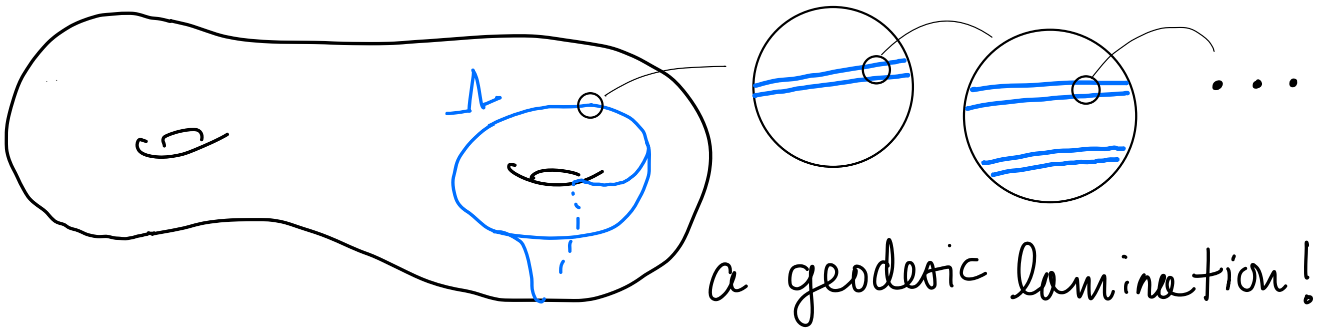
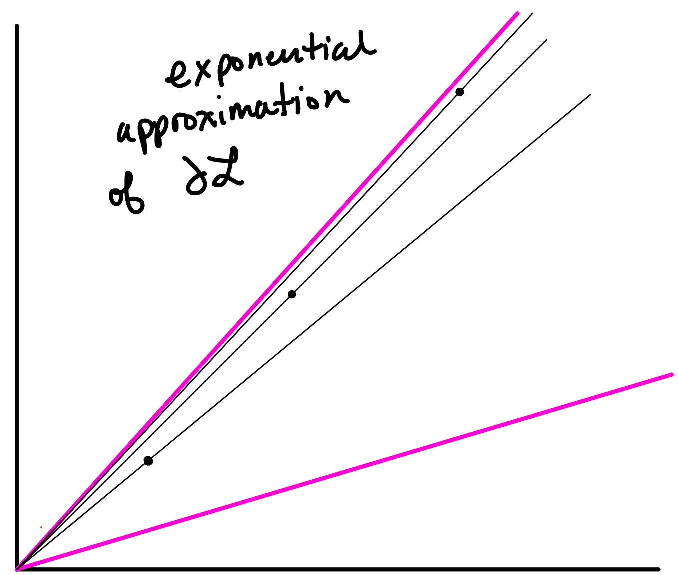


$\underline{\text{Cor}}$ (Thurston's Simple Theorem): ∃ (γ_n) simple closed curves with $\dfrac{\lambda_{\rho_2}(\gamma_n)}{\lambda_{\rho_1}(\gamma_n)} \to C_\lambda$

Laminations have exponential self recurrence.

$$\Downarrow$$

Cor 2: $$\left| \lambda_{\rho_2}(\gamma_n) - C_\lambda \lambda_{\rho_1}(\gamma_n) \right| = O\left(e^{-\alpha \lambda_{\rho_1}(\gamma_n)}\right)$$



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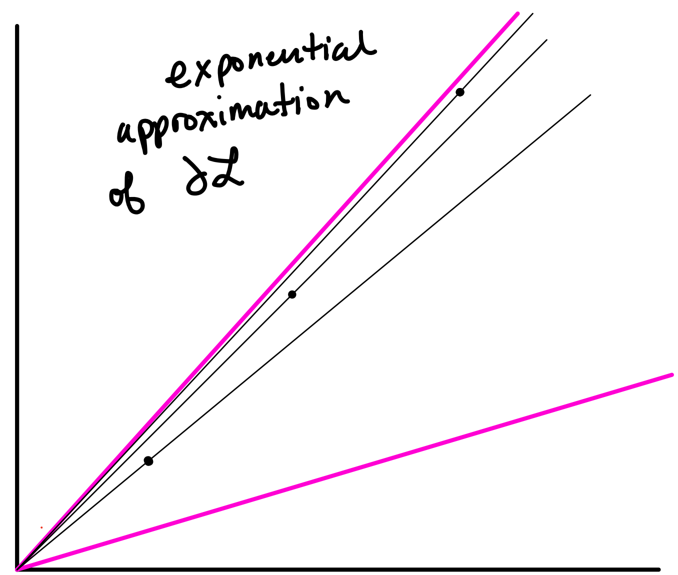
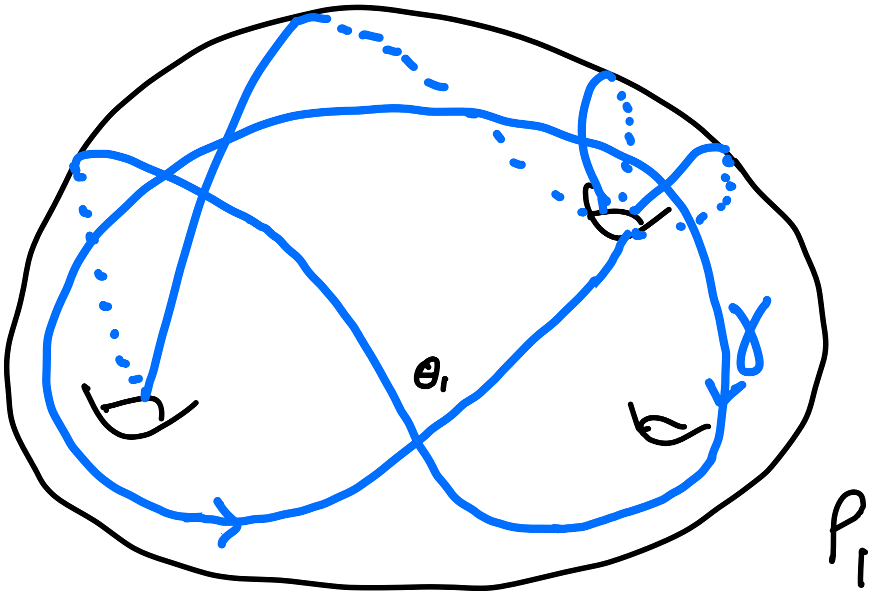
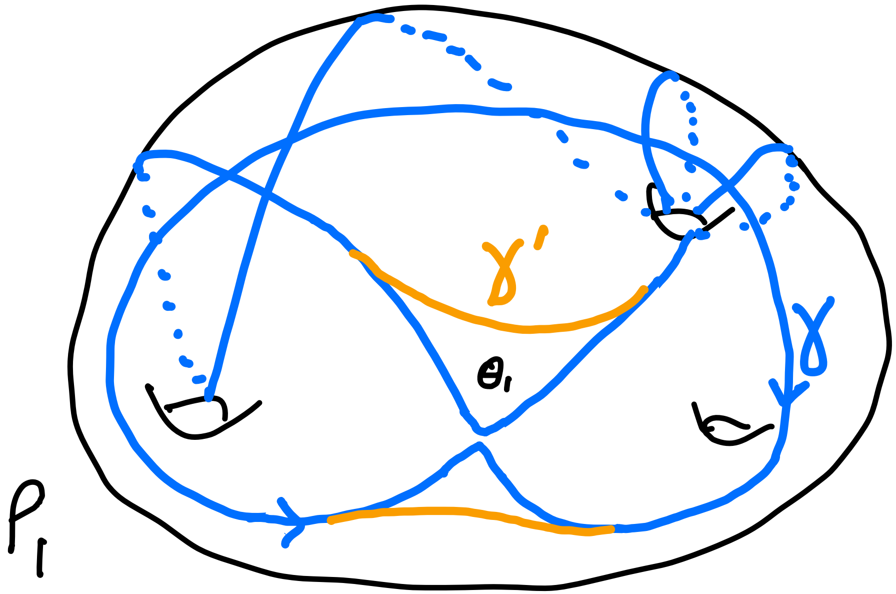


Recurring theme: Extremizers are dynamically well behaved.

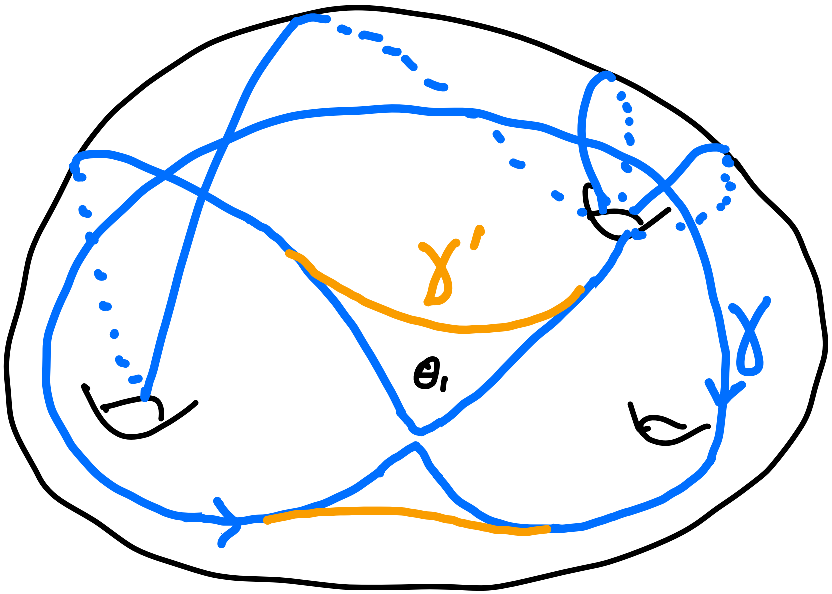


θ_1
γ
P_1



$$\lambda_{\rho_1}(\gamma') - \lambda_{\rho_1}(\gamma) = 4 \ln \sin \frac{\theta_1}{2} + o(1)$$

$$=: \text{slack}_1$$



ρ_1

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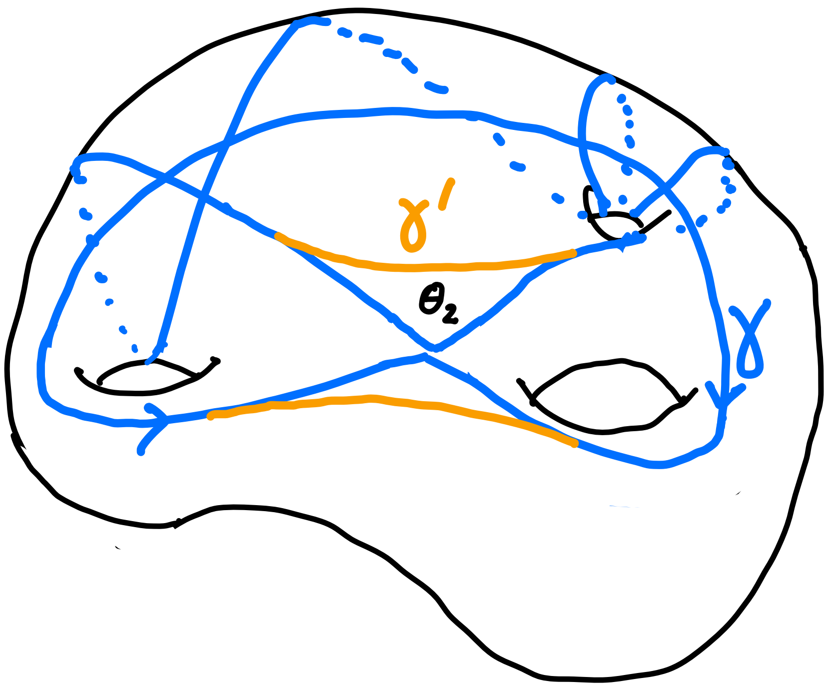
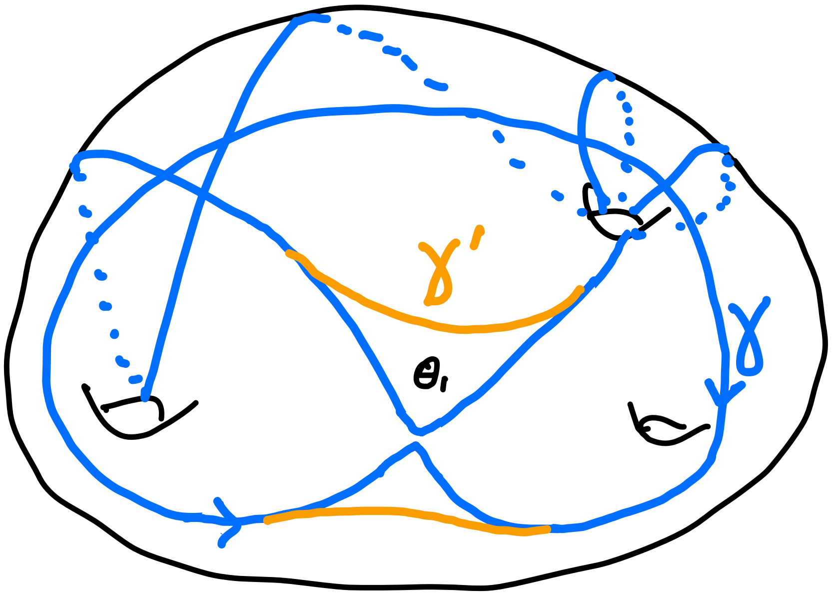
$=:$ slack$_2$



ρ_2



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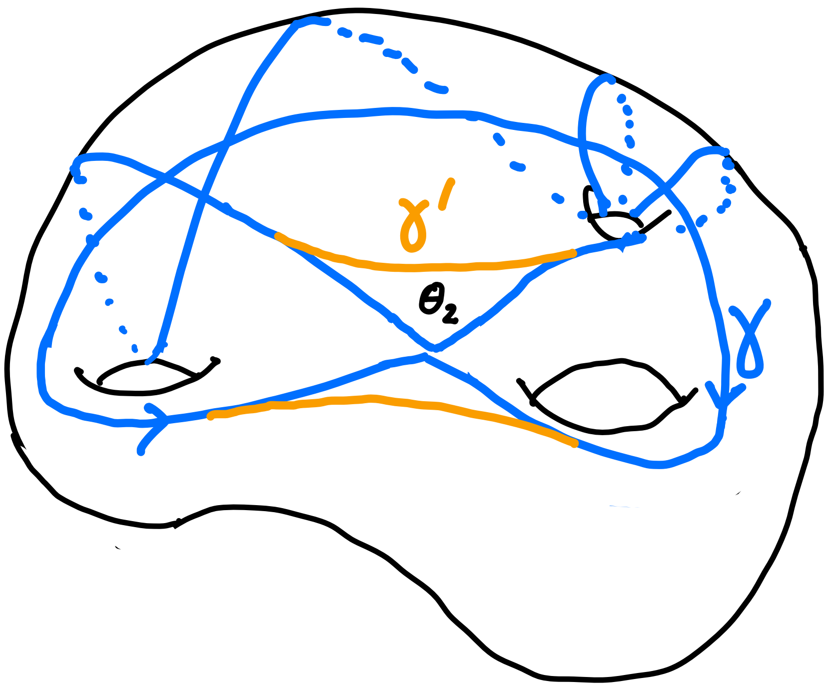
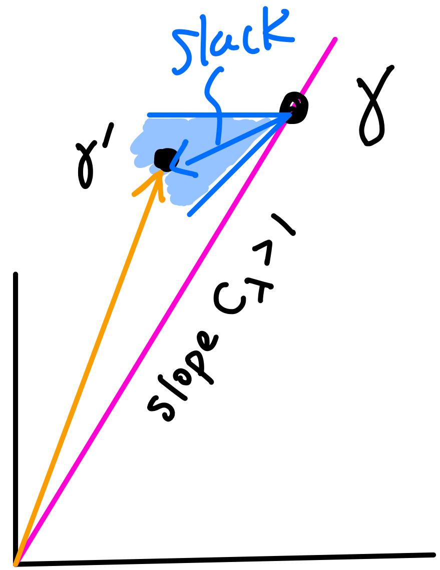
$=:$ slack_2

if sup realized: $C_\lambda = \dfrac{\lambda_{\rho_2}}{\lambda_{\rho_1}}(\gamma)$

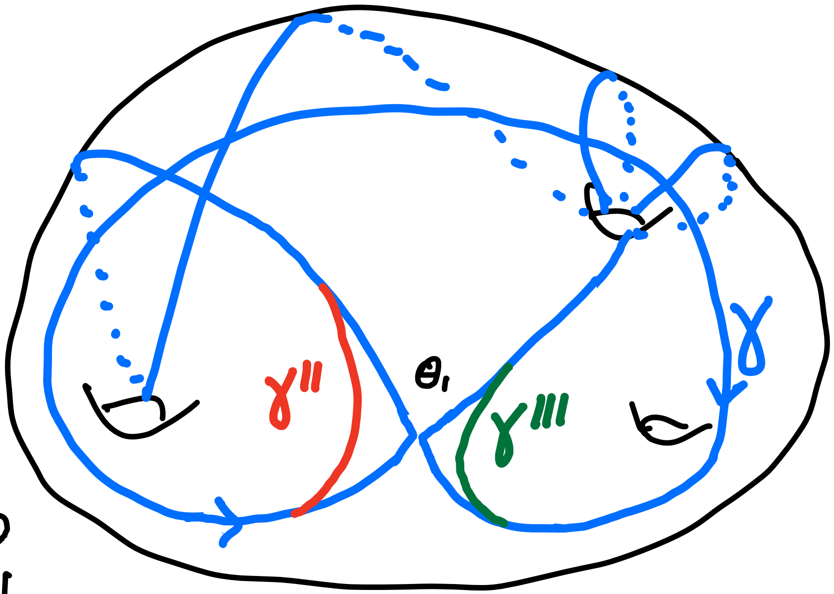
if $\theta_1 \leq \theta_2$



ρ_2



Contradiction!



ρ_1

$$\lambda_{\rho_1}(\gamma'') + \lambda_{\rho_1}(\gamma''') - \lambda_{\rho_1}(\gamma) = 4 \ln \cos \frac{\theta_1}{2} + o(1)$$

$$=: \text{slack}_1$$

$$\lambda_{\rho}(\gamma'') + \lambda_{\rho}(\gamma''') - \lambda_{\rho}(\gamma) = 4 \ln \cos \frac{\theta_2}{2} + o(1)$$

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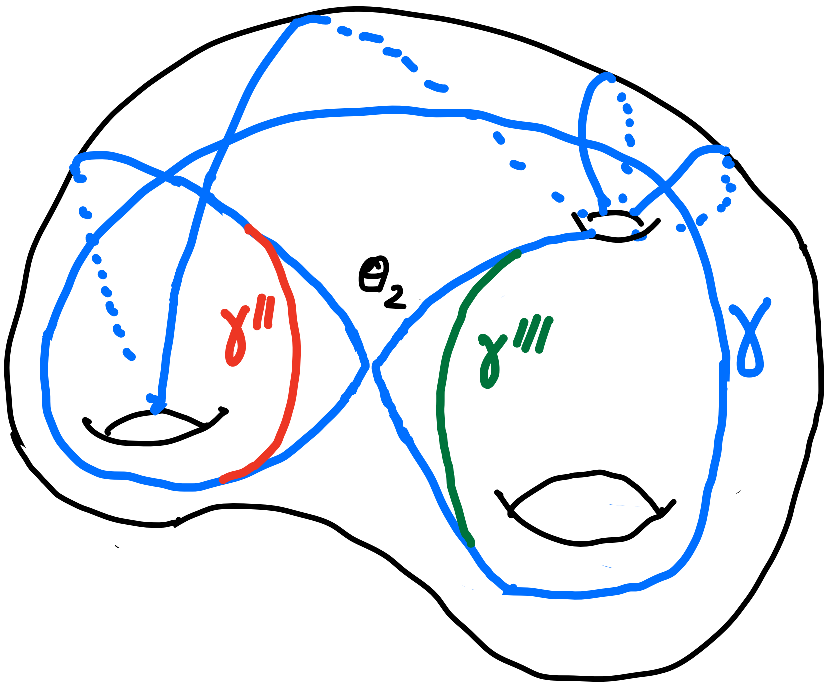
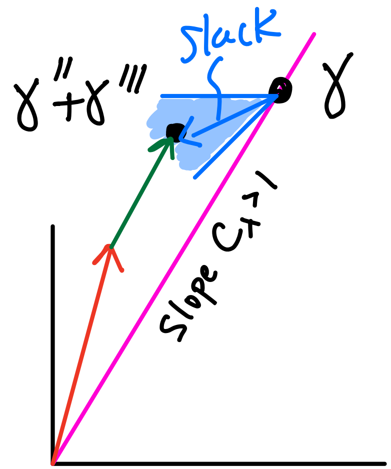


ρ_2



Contradiction!

if sup not realized: $C_\lambda = \lim\limits_{n\to\infty} \frac{\lambda_{\rho_2}}{\lambda_{\rho_1}}(\gamma_n)$.

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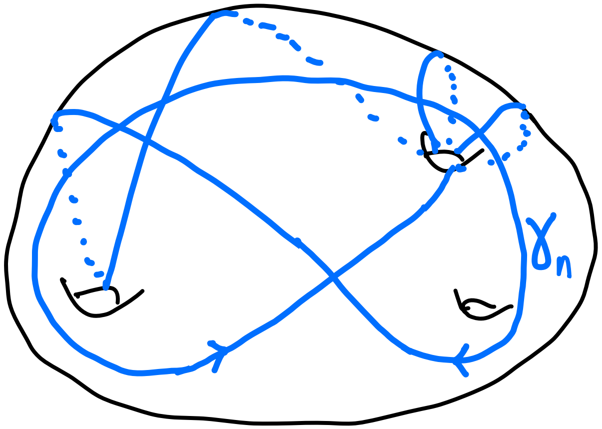

$\gamma_n \rightsquigarrow$ length measure μ_n along γ_n in T^1S.

$\left(\frac{1}{\lambda_n}\right)\mu_n \longrightarrow \mu_\infty$ in Curr(S) := geodesic flow invariant measures on T^1S

geodesic currents.

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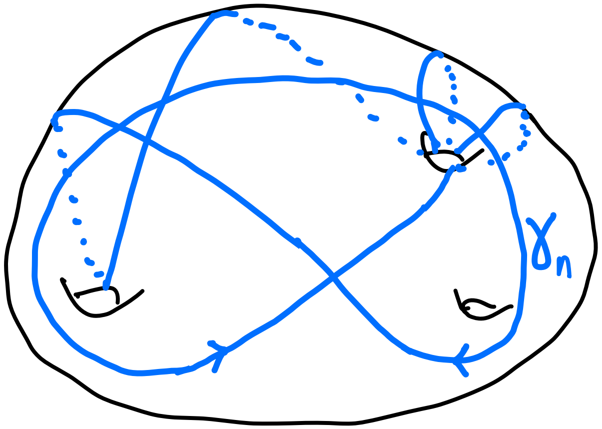

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Basic Facts: Curr(S)

- $\mathbb{P}\,\mathrm{Curr}(S)$ compact
- weighted closed curves are dense
- length function extends $\lambda_\rho : \mathrm{Curr}(S) \to \mathbb{R}^+$

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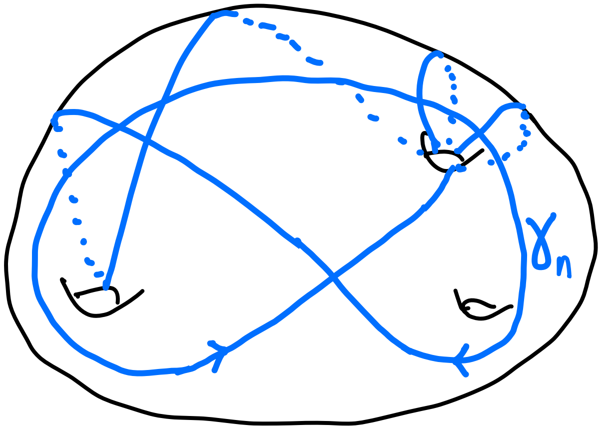

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$$\begin{array}{c} \mathrm{Curr}(S) \\ \cup \end{array}$$

measured laminations $\mathcal{ML}(S) \ni \mu \Leftrightarrow$ supp(μ) is a geodesic lamination

$$C_\lambda = \lim_{n\to\infty} \frac{\lambda_{\rho_2}}{\lambda_{\rho_1}}(\gamma_n).$$

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\gamma_n = \mu_\infty \in \mathrm{Curr}(S)$$

$$\frac{\lambda_{\rho_2}(\mu_\infty)}{\lambda_{\rho_1}(\mu_\infty)} = C_\lambda$$

measured laminations

Claim: $\mu_\infty \in \mathcal{ML}(S)$, i.e. $\mathrm{supp}(\mu)$ is simple.

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measured laminations

$\boxed{\underline{\text{Claim}}:}$ if each γ_n is a record breaker, then $\mu_\infty \in \mathcal{ML}(S)$, i.e. $\mathrm{supp}(\mu)$ is simple.

$$\frac{\lambda_{\rho_2}}{\lambda_{\rho_1}}(\gamma_n) \geq \frac{\lambda_{\rho_2}}{\lambda_{\rho_1}}(\eta) \quad \text{for all shorter } \eta \ (\text{wrt } \rho_1)$$

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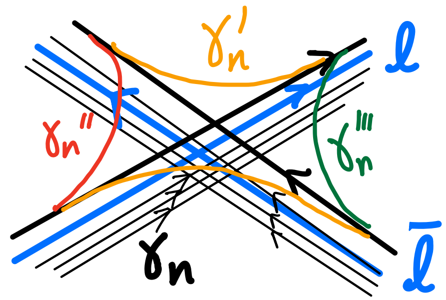
$\hookrightarrow \frac{\lambda_{\rho_2}}{\lambda_{\rho_1}}(\gamma_n) \geq \frac{\lambda_{\rho_2}}{\lambda_{\rho_1}}(\eta)$ for all shorter η (wrt ρ_1)

Pf: assume $\mathrm{supp}(\mu_\infty) \ni \ell, \bar{\ell}$ two crossing geodesics $\begin{cases} \text{angle } \theta_1 \text{ in } \rho_1 \\ \text{angle } \theta_2 \text{ in } \rho_2 \end{cases}$

γ_n crosses itself many times

resolve one crossing



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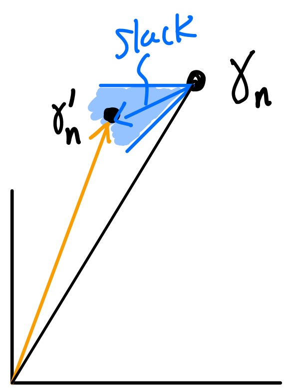

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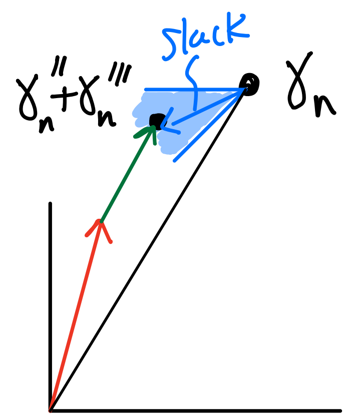

d-tuples $(\rho_1, \ldots, \rho_d)$ of hyperbolic structures

$$\underbrace{(\rho_1, \ldots, \rho_d)}_{\rho} : \pi_1 S \longrightarrow \underbrace{PSL_2\mathbb{R} \times \ldots \times PSL_2\mathbb{R}}_{\mathbb{G}}$$

$$\begin{aligned} \lambda_\rho : [\pi_1 S] &\longrightarrow (\mathbb{R}^+)^d \\ \gamma &\longmapsto (\lambda_{\rho_1}(\gamma), \ldots, \lambda_{\rho_d}(\gamma)) \end{aligned}$$

vector valued length spectrum

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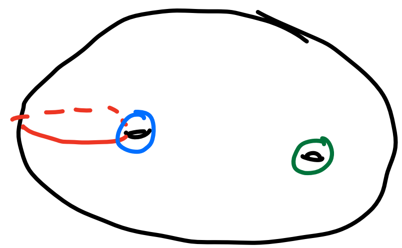
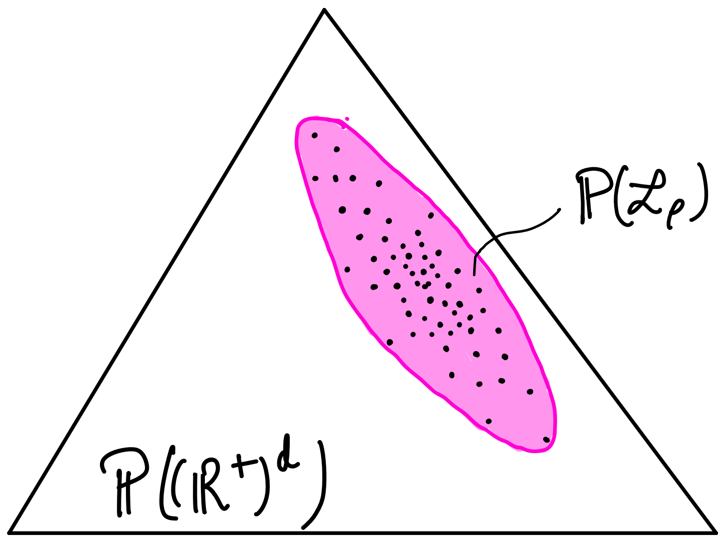
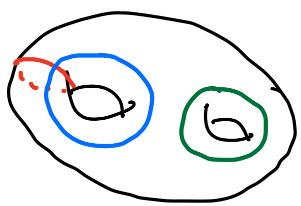
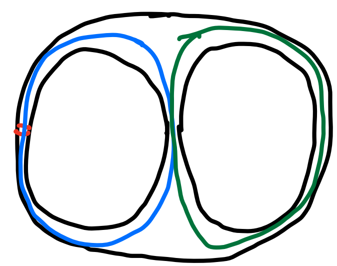


Defn (Benoist) : The limit cone

$$\mathcal{L}_\rho := \overline{\mathbb{R}_+ \lambda_\rho([\pi_1 S])}$$

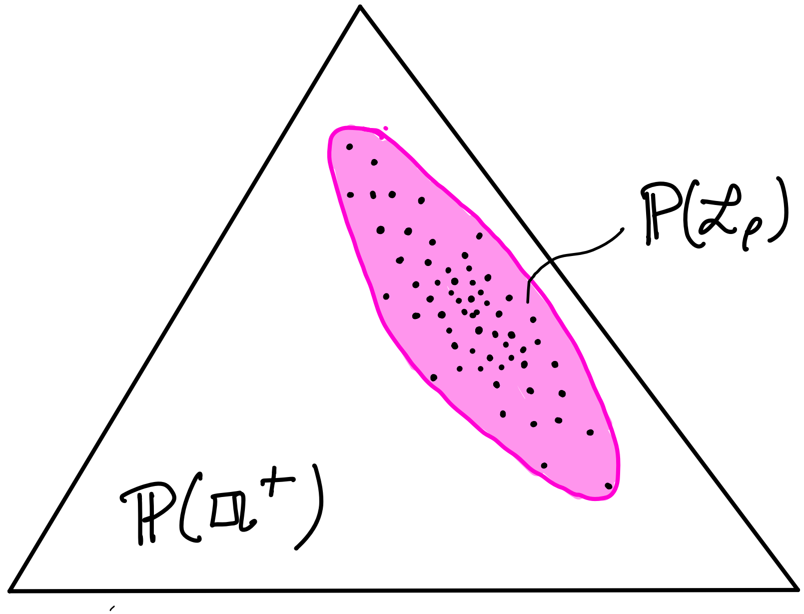






$\rho : \underset{\text{f.g.}}{\Gamma} \longrightarrow \underset{\text{semi-simple}}{G}$ injective, discrete

$\boxed{\underline{\text{Def}^n} : \mathcal{L}_\rho := \overline{\mathbb{R}_+ \lambda_\rho([\Gamma])} \quad \text{limit cone}}$



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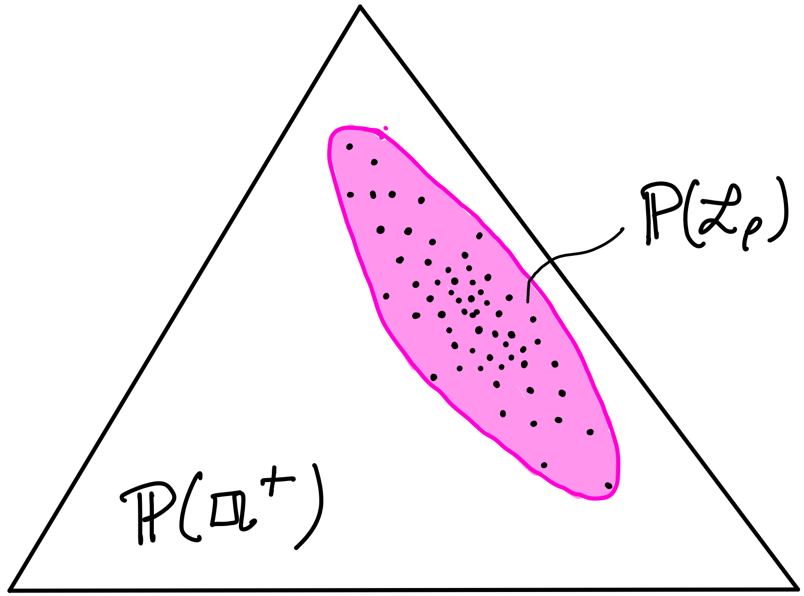

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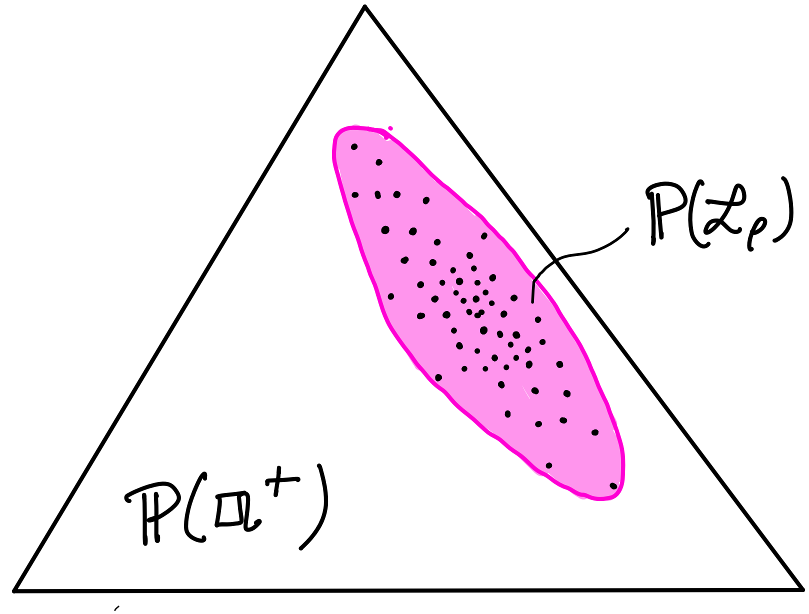

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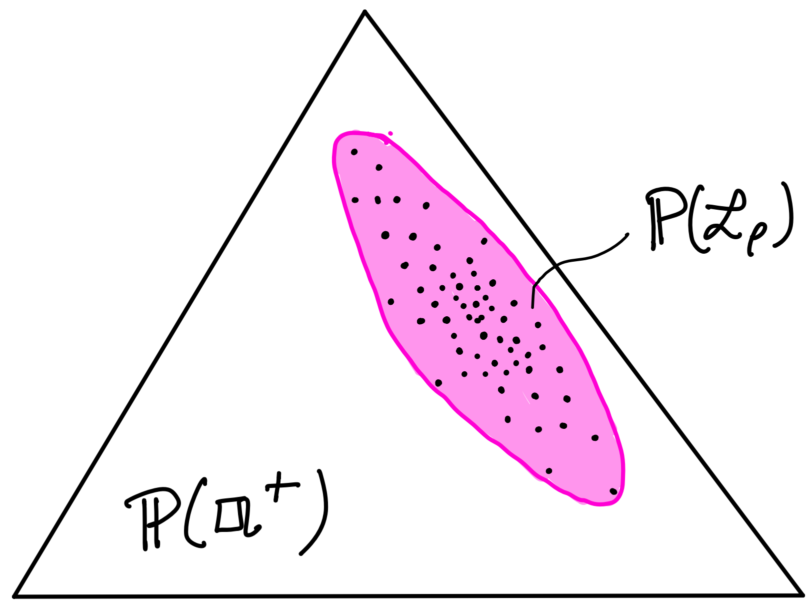

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Our goal: Understand $\partial \mathcal{L}_\rho$ when $\Gamma = \pi_1 S$, $\rho = (\ell_1, \ldots, \ell_d)$

① How well is $\partial \mathcal{L}_\rho$ approximated by $\lambda_\rho([\pi_1 S])$?

② Which closed curves / currents see the boundary?

Main Theorem 1 (D - Guéritaud - Kassel, in progress)

$\rho = (\rho_1, \ldots, \rho_d)$ d-tuple of hyperbolic structures on S, with limit cone $\mathcal{L}_\rho \subset (\mathbb{R}^+)^d$.

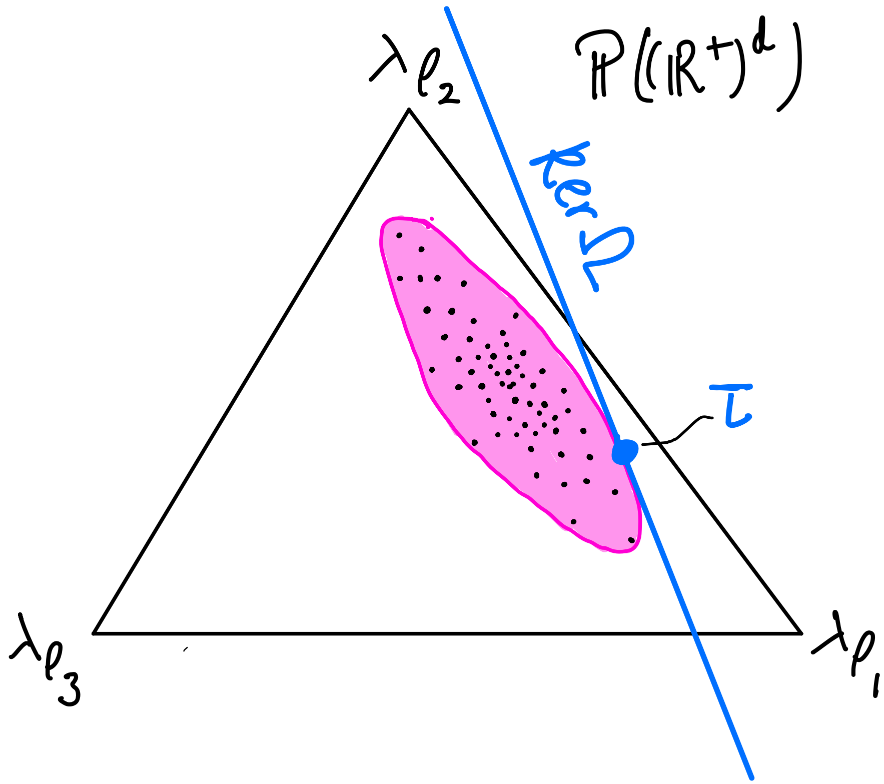
$\Omega = (\omega_1, \ldots, \omega_d) \in (\mathbb{R}^d)^*$ with $\Omega(\mathcal{L}_\rho) \geq 0$, $\ker \Omega \cap \partial \mathcal{L}_\rho = \mathbb{R}^+ \tau \in (\mathbb{R}^+)^d$.

supporting hyperplane

If $\omega_1 < 0 < \omega_2, \ldots, \omega_d$ and $\omega_1 + \ldots + \omega_d > 0$,

then $\tau = \lambda_\rho(\mu)$ for

μ a measured lamination.



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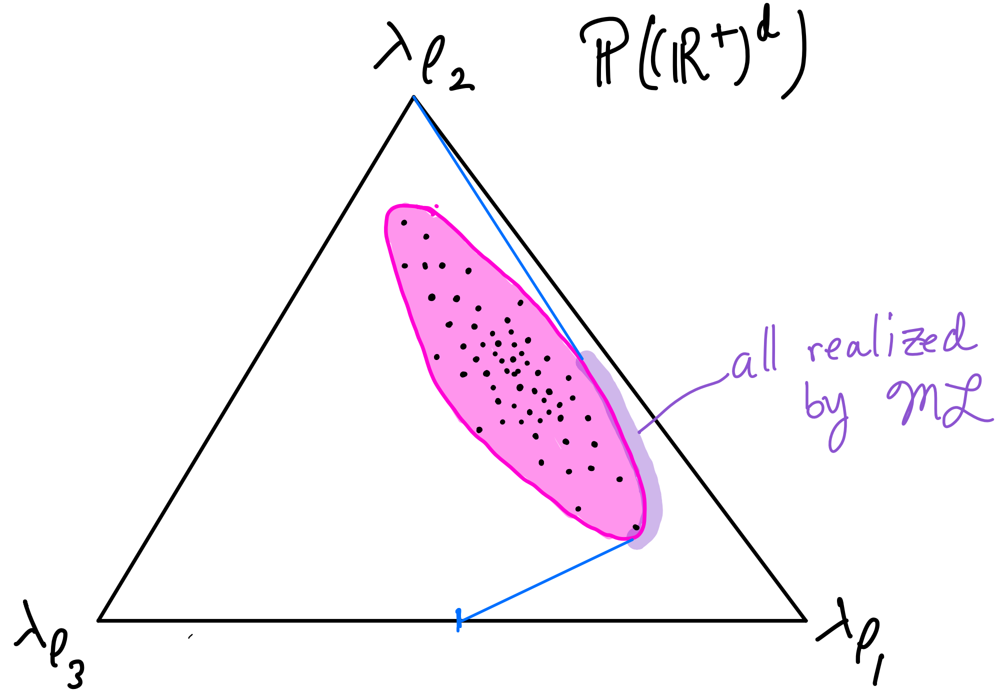

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If τ_i unique minimum coordinate of $\tau = (\tau_1, \dots, \tau_d)$ and $\omega_i \neq 0$, then:

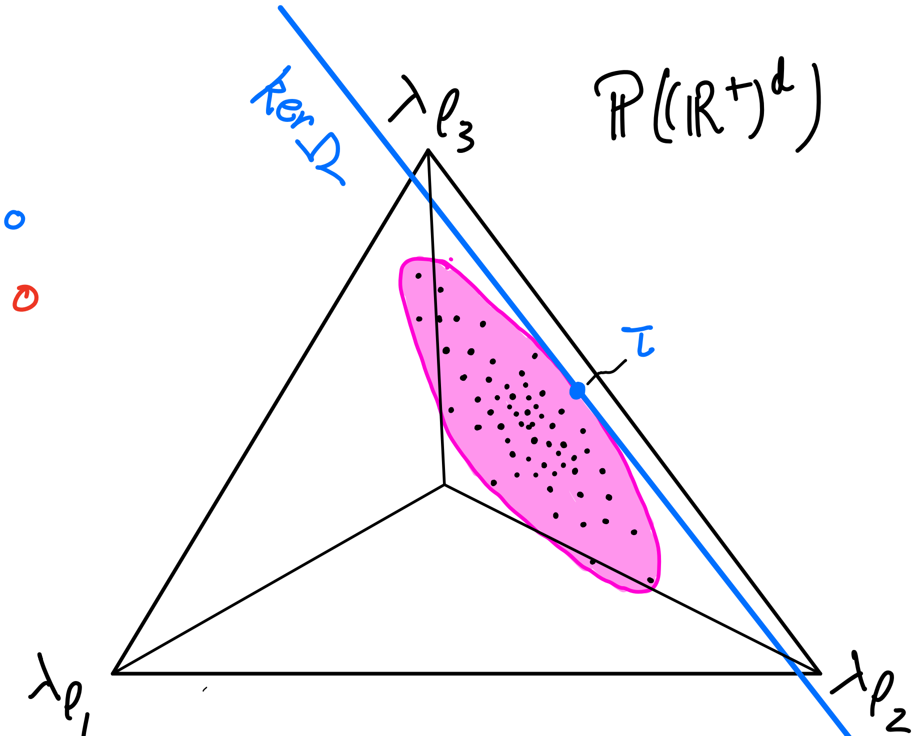
(A) $\tau = \lambda_\rho(\mu)$ for $\mu \in Curr(S)$ a

- measured quasi-lamination if $\omega_i > 0$
- measured anti-lamination if $\omega_i < 0$

and

(B) $\exists \; [\gamma_n] \to \infty$ in $[\pi_1 S]$ s.t.

$$\Omega(\lambda_\rho(\gamma_n)) = O(\exp(-k \|\gamma_n\|))$$



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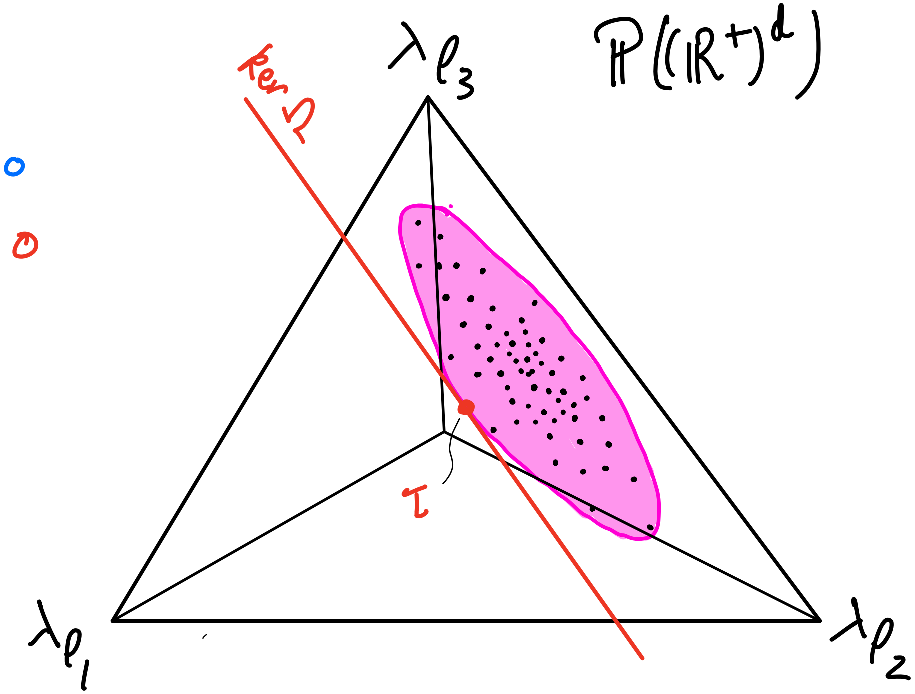

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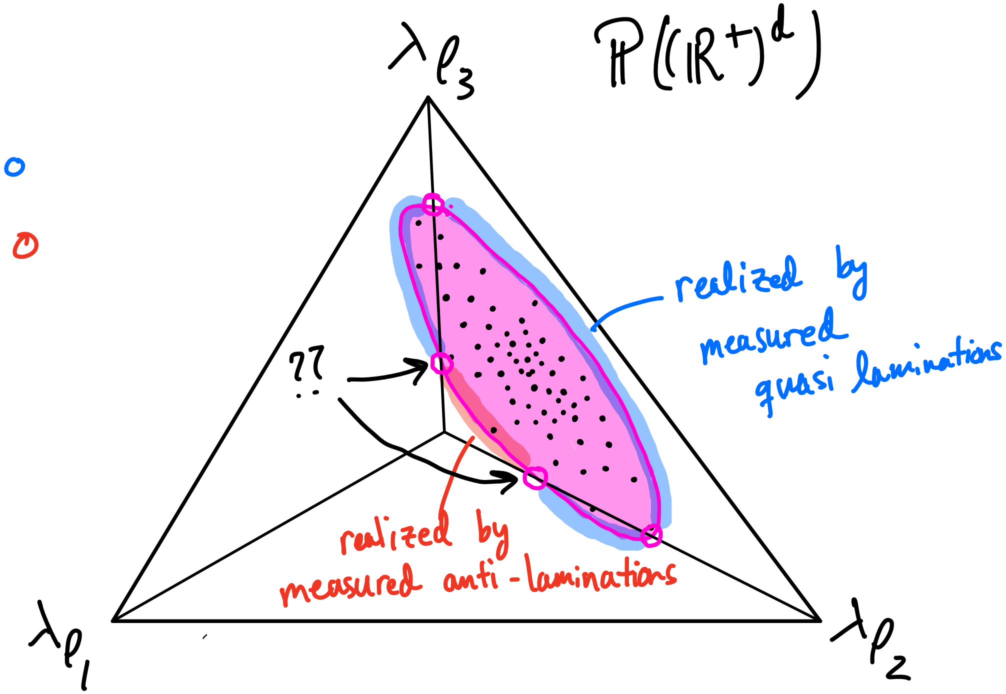

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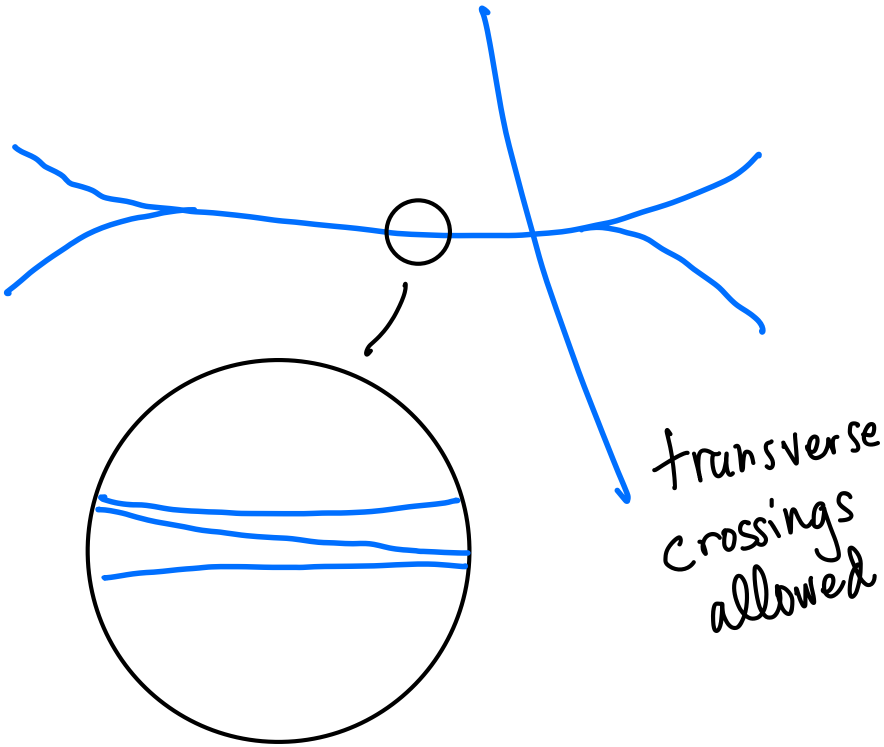
Fix a reference hyperbolic metric on S.

Def^n A measured $\begin{matrix}\text{quasi}\\ \text{anti}\end{matrix}$ lamination is $\mu \in \mathrm{Curr}(S)$ with $\Lambda = \mathrm{supp}(\mu)$ a

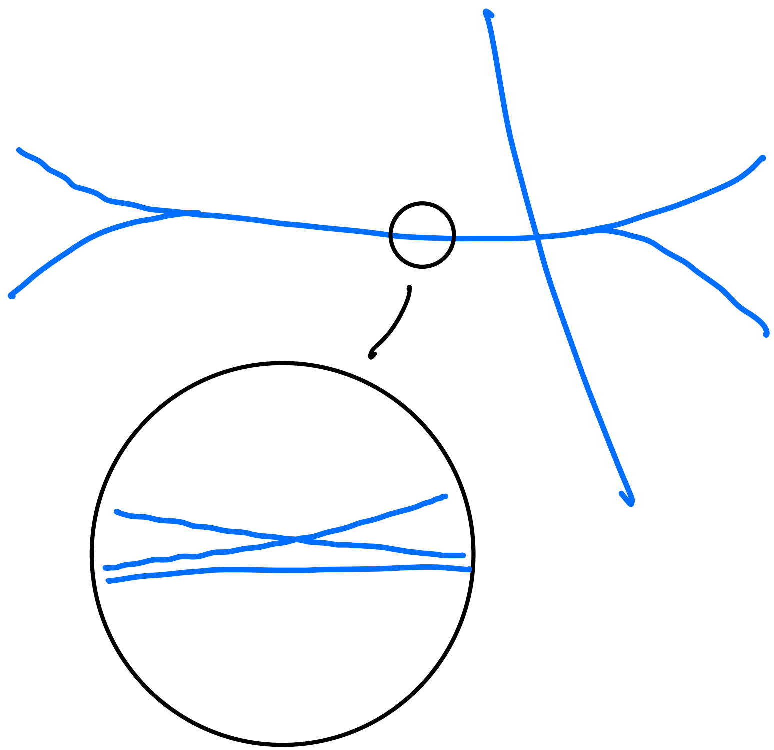
quasi-lamination:

$\exists\, \varepsilon > 0$ s.t. any flow segments $\ell, \bar{\ell} \in \Lambda$ that are ε-close in T^1S are disjoint on S.



anti-lamination:

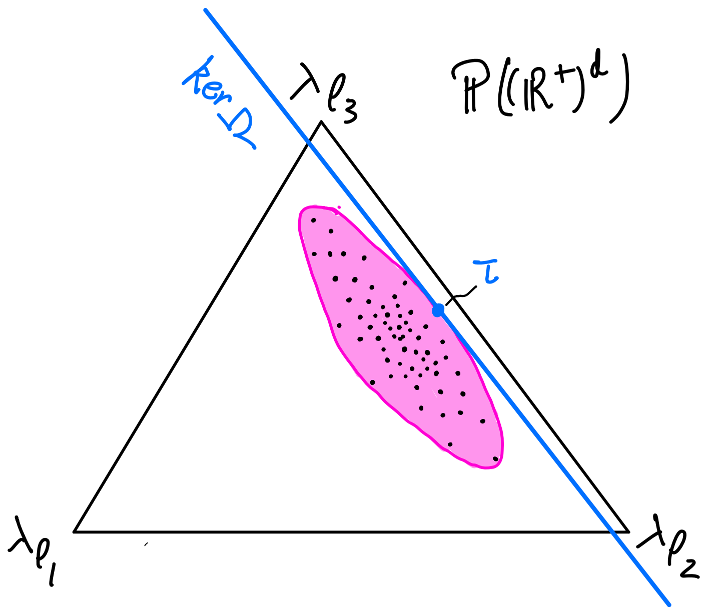
$\exists\, \varepsilon > 0$ s.t. any flow segments $\ell, \bar{\ell} \in \Lambda$ (non-asymptotic) that are ε-close in T^1S must cross.



proof ideas Main theorem 2 Ⓐ

for $\ell, \bar{\ell}$ crossing geodesics on $\tilde{S}$, angle θ_i in hyp. structure ρ_i

$$\overrightarrow{slack}(\ell, \bar{\ell}) := \left(4 \log \cos \tfrac{\theta_1}{2}, \ldots, 4 \log \cos \tfrac{\theta_d}{2}\right)$$



$$\lim_{n \to \infty} t_n [\gamma_n] = \mu \in Curr(S) \qquad \lambda\!\!\lambda_\rho(\mu) = \tau$$

Choose $[\gamma_n]$ record breakers:

$$\frac{\Omega(\lambda\!\!\lambda_\rho([\gamma_n]))}{\lambda_{\rho_1}([\gamma_n])} \leq \frac{\Omega(\lambda\!\!\lambda_\rho([\eta]))}{\lambda_{\rho_1}([\eta])} \quad \text{for all shorter } \eta$$

$\boxed{\text{Lemma 1: for any } \ell, \bar{\ell} \in supp(\mu) \text{ that cross, } \Omega(\overrightarrow{slack}(\ell, \ell')) > 0}$

$\boxed{\text{Lemma 2: } \theta_j \approx \theta_i^{\tau_j/\tau_i}. \Rightarrow \text{if } \tau_i \text{ smallest, and } \theta_i \ll 1, \ \overrightarrow{slack}(\ell, \bar{\ell}) \text{ points in } -e_i \text{ direction.}}$

Proof ideas Main Thm 2.Ⓑ: exponential approximation

Quasi/anti laminations have exponential self recurrence.

$\boxed{\text{Recurring theme: Extremizers are dynamically well behaved.}}$

Analog of Lipschitz maps:

Flat flow maps $\left\{ \begin{array}{l} \Phi : T^1\tilde{S} \longrightarrow (\mathbb{H}^2)^d \qquad (\rho_1, \dots, \rho_d) \text{ equivariant} \\ \quad\ \downarrow \\ \quad\ \ell \rightsquigarrow \quad \underbrace{\ell_1 \times \dots \times \ell_d \ \text{product of geodesics}}_{\text{a flat!}} \\ \text{flow line} \end{array} \right.$

<u>Optimize</u>: