Record breakers, Slack calculus, and the Benoist limit cone

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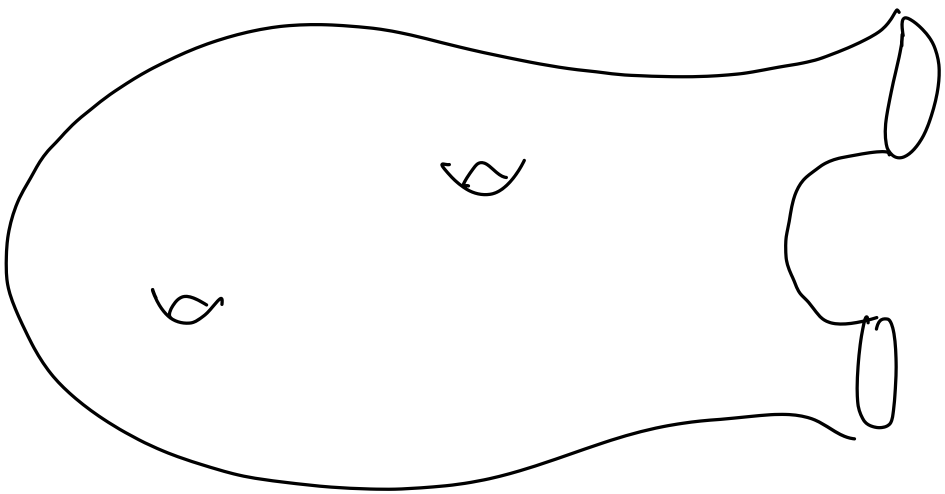
$S = {}_{\rho}\backslash \mathbb{H}^2$ hyperbolic surface

$\rho : \pi_1 S \longrightarrow G = PSL_2\mathbb{R}$



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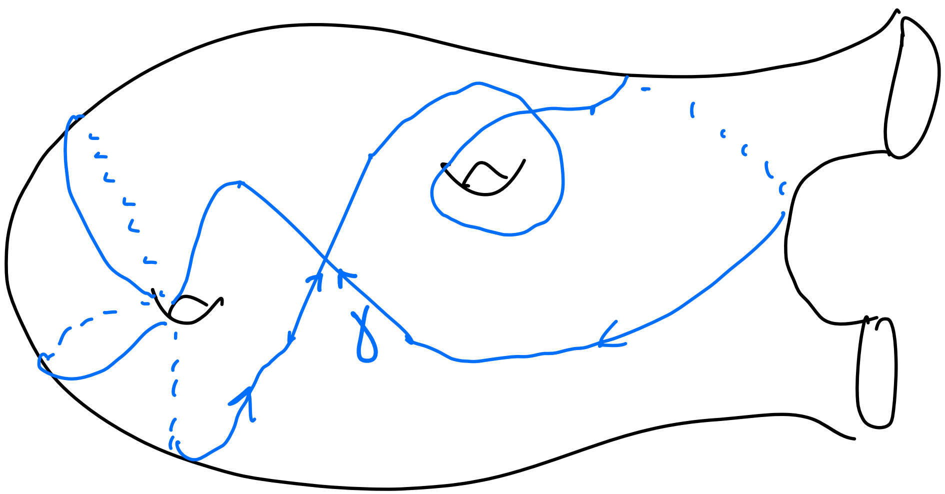
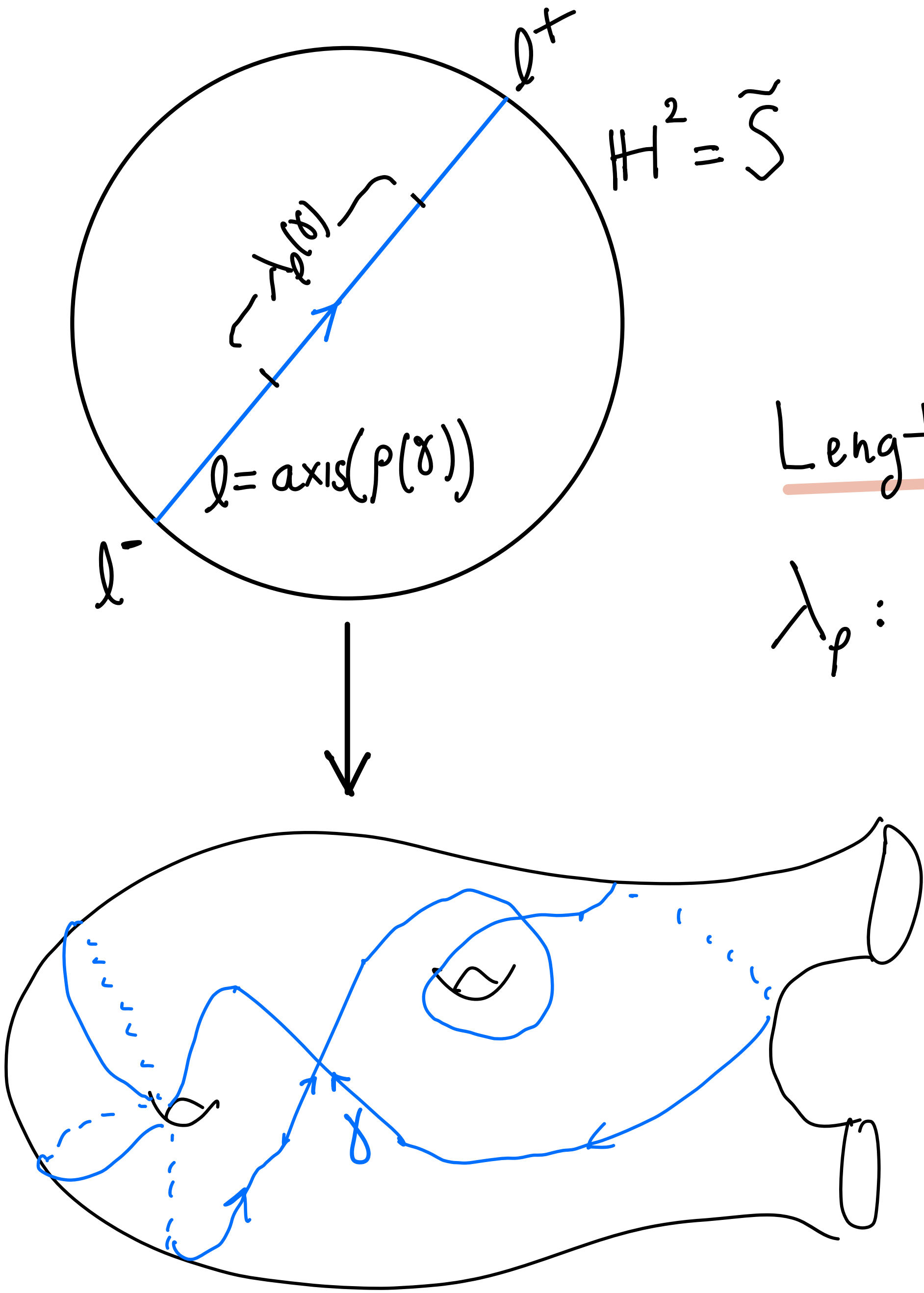
$\lambda_\rho : [\pi_1 S] \longrightarrow \mathbb{R}_+$

$[\gamma] \longmapsto$ length of geodesic representative of $[\gamma]$





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$= \lambda(\rho(\gamma))$

$\lambda : G \to \mathbb{R}_+$

$g \mapsto tl_{\mathbb{H}^2}(g)$

$= 2 \log$ top |eigenvalue|

$\mathbb{G}$ = semi-simple, rank $d \geq 2$

$\mathbb{G} \curvearrowright \mathbb{X} = \mathbb{G}/K$ symmetric space

$\mathbb{G} = \begin{cases} SL_{d+1}\mathbb{R} \\ (PSL_2\mathbb{R})^d \end{cases}$

geodesics in $\mathbb{H}^2$ $\rightsquigarrow$ maximal flats $F \cong \mathbb{R}^d$ in $\mathbb{X}$

$\mathrm{Stab}_{\mathbb{G}}(F) \cong W \ltimes \mathbb{R}^d$

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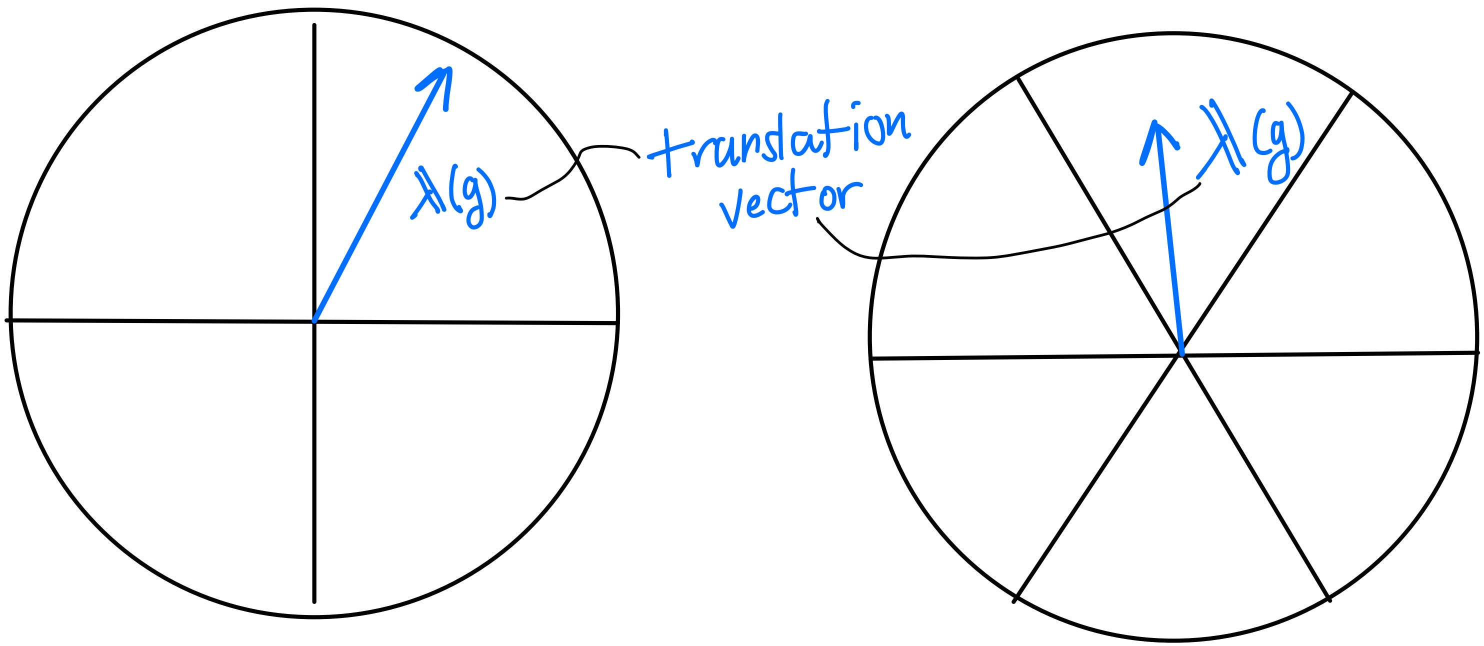

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directed geodesics in $\mathbb{H}^2$ $\rightsquigarrow$ directed maximal flats $F \cong \mathbb{R}^d$ in $\mathbb{X}$

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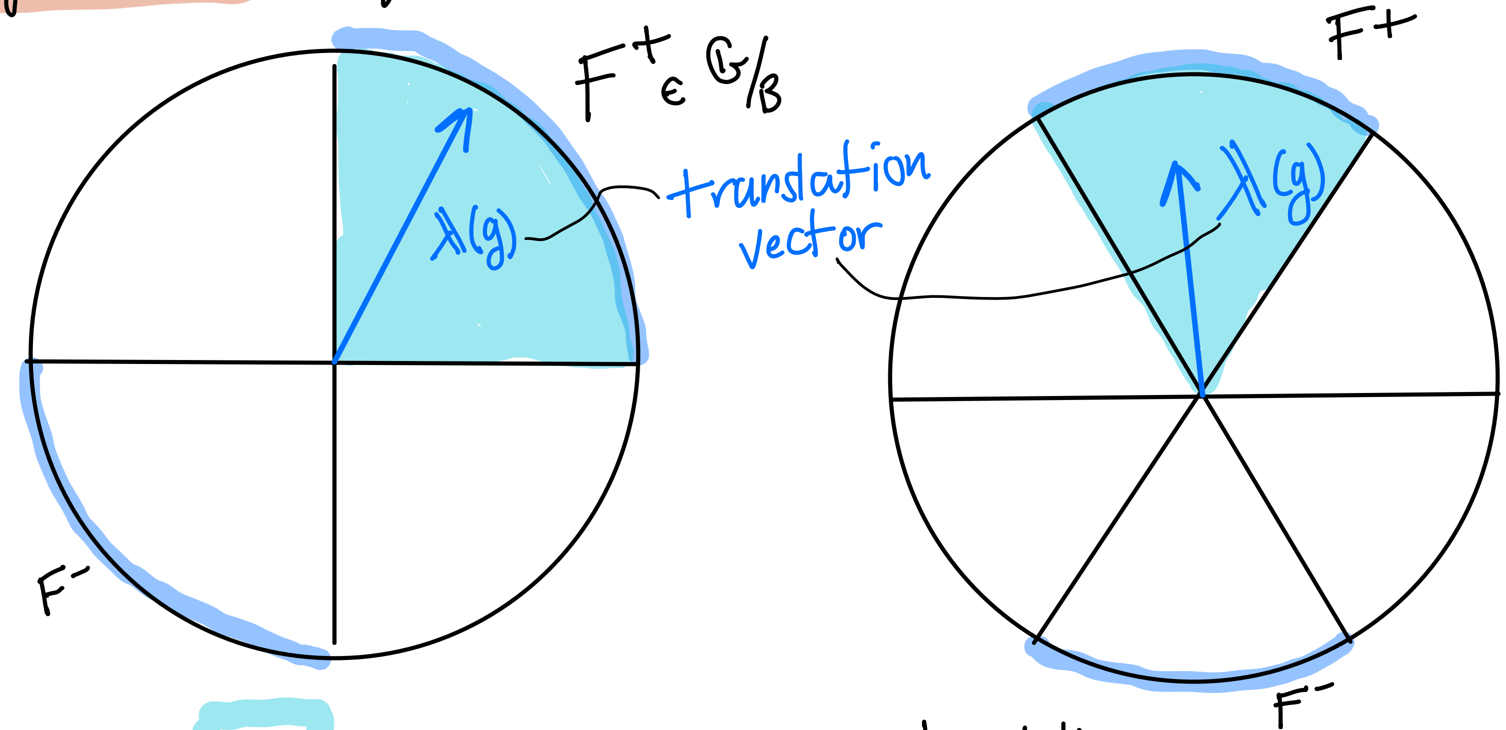


Regard $\lambda(g) \in \boxed{\mathfrak{a}^+}$ Positive Weyl chamber $\subset \mathfrak{a}$ = translation group of standard maximal flat

$\rho : \underset{\text{f.g.}}{\Gamma} \longrightarrow G$ semisimple rank $d \geq 2$ $\begin{cases} SL_{d+1}\mathbb{R} \\ (PSL_2\mathbb{R})^d \end{cases}$

discrete faithful

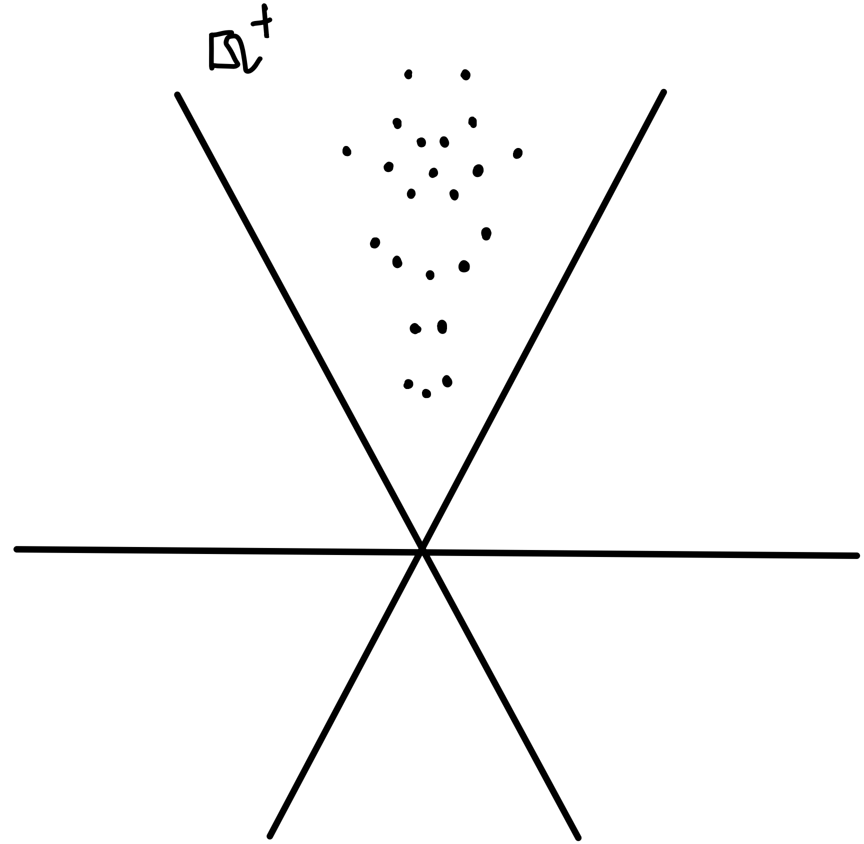
Vector valued Length Spectrum $\lambda_\rho([\gamma]) = \lambda(\rho(\gamma))$

$\lambda : G \to \mathfrak{a}^+ = \begin{cases} \left\{(\lambda_1, \ldots, \lambda_{d+1}) : \begin{matrix} \lambda_1 \geq \cdots \geq \lambda_{d+1} \\ \lambda_1 + \cdots + \lambda_{d+1} = 0 \end{matrix}\right\} \\ \mathbb{R}^d_+ \end{cases}$

log |eigenvalues|

$(\lambda_{PSL_2\mathbb{R}}, \ldots, \lambda_{PSL_2\mathbb{R}})$

Jordan Projection



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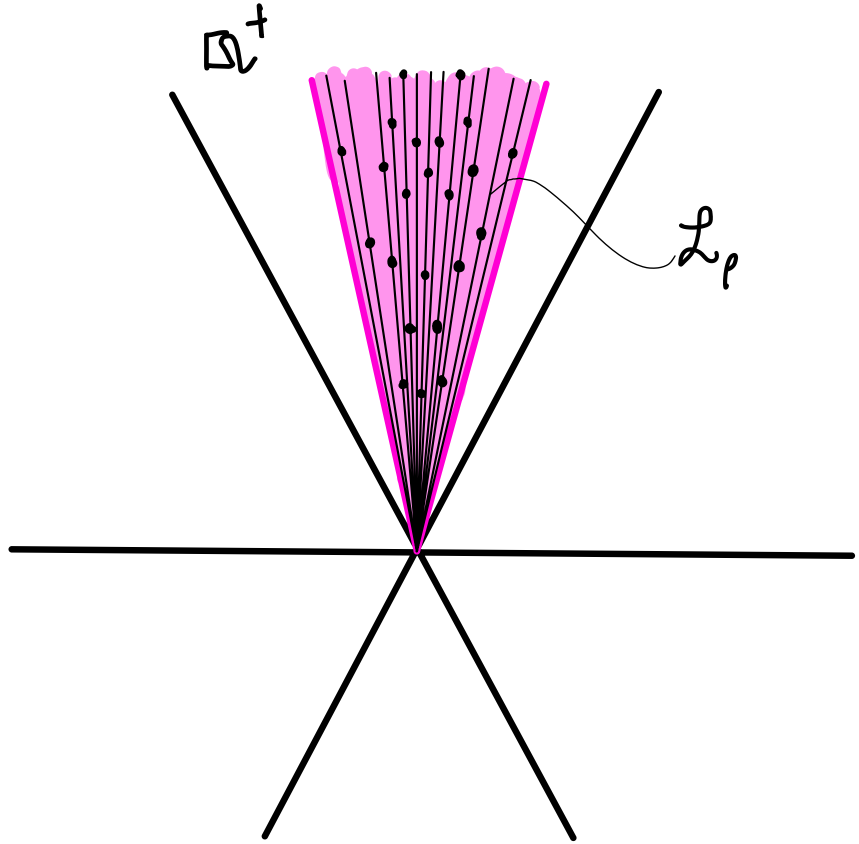

$$\boxed{\underline{\text{Def}^n}: \mathcal{L}_\rho := \overline{\mathbb{R}_+ \lambda_\rho([\Gamma])} \text{ the limit cone.}}$$

$\rho : \underset{\text{f.g.}}{\Gamma} \longrightarrow \underset{\text{s.s.}}{G}$ injective, discrete

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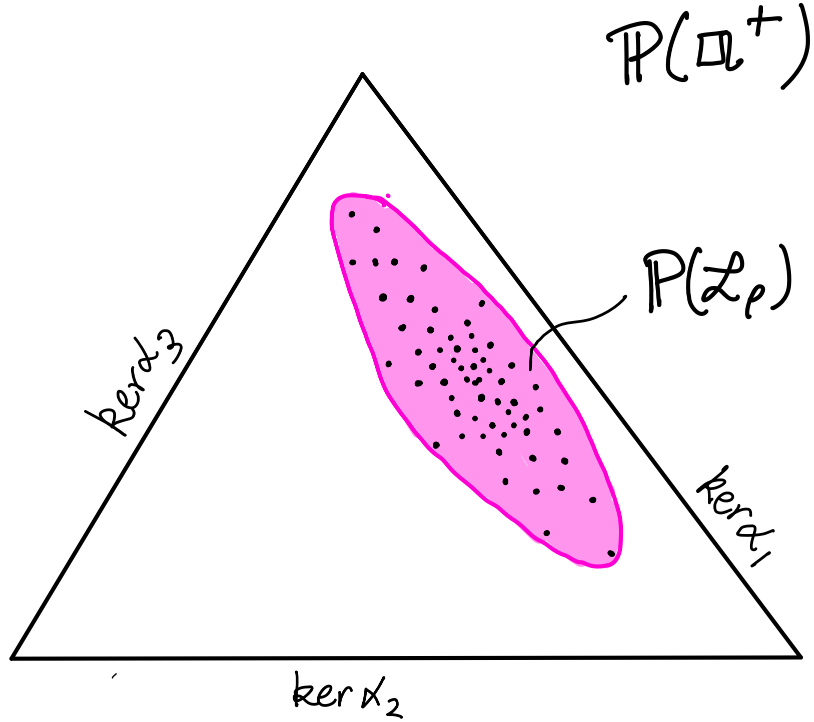

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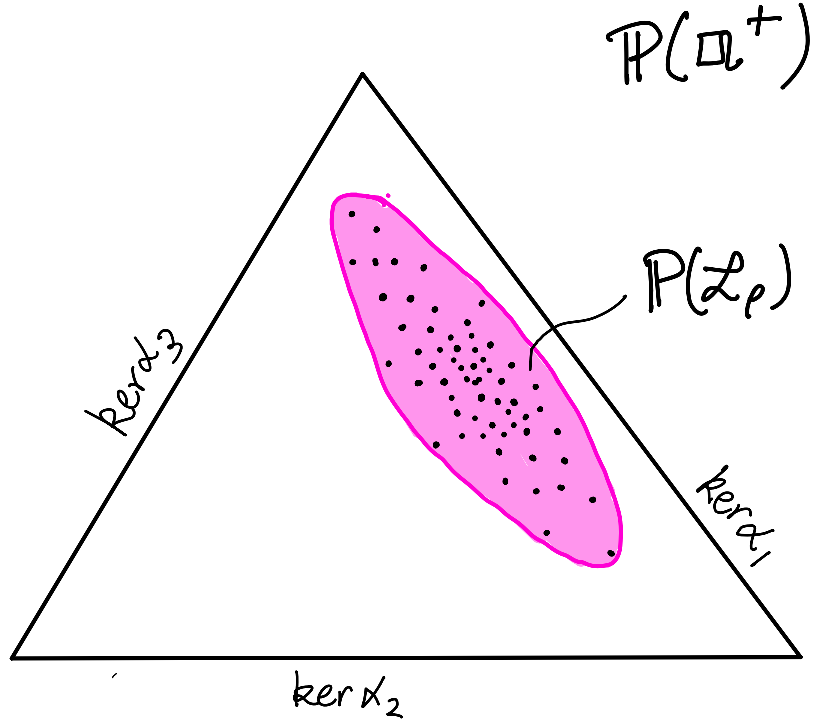

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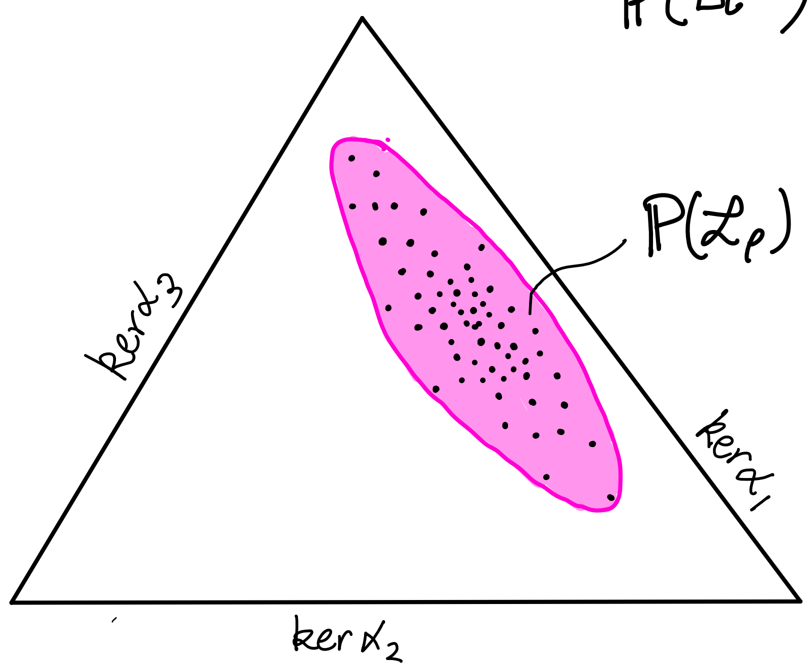

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- Approximation properties: $\forall$ closed subcone $C \subset \mathring{\mathcal{L}}_\rho$, $\varepsilon > 0$, $\lambda_\rho([\pi_1 S])$ is ε-dense in $C \setminus K$, for $K = K(C, \varepsilon)$ compact. (Benoist)

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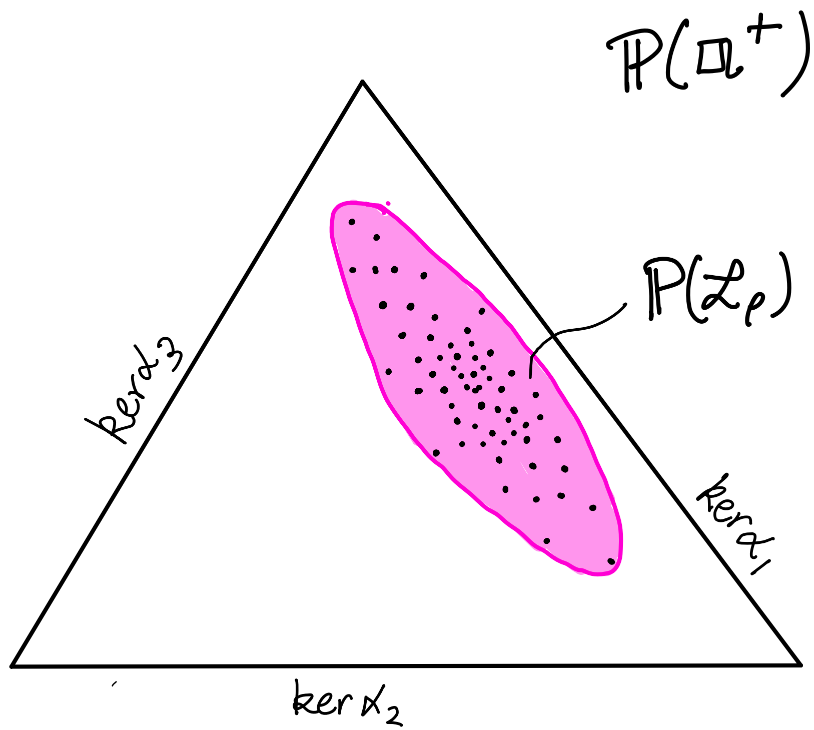

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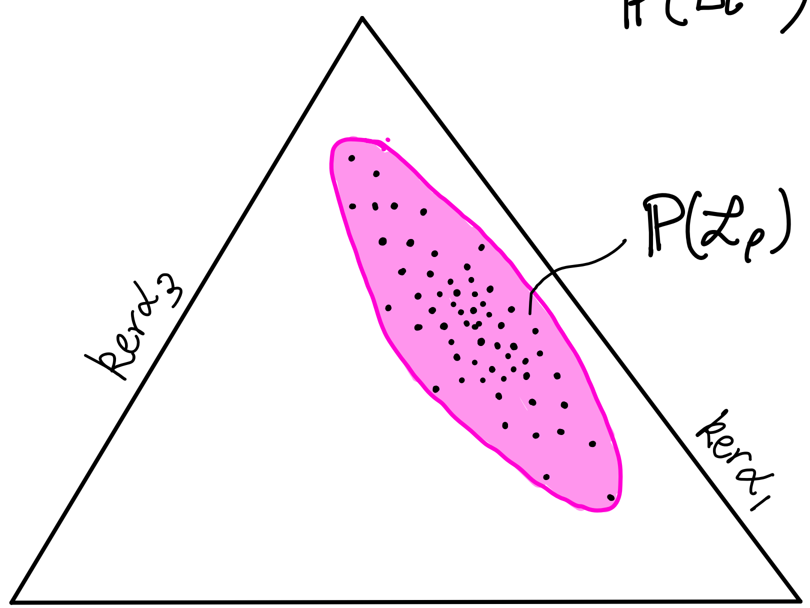

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Our goal: Understand $\partial\mathcal{L}_\rho$ when $\Gamma = \pi_1 S$

① How well is $\partial\mathcal{L}_\rho$ approximated by $\lambda_\rho([\pi_1 S])$?

② What elements/directions in $\pi_1 S$ see the boundary?

$\mathbb{G} = PSL_2\mathbb{R} \times PSL_2\mathbb{R}$

$\rho = (\rho_1, \rho_2) : \pi_1 S \longrightarrow \mathbb{G}$

$S \xrightarrow{\cong} \rho_1 \backslash \mathbb{H}^2$

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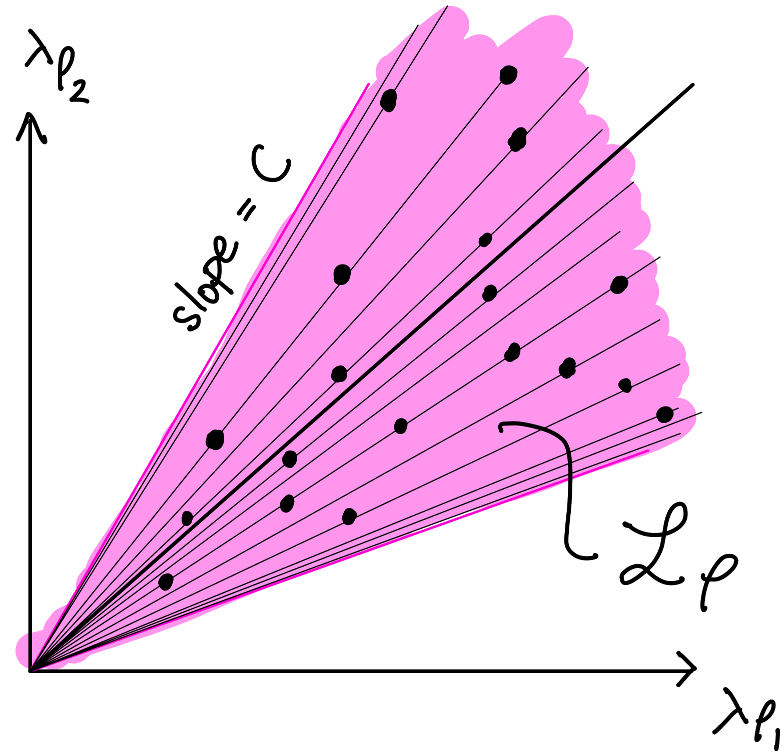

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Thurston, Kassel, Guéritaud-Kassel:

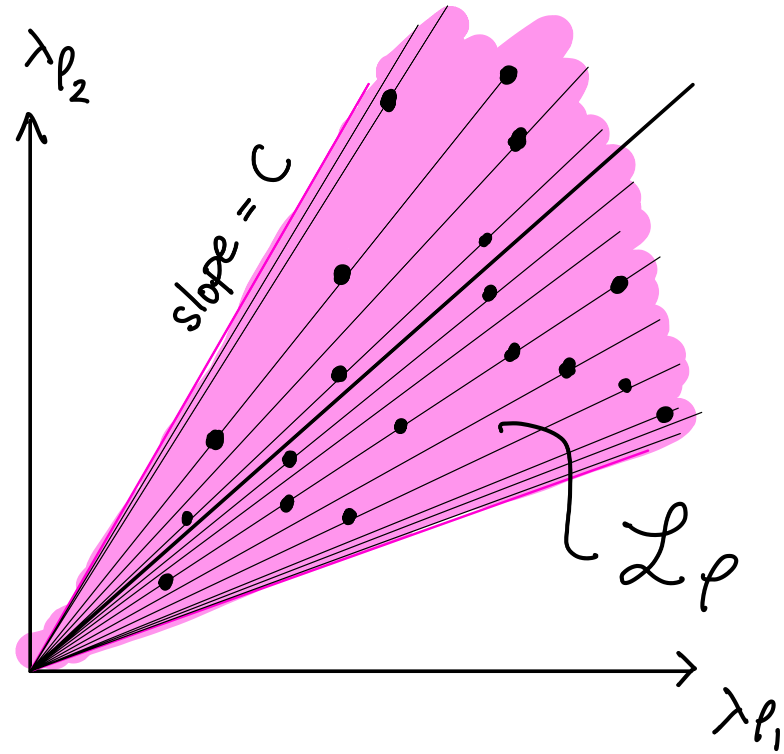
if $C > 1$, then

$$C = \inf \left\{ \mathrm{Lip}(f) : \begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ \rho_1\backslash\mathbb{H}^2 \arrow[rr, "f"] & & \rho_2\backslash\mathbb{H}^2 \end{tikzcd} \right\}$$

and $\exists$ optimal f that stretches by C along

a geodesic lamination $\Lambda \subset \rho_1\backslash\mathbb{H}^2$

a closed subset foliated by geodesics



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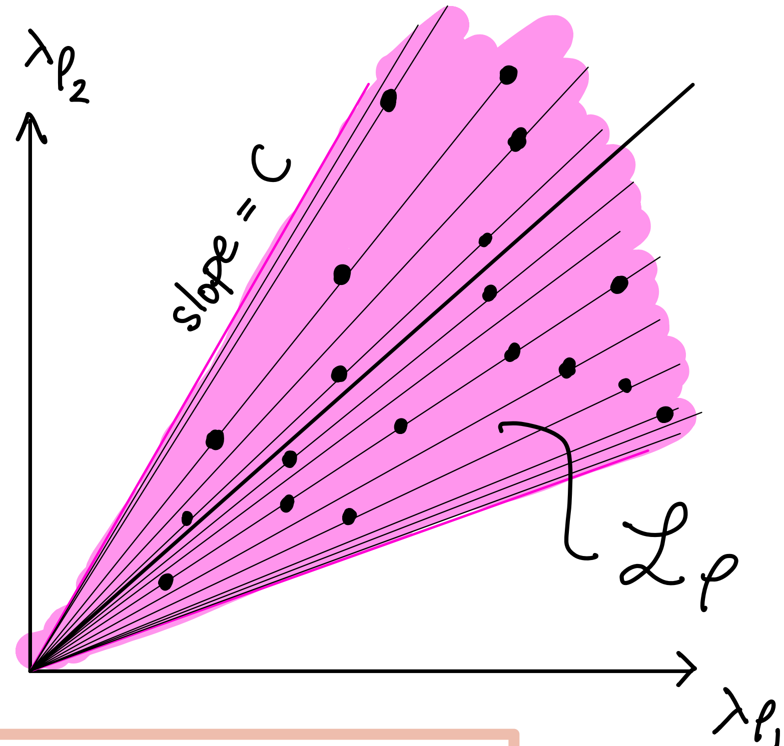
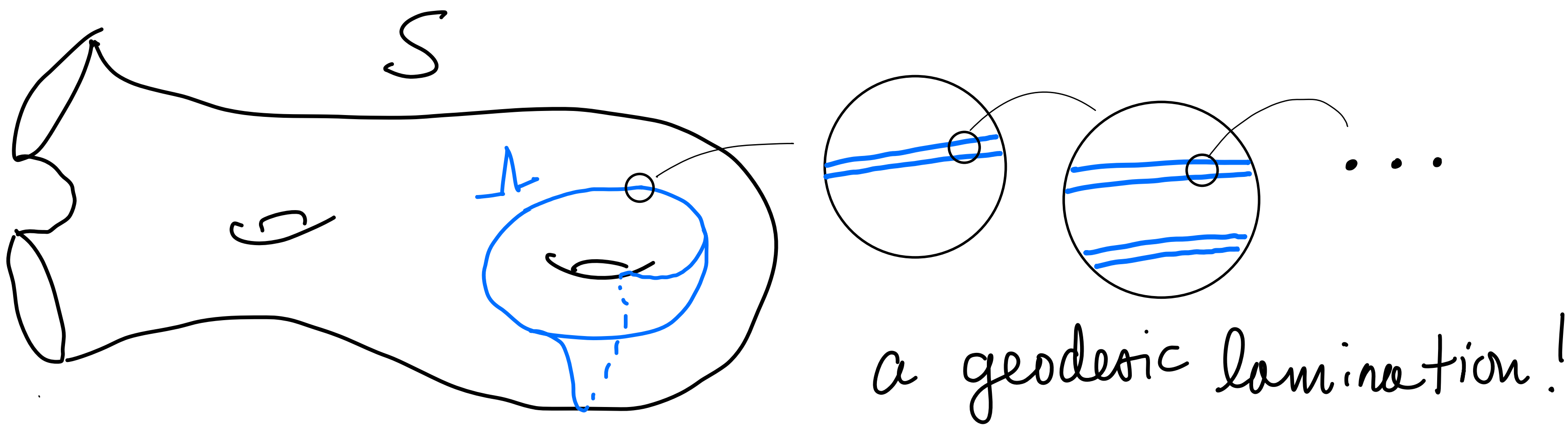


Cor:

① $\exists\ (\gamma_n)$ s.t. $\mathrm{dist}\left(\lambda_\rho(\gamma_n), \partial^{top}\mathcal{L}_\rho\right) = O\left(\exp(-c\,\lambda_{\rho_1}(\gamma_n))\right)$

② each γ_n can be chosen simple.



Fact: Laminations have good recurrence properties.

Digression on Geodesic Currents

if $\rho : \pi_1 S \longrightarrow \mathbb{G}$ is nice (B-Anosov), then:

λ_ρ extends to the space $\mathrm{Curr}(S)$ of geodesic currents on S

$$\boxed{\text{Fact}: \quad \mathcal{L}_\rho = \overline{\mathbb{R}_+ \lambda_\rho([\pi_1 S])} = \lambda_\rho(\mathrm{Curr}(S))}$$

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Recall: $\mathcal{G}(\tilde{S}) = \partial\pi_1 S \times \partial\pi_1 S \setminus \Delta$ space of geodesics in $\tilde{S}$

$\mathrm{Curr}(S)$ = space of $\pi_1 S$-invariant, locally finite, Borel measures μ on $\mathcal{G}(\tilde{S})$.

Fact:
- weighted closed curves $\hookrightarrow$ $\mathrm{Curr}(S)$ dense.

measured laminations

- $\overline{\text{weighted simple closed curves}} = \mathcal{ML}(S) = \{\mu \in \mathrm{Curr}(S) : \mathrm{supp}(\mu) = \text{geodesic lamination}\}$

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in $\mathbb{G} = PSL_2\mathbb{R} \times PSL_2\mathbb{R}$ setting, if $\tau = (1, c) \in \partial^{top}\mathcal{L}_\rho$ and $c > 1$, then

$\exists\ \mu \in \mathcal{ML}(S)$ with $\lambda_\rho(\mu) = \tau$

Example (D-Stecker): $\exists$ a $\rho : \pi_1 S \longrightarrow SL_3\mathbb{R}$ Hitchin

such that $\forall\ \mu \in \mathrm{Curr}(S)$ with $\lambda_\rho(\mu) \in \partial \mathcal{L}_\rho$, $\mu = t\gamma$ for some closed curve γ with self crossings.

G = real split semi-simple, rank $= d$ e.g. $G = \begin{cases} SL_{d+1}\mathbb{R} \\ (PSL_2\mathbb{R})^d \end{cases}$

Main theorem (D-Guéritaud-Kassel, in progress)

Let $\rho: \pi_1 S \longrightarrow G$ a positive representation, with limit cone $\mathcal{L}_\rho \subset \mathfrak{a}^+$.

Let $\Omega = \sum_{i=1}^{d} c_i \omega_i \in \mathfrak{a}^*$ with $\Omega(\mathcal{L}_\rho) \geq 0$
weights $\ker\Omega \cap \mathcal{L}_\rho = \mathbb{R}_+ \tau$.

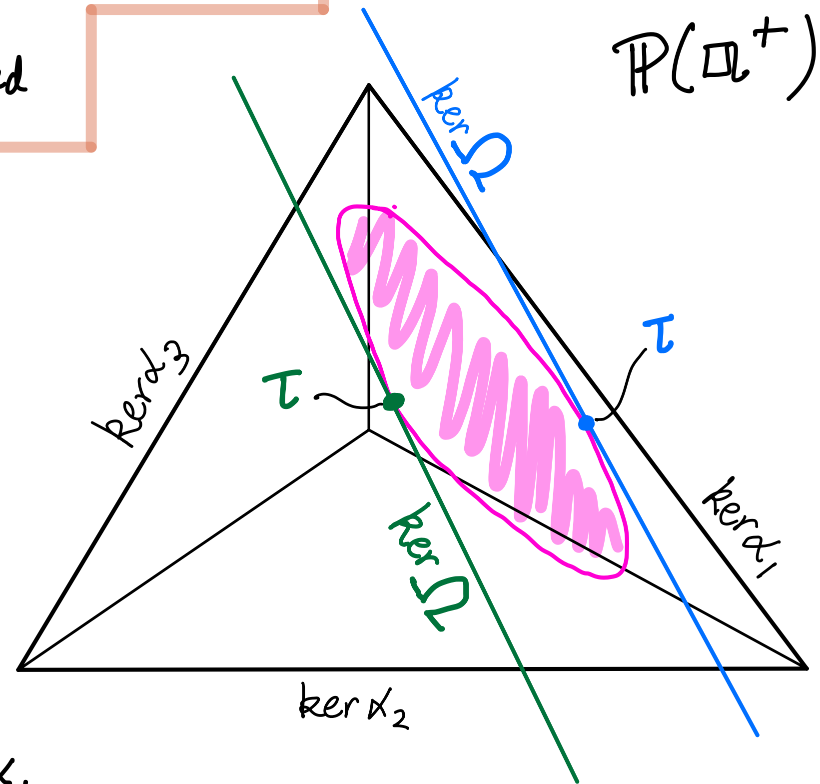
If $(\alpha_1(\tau), \ldots, \alpha_d(\tau))$ is uniquely minimized by $\alpha_j(\tau)$ and $c_j \neq 0$, then
roots

① $\exists\ \gamma_n \to \infty$ in $[\pi_1 S]$ s.t.

$$\Omega(\lambda_\rho(\gamma_n)) = O(\exp(-c\|\gamma_n\|))$$

② $\tau = \lambda_\rho(\mu)$ for $\mu \in Curr(S)$ a

- measured quasi-lamination if $c_j > 0$
- measured anti-lamination if $c_j < 0$.



$\bullet\ \alpha_1$

Fix a reference hyperbolic metric on S.

Def^n A measured $\begin{matrix}\text{quasi}\\ \text{anti}\end{matrix}$ lamination is $\mu \in Curr(S)$ with $\Lambda = supp(\mu)$ a

quasi-lamination:	anti-lamination:

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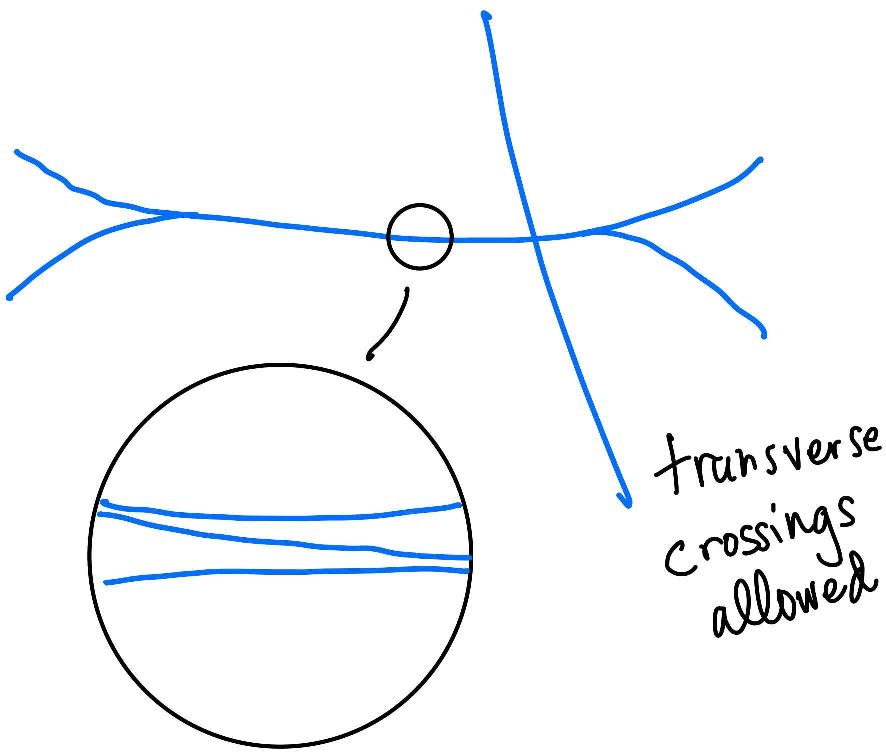
quasi-lamination:

$\exists\, \varepsilon > 0$ s.t. any $\ell, \ell' \in \Lambda \subset \mathcal{G}(\tilde{S})$ that pass ε close in $T^1\tilde{S}$, are disjoint.



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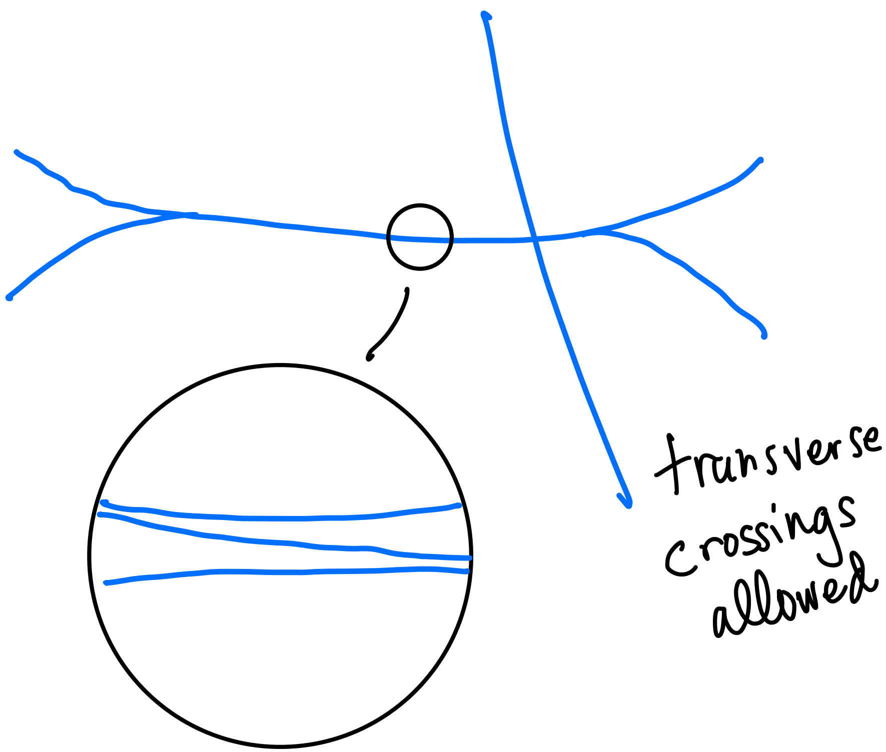
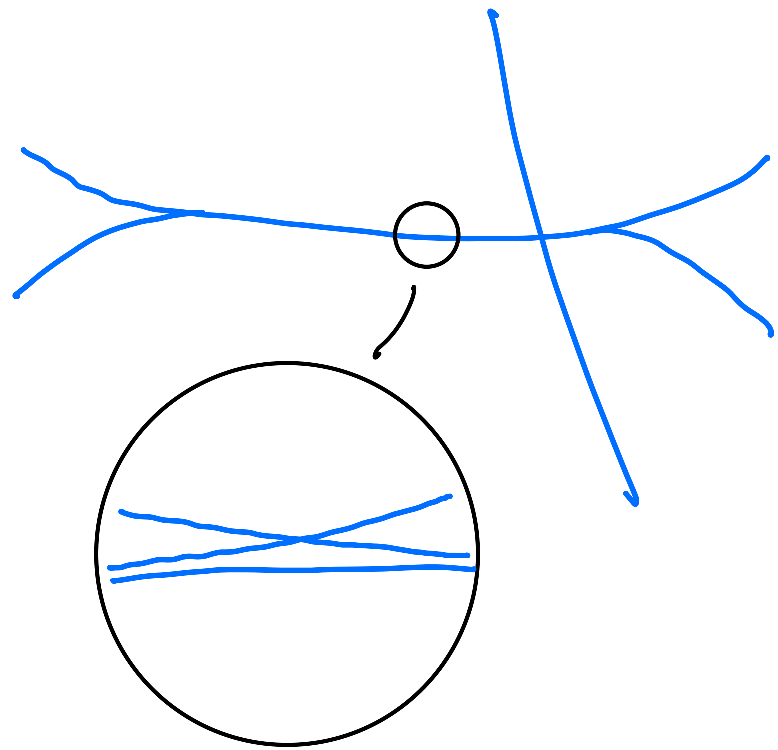
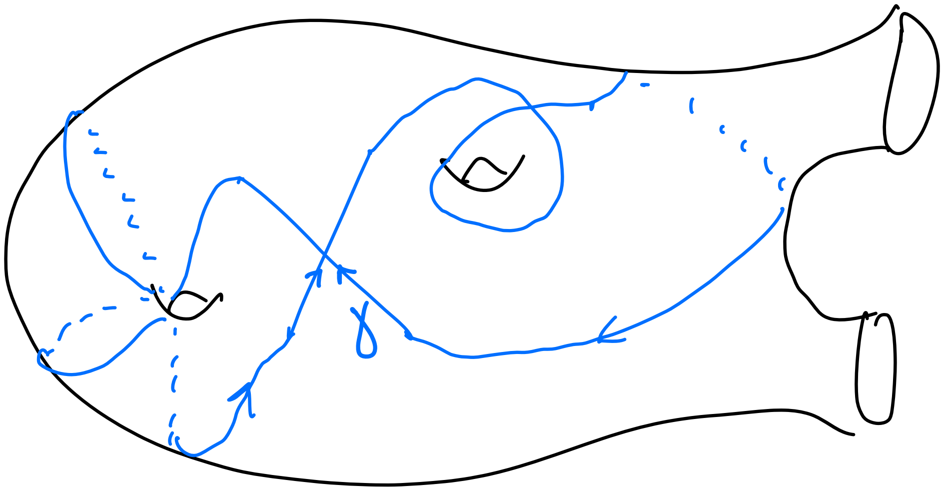


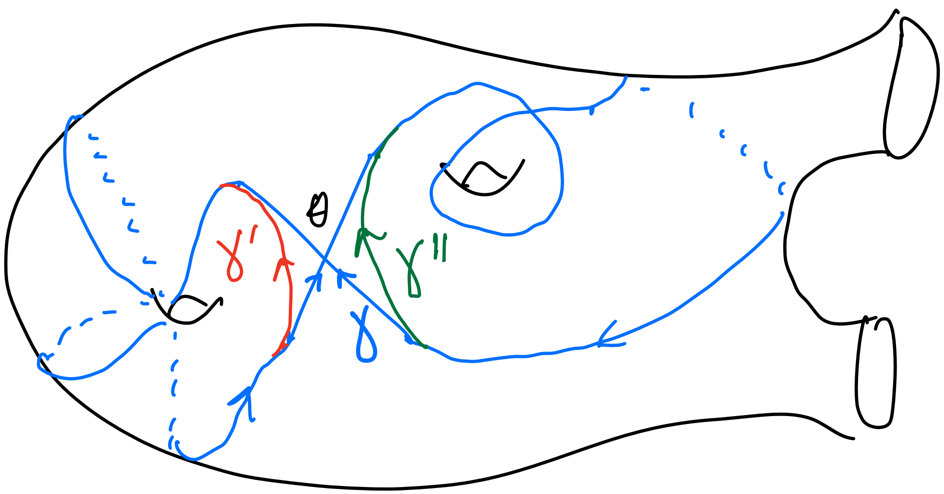
anti-lamination:

$\exists\, \varepsilon > 0$ s.t. any $\ell, \ell' \in \Lambda$ non-asymptotic that pass ε-close in $T^1\tilde{S}$, must cross.





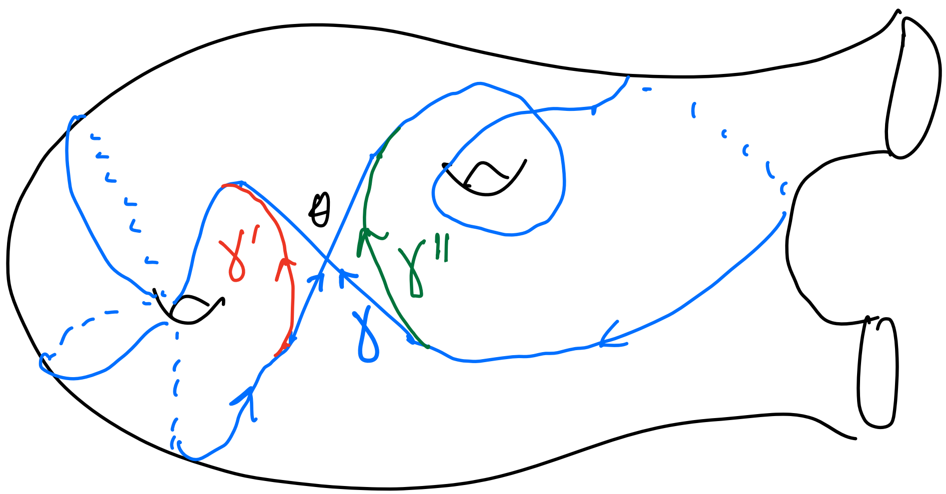
Back to $PSL_2\mathbb{R}$ . . .



Back to $PSL_2\mathbb{R}$ ...

$$\boxed{\lambda_\rho(\gamma') + \lambda_\rho(\gamma'') = \lambda_\rho(\gamma) + 4\ln\cos\frac{\theta}{2} + o(1)}$$

slack formula!



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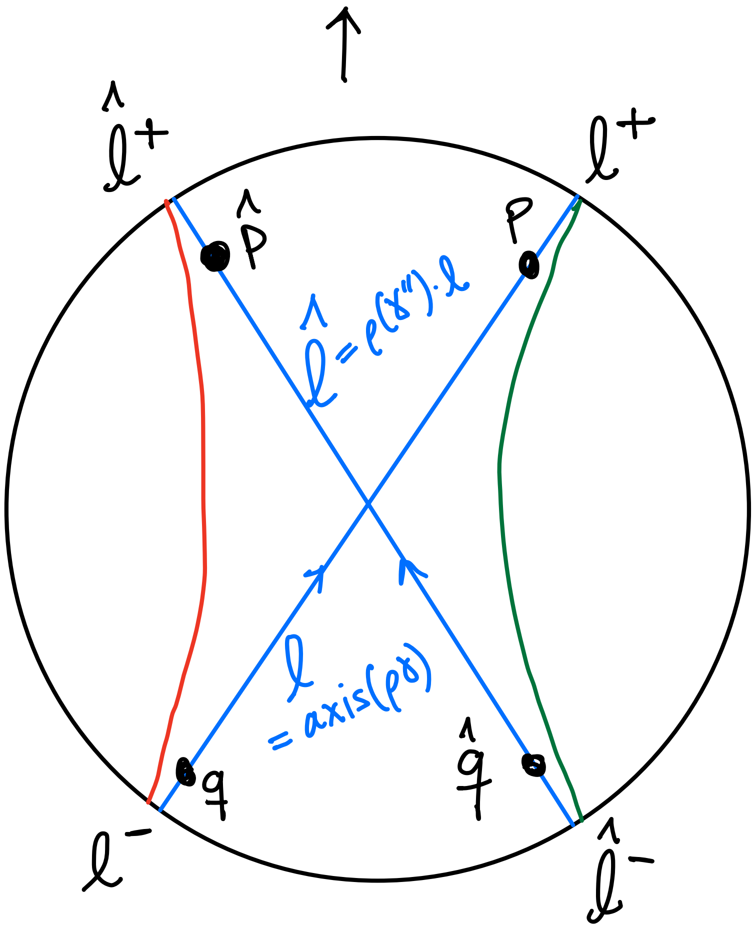


$$\lim_{\substack{p\to\ell^+ \quad \hat{p}\to\hat{\ell}^+ \\ q\to\ell^- \quad \hat{q}\to\hat{\ell}^-}} d(q,\hat{p}) + d(\hat{q},p) - d(q,p) - d(\hat{q},\hat{p})$$

$$= 2\ln\,[\ell^-, \hat{\ell}^-; \hat{\ell}^+, \ell^+]$$

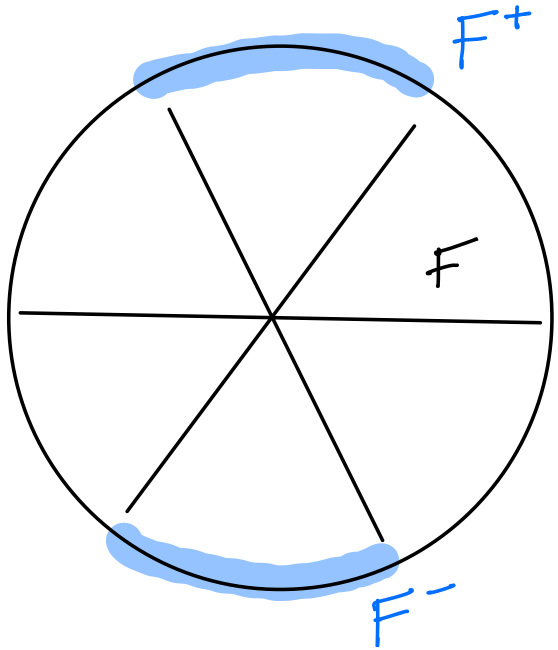
$$= 4\ln\cos\frac{\theta}{2}$$

Def^n: for ℓ, ℓ' directed geodesics in $\mathbb{H}^2$

$$\mathrm{slack}(\ell, \hat{\ell}) := 4\ln\cos\frac{\theta}{2}$$

Slacks in higher rank

directed geodesic in $\mathbb{H}^2$ ⇝ directed maximal flat in $\mathbb{X} = \mathbb{G}/\mathbb{K}$



"flag manifold"

$F^+, F^- \in \mathbb{G}/\mathbb{B}$

two opposite chambers in $\partial_\infty F$

Given directed flats F, $\hat{F}$, define:

cross ratio (Benoist)

$$\mathrm{slack}(F,\hat{F}) = [F^-, \hat{F}^-; \hat{F}^+, F^+] \in \mathfrak{a}$$

vector valued slack

$$= \lim_{\substack{p \to F^+ \\ q \to F^-}} \ \lim_{\substack{\hat{p} \to \hat{F}^+ \\ q \to \hat{F}^-}} \vec{d}(\hat{p}, q) + \vec{d}(p, \hat{q}) - \vec{d}(p, q) - \vec{d}(\hat{p}, \hat{q})$$

$\mathbb{G} = SL_{d+1}\mathbb{R}$

$$[A,B;C,D] = \sum_{i=1}^{d} \frac{(A^{d-i} \wedge D^i)(B^{d-i} \wedge C^i)}{(A^{d-i} \wedge C^i)(B^{d-i} \wedge D^i)} \alpha_i$$

$\mathbb{G} = (PSL_2\mathbb{R})^d$

$$[A,B;C,D] = \left([A^1, B^1; C^1, D^1], \ldots, [A^d, B^d; C^d, D^d]\right)$$

Positive Representations

$S =$ compact, oriented surface

$G =$ split real semi simple $= \begin{cases} SL_{d+1}\mathbb{R} \\ (PSL_2\mathbb{R})^d \end{cases}$

Def^n (Fock-Goncharov)

$\rho : \pi_1 S \to G$ is positive if $\exists$ a ρ-equivariant boundary map $\xi : \partial_\infty \pi_1 S \to G/B$ (full flags)

taking positively oriented triples to positive triples of flags (Lusztig)

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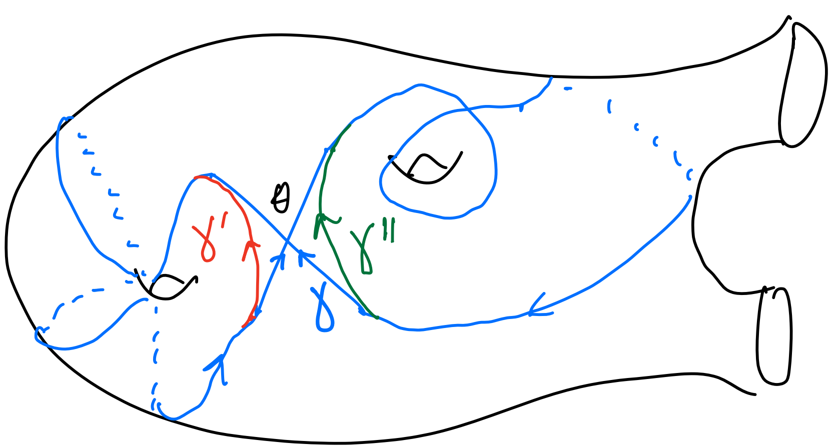
$G =$ split real semi simple $= \begin{cases} SL_{d+1}\mathbb{R} \\ (PSL_2\mathbb{R})^d \end{cases}$

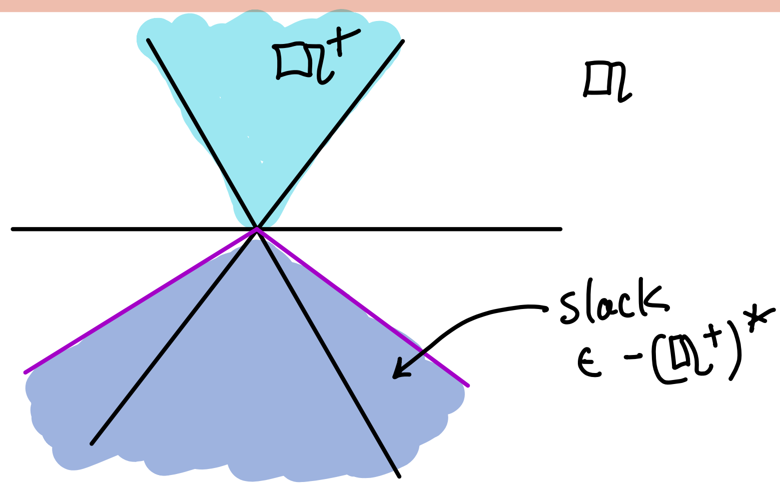
Def^n (Fock-Goncharov)

$\rho : \pi_1 S \to G$ is positive if $\exists$ a boundary map $\xi : \partial_\infty \pi_1 S \to G/B$ (ρ-equivariant; G/B = full flags) taking positively oriented triples to positive triples of flags (Lusztig)

Fact : each $\rho(\gamma)$ is regular $\rightsquigarrow$ invariant directed flat $F(\rho(\gamma))$

$$\boxed{\textbf{Slack formula} \quad \lambda_\rho(\gamma') + \lambda_\rho(\gamma'') - \lambda_\rho(\gamma) = \mathrm{slack}\big(F(\rho(\gamma)),\ \rho(\gamma'')F(\rho(\gamma))\big) + o(1)}$$

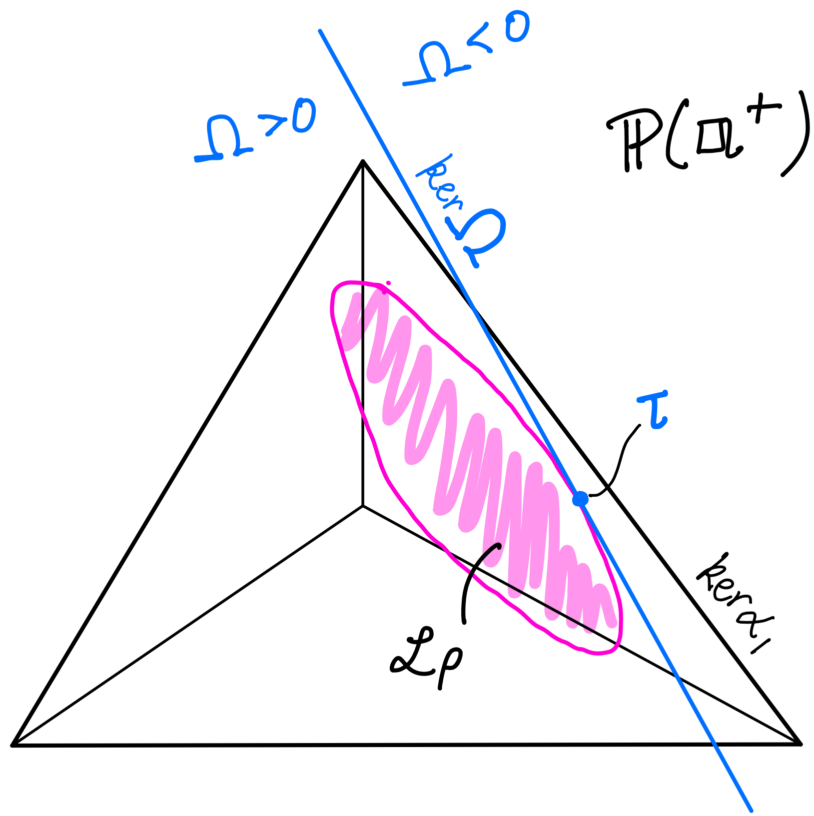




proof ideas Main theorem ②

- Fix $\varphi \in \mathfrak{a}^*$ $\varphi(\mathcal{L}_\rho - 0) > 0$
- Let $(\gamma_n) \subset \pi_1 S$ s.t. $t_n \lambda_\rho(\gamma_n) \to \tau$

  $t_n \gamma_n \to \mu \in Curr(S)$



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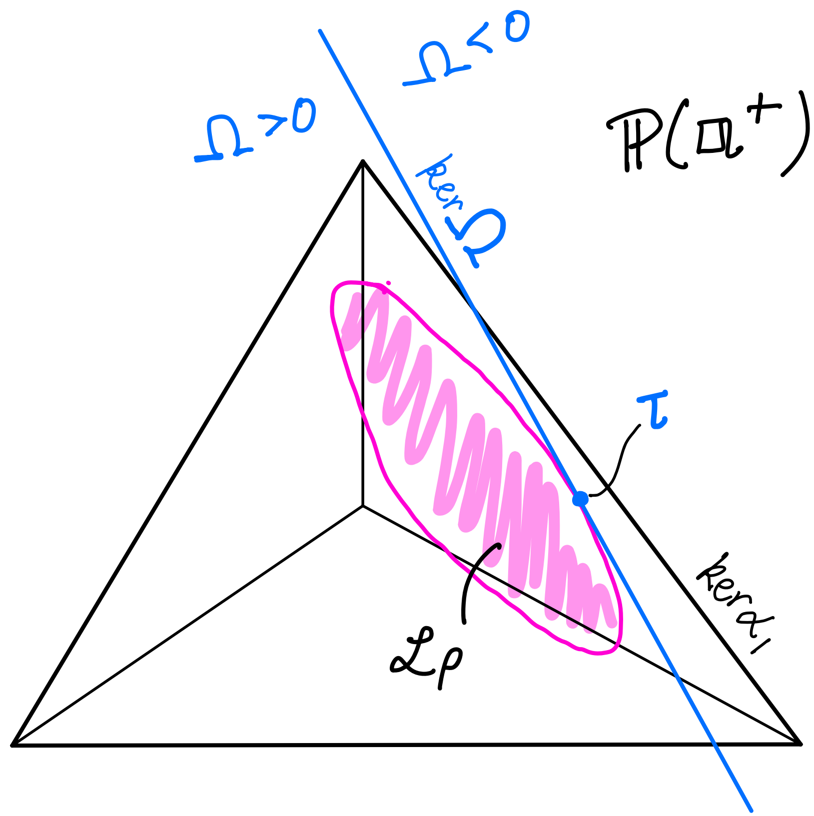

Lemma 1 : $\forall$ $\ell, \hat{\ell} \in supp(\mu)$ crossing

$$\Omega(slack(\ell, \hat{\ell})) \geq 0$$

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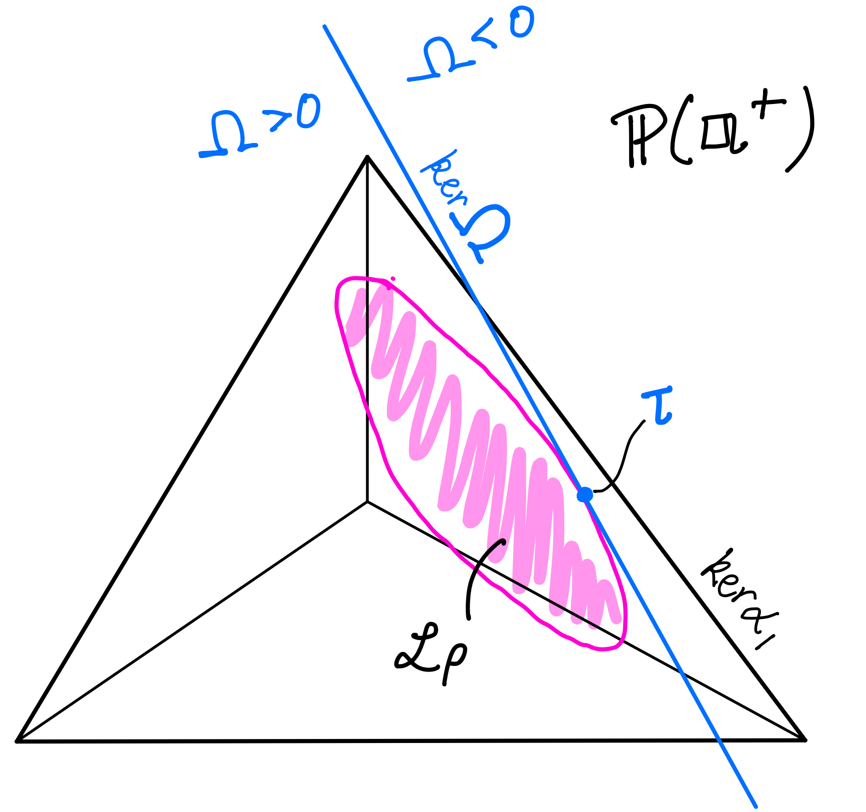

Lemma 1 : $\forall$ $\ell, \hat{\ell} \in supp(\mu)$ crossing

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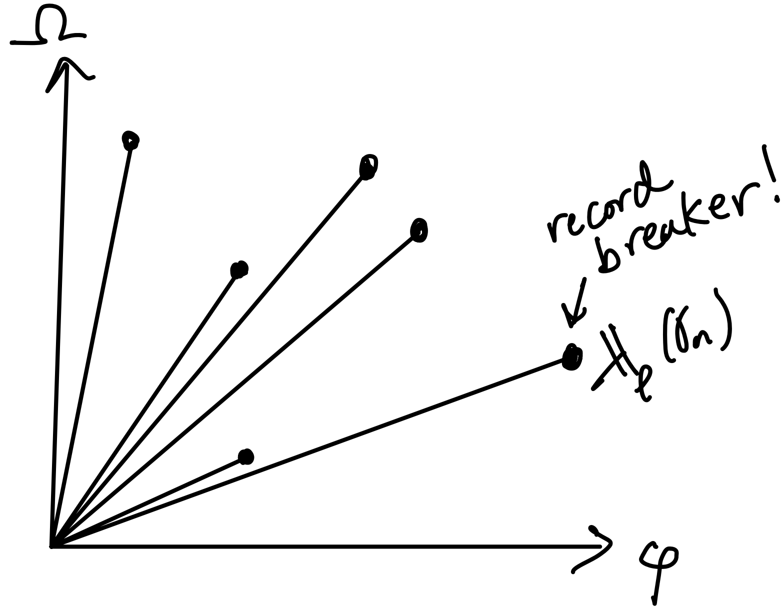
provided each γ_n is a record breaker :

$$\delta(\gamma_n) < \delta(\gamma) \quad \text{whenever} \quad \varphi(\gamma) < \varphi(\gamma_n)$$

where $\delta(\gamma) = \dfrac{\Omega(\rho(\gamma))}{\varphi(\rho(\gamma))}$



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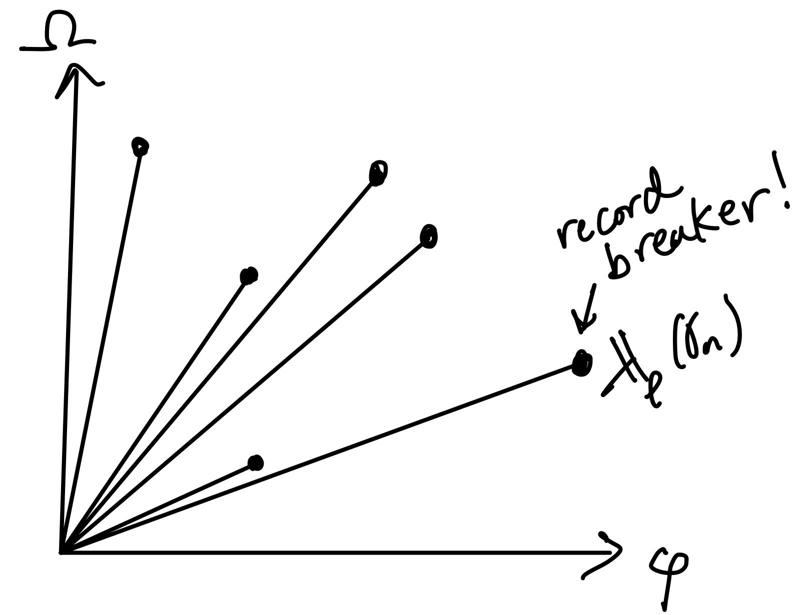

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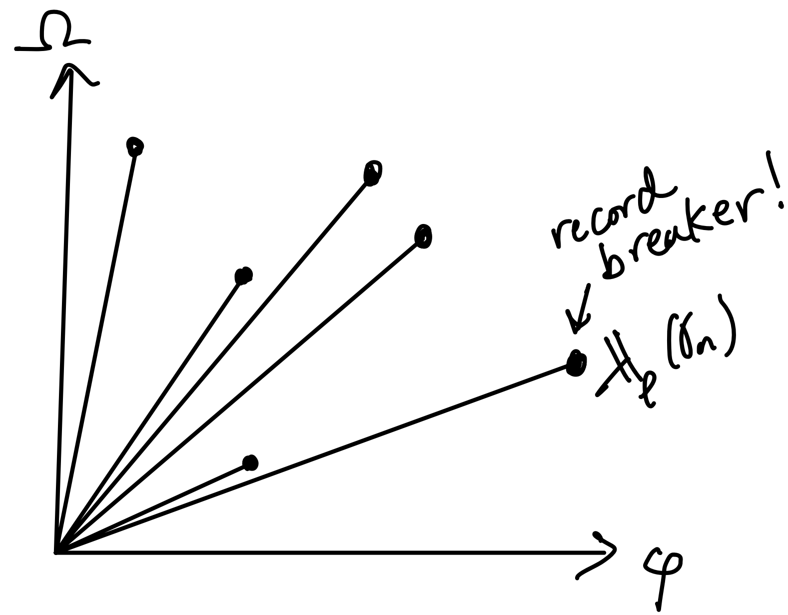

Pf : each γ_n crosses itself :

$\gamma_n = \gamma_n' \cdot \gamma_n''$

with $\mathrm{axis}(\gamma_n) \to \ell$

$\gamma_n'' \cdot \mathrm{axis}(\gamma_n) \to \hat{\ell}$

Lemma 1: $\forall\ \ell, \hat{\ell} \in \mathrm{supp}(\mu)$ crossing

$$\Omega(\mathrm{slack}(\ell, \hat{\ell})) \geqslant 0$$

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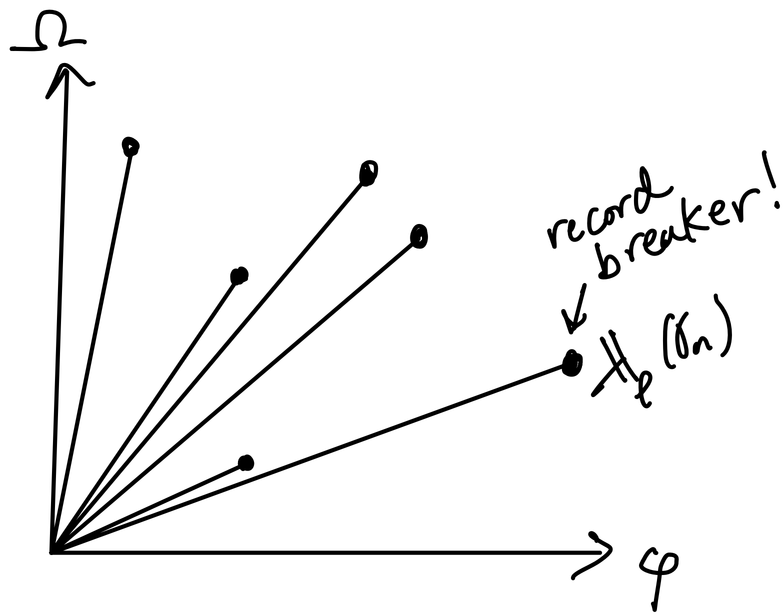

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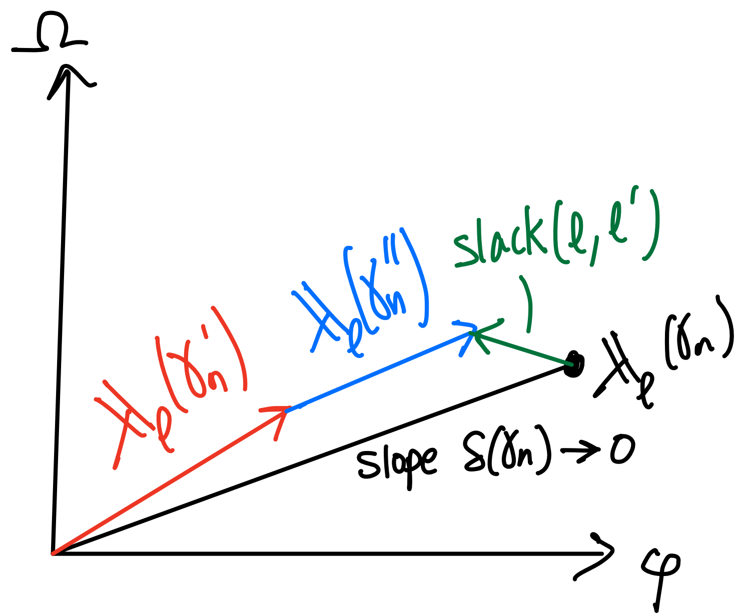
$$\mathrm{slack}(F(\gamma_n), \gamma_n'' F(\gamma_n)) \to \mathrm{slack}(\ell, \hat{\ell})$$

$$\|$$

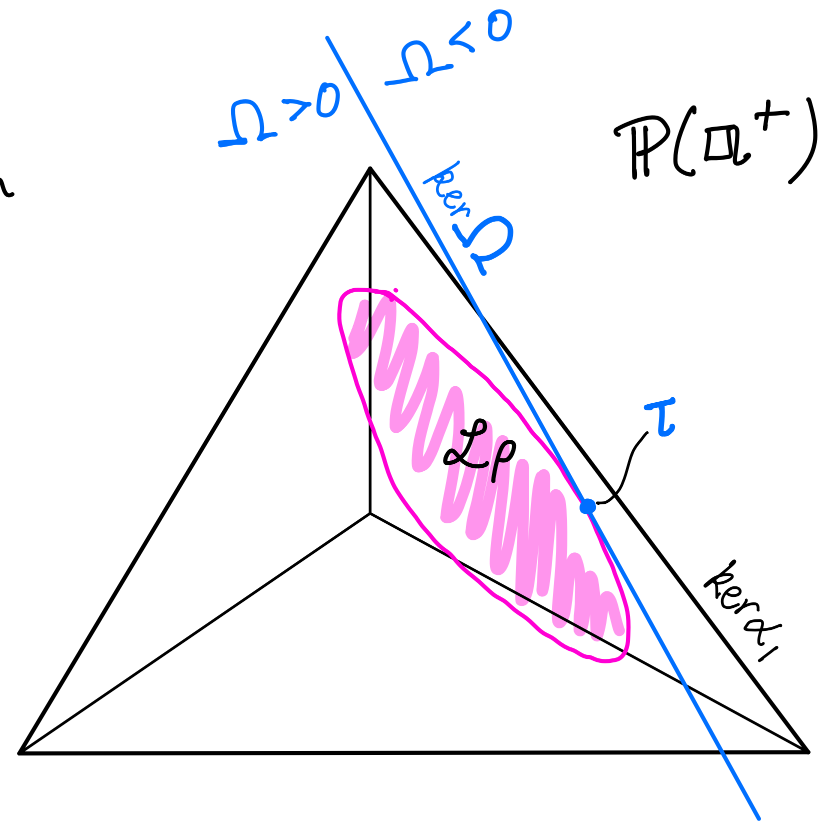
$$\lambda_\rho(\gamma_n') + \lambda_\rho(\gamma_n'') - \lambda_\rho(\gamma) + o(1)$$



$\mu \in \mathrm{Curr}(S)$ rescaled limit of record breakers (γ_n), $\lambda_\rho(\mu) = \tau$.

Lemma 2: $\exists$ $\varepsilon = \varepsilon(\theta)$ such that if $\ell \in \mathrm{supp}(\mu)$ and $\hat{\ell}$ crosses ℓ with angle $\theta \ll 1$, then

$$\boxed{\mathrm{slack}(\ell, \ell') \text{ "}\sim\text{" } -\sum_{i=1}^{d} \varepsilon^{\alpha_i(\tau)} \alpha_i}$$



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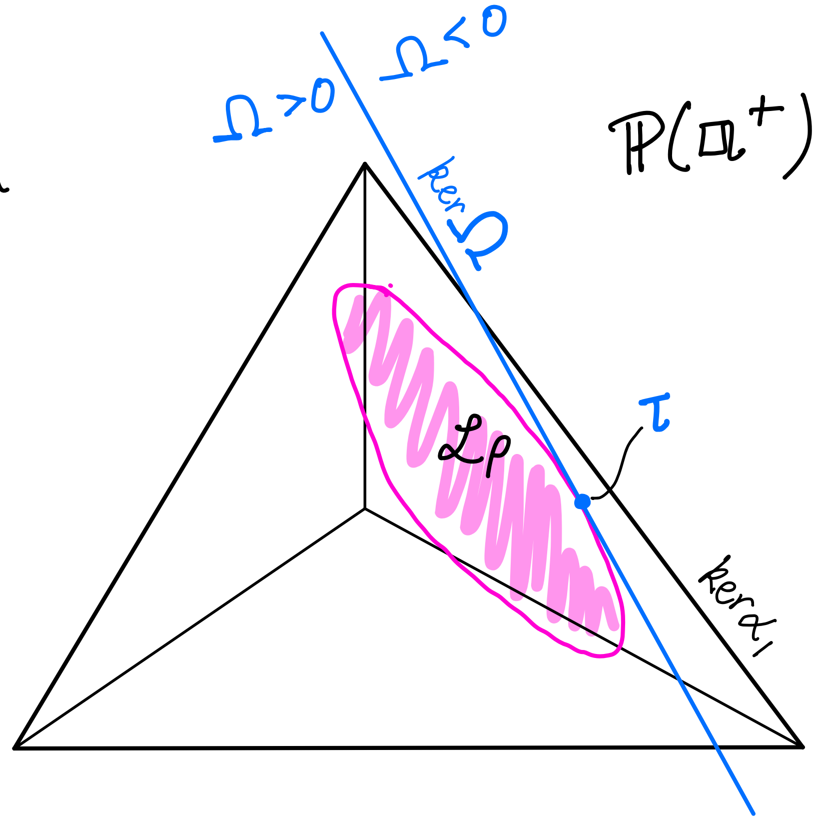

if $(\alpha_i(\tau))$ minimized by α_j then

$$\mathrm{slack}(\ell, \hat{\ell}) \sim -C\, \varepsilon^{\alpha_j(\tau)} \alpha_j$$

$$\Omega(\mathrm{slack}(\ell, \hat{\ell})) < 0 \quad \text{if } \Omega(\alpha_j) > 0$$

Lemma 1 $\Rightarrow$ crossing not allowed!

$\mu \in \mathrm{Curr}(S)$ rescaled limit of record breakers (γ_n), $\lambda_\rho(\mu) = \tau$.

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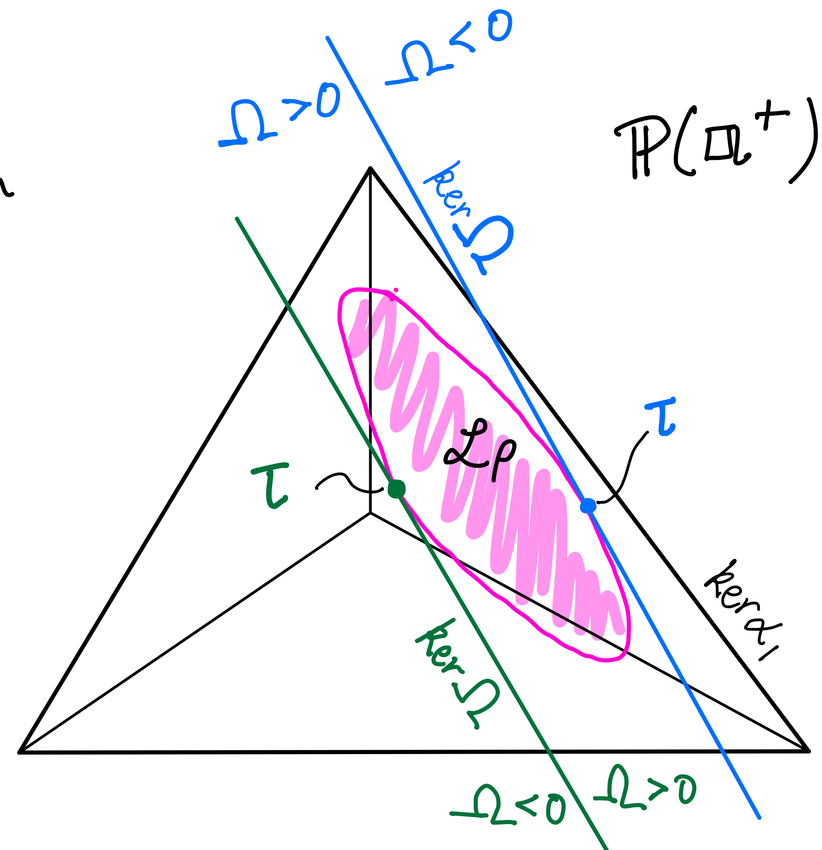

if $(\alpha_i(\tau))$ minimized by α_j then

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Lemma 1

$$\Omega(\mathrm{slack}(\ell, \hat{\ell})) \begin{cases} <0 & \text{if } \Omega(\alpha_j) > 0 \\ >0 & \text{if } \Omega(\alpha_j) < 0 \end{cases}$$

$\Rightarrow$ crossing not allowed!

$\Rightarrow$ crossing required !

Proof ideas Main theorem 1

$G = PSL_2\mathbb{R} \times PSL_2\mathbb{R}$

$\rho = (\rho_1, \rho_2)$

Lipschitz maps

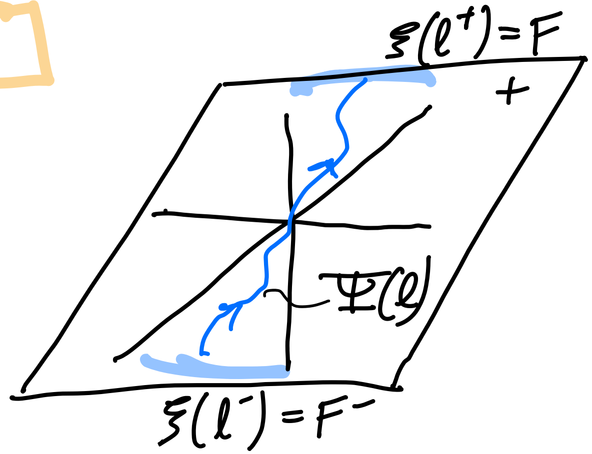
$f : \mathbb{H}^2 \to \mathbb{H}^2$, $\circlearrowleft \rho_1$, $\circlearrowleft \rho_2$

any G s.s.

flat flow maps

$\Psi : T^1\tilde{S} \to X$

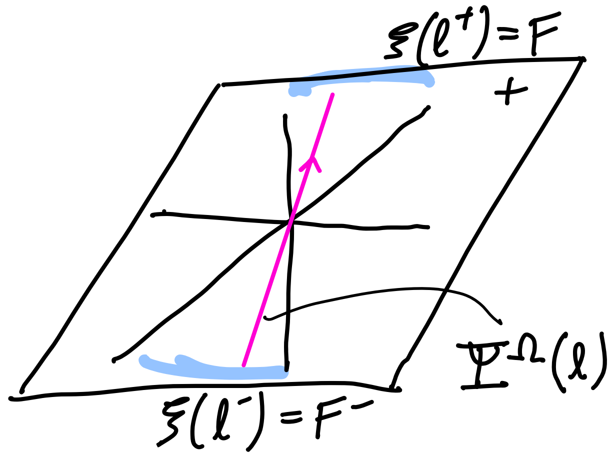
ℓ geodesic $\to$ $F(\ell^-, \ell^+)$ flat



optimal Lipschitz map stretching along geodesic lamination Λ

for each Ω supporting $\mathcal{L}_\rho$ find ironed Ψ^Ω : let μ the associated measured Q/A - lamination :

for $\ell \in \mathrm{supp}(\mu)$,
$\Psi(\ell) \subset$ hyperplane parallel to $\ker \Omega$.



Thank you