# SHEAVES IN SMOOTH TOPOLOGY

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Abstract. These notes are from a talk given at UT in a graduate differential topology class, and are meant to be self-contained. The Goal is to introduce sheaves and how they relate to the category of smooth manifolds. Sheaves are a systematic way to keep track of a bevy of data within one object. Considering the historicity of sheaves we will briefly go over Grothendieck's contribution to the field of algebraic geometry and how his foundational work [EGA] has had a long-standing impact on the whole of mathematics. We will be mainly pulling from Global Calculus By Ramanan [Ram], and Algebraic Geometry By Hartshorne[Ram].

## 1. Sheaves

Historically, the terminology used to developed the theory of schemes was inspired by the agrarian social-political movement of the time. With that context we start with our first definition.

**Definition 1.1.** A **smooth function element** on $M$ is an ordered pair $(f, U)$, where $U$ is an open subset of $M$ and $f : U \to \mathbb{R}$ is a smooth function.[Lee]

**Definition 1.2.** Given a point $p \in M$, let us define an equivalence relation on the set of all smooth function elements whose domains contain $p$ by setting $(f, U) \sim (g, V)$ if $f \equiv g$ on some neighborhood of $p$. The equivalence class of a function element $(f, U)$ is called the **germ of $f$ at $p$**.[Lee]

The set of all germs of smooth functions at $p$ is denoted by $C_p^\infty(M)$. Now continuing with our agrarian terminology,

**Definition 1.3.** Let $X$ be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups on $X$ consists of the data

(a) for every open subset $U \subseteq X$, an abelian group $\mathcal{F}(U)$, and
(b) for every inclusion $V \subseteq U$ of open subsets of $X$, a homomorphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$,

subject to the condtions

(0) $\mathcal{F}(\varnothing) = 0$,
(1) $\rho_{UU}$ is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and
(2) if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

[Har]

**Examples 1.4.** Examples of abelian groups assigned to each open subsets of $X$,

(1) $\mathbb{R}$ valued functions on $U$,

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(2) Differential forms on $U$
(3) Vector Fields on $U$
(4) Holomorphic functions on $U$,
(5) Constant functions on $U$

*Remark* 1.5. If $\mathcal{F}$ is a presheaf on $X$ then $\mathcal{F}(U)$ is referred to as the section of the presheaf $\mathcal{F}$ over the open set $U$. Where the sections are elements in $\Gamma(U, \mathcal{F})$. Also, restriction maps $\rho_{UV}(s) = s|_V$

**Definition 1.6.** A presheaf $\mathcal{F}$ on a topological space $X$ is a sheaf if it satisfies the following conditions:

(3) if $U$ is an open set, if $\{V_i\}$ is an open covering of $U$, and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all $i$, then $s = 0$.
(4) if $U$ is an open set, if $\{V_i\}$ is an open covering of $U$, and if we have elemnts $s_i \in \mathcal{F}(V_i)$ for each $i$, with the property that for each $i, j, s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$, then there is an element $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each $i$.

[Har]

For a fixed $x$, one says that elements $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ are equivalent if there exists $W : x \in W$ where $W \subseteq U \cap V$ has $\rho_{WU}(f) = \rho_{WV}(g)$, (both elements of $\mathcal{F}(W)$).
These equivalence classes form the stalk $\mathcal{F}_x$ at $x$ of the presheaf. Where this equivalence relation is the abstraction of the germ equivalence.

**Definition 1.7.** The stalk of $\mathcal{F}$ at $x$ is defined as

$$\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$$

The direct limit is indexed over all the open sets containing $x$, with order relation induced by reverse inclusion.[Har]

*Remark* 1.8. In some contexts it is best to think
1) if **Germ** vanishes at $p$ then the **Germ** represents tangent vector at $p$,
2) a **Stalk** as containing the maximal ideal of all the **Germs** that vanish at $p$, and from here you can quotient by the **Germs** which vanish with order two, and
3) a **Sheaf** as all the collection of all the data of the manifold from which you can extract the tangent bundle out of.[Har]

**Example 1.9.** Let $X$ be a topological space, and A an abelian group. Define the *constant sheaf* $\mathcal{A}$ on $X$ determined by $A$ as follows. We give $A$ the discrete topology, and for any open set $U \subset X$, let $\mathcal{A}(U)$ be the group of continuous maps $U$ into $A$. Then with the usual restriction maps we obtain $\mathcal{A}$ the constant sheaf.[Har]

*Remark* 1.10. Note that for every connected open set $U$, $\mathcal{A}(U) = A$.

*Remark* 1.11. You can define a presheaf of rings, sets, or abelian groups. This is done in the obvious way i.e., Define the presheaf with values in any fixed category $\mathfrak{C}$, by replacing the word abelian group in the definition of presheaf by the object in $\mathfrak{C}$ you'd like.

**Example 1.12.** Let $X$ be a topological space and for every open subset $U \subset X$ let $\mathscr{O}(U)$ be the ring $C^\infty(U)$. For each $V \subset U$, let $\rho_{UV} : \mathcal{A}(U) \to \mathcal{A}(V)$ be the restriction map. Then $\mathscr{O}$ is the sheaf of rings on $X$.

**Definition 1.13.** If $\mathcal{F}, \mathcal{G}$ are presheaves on $X$, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of abelian groups $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U$, such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

is commutative, where $\rho$, and $\rho'$ are the restriction maps in $mcf$ and $\mathcal{G}$. If $mcf$, and $\mathcal{G}$ are sheaves on $X$, we use the same definition for morphism. [Har]

*Remark* 1.14. A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves on $X$ induces a morphism $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ on the stalks, for any point $p \in X$.

**Proposition 1.15.** *$\varphi$ is a isomorphism of sheaves over $X$ if and only if the map induced on the stalks in a isomorphism for every point $p \in X$.* [Har]

We now introduce the concept of local rings.

**Definition 1.16.** A ring is *local* if it has a unique maximal ideal.

**Example 1.17.** (A) Any field.
(B) $\{a/b \in \mathbb{Q} \mid p \nmid b\}$.
(C) More generally, suppose $A$ is a commutative ring and $\mathfrak{p} \subset A$ is a prime ideal. Let $S = A - \mathfrak{p}$. Then,

$$A_{\mathfrak{p}} \coloneqq S^{-1}A = \{a/b \mid b \notin \mathfrak{p}\}.$$

**Proposition 1.18** (Locality Criterion)**.** *A ring $A$ is local with maximal ideal $\mathfrak{m}$ if, and only if, $\mathfrak{m}$ is an ideal such that every element in $A \setminus \mathfrak{m}$ is a unit.*

**Definition 1.19.** A *ringed space* is a topological space $X$ along with a sheaf of rings $\mathscr{O}$ on $X$. We say $(X, \mathscr{O})$ is a *locally ringed space* if the stalks $\mathscr{O}_p$ are local for all $p \in X$.

**Example 1.20** (Relevant Example)**.** Recall the example $(M, \mathscr{O})$ of a manifold $M$ with $\mathscr{O}(U)$ the smooth functions $U \to \mathbb{R}$. Observe that $\mathscr{O}(U)$ is a ring for each $U$, and the restriction maps are ring homomorphisms. Hence, $(M, \mathscr{O})$ is a ringed space.

We claim $(M, \mathscr{O})$ is a locally ringed space. For $p \in M$, the stalk at $p$ is

$$\mathscr{O}_p = \{[(U, f)] \colon p \in U\}.$$

Let

$$\mathfrak{m}_p = \{[(U, f)] \colon p \in U, f(p) = 0\} \subset \mathscr{O}_p.$$

Clearly $\mathfrak{m}_p$ is an ideal in $\mathscr{O}_p$. Suppose $[(U, f)] \in \mathscr{O}_p \setminus \mathfrak{m}_p$. Then, $f \neq 0$ on an open neighborhood $W \ni p$, so $f|_W$ has an inverse $g \colon W \to \mathbb{R}$. We see

$$[(U, f)] \cdot [(W, g)] = [(W, f|_W)] \cdot [(W, g)] = [(W, f|_W g)] = [(W, 1)] = 1.$$

Thus, $[(U, f)]$ is a unit. Hence, $\mathscr{O}_p$ is local with maximal ideal $\mathfrak{m}_p$.

*Remark* 1.21. Let $\mathbf{Man}^\infty$ be the category of smooth manifolds with morphisms given by smooth maps. As we have shown, manifolds can also be thought of as locally ringed spaces. It is natural to ask how much information about a manifold is encoded in its sheaf of smooth functions. The answer to this question is *all*. Explicitly, we would like to describe a functor from $\mathbf{Man}^\infty$ to **LRS**, the category of locally ringed spaces. However, we first need to describe the morphisms of **LRS**.

**Definition 1.22.** Suppose $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are two locally ringed spaces. A *morphism of locally ringed spaces* $(f, f^\sharp)$ is the data of

(A) A continuous map $f\colon X \to Y$
(B) A morphism of sheaves $f^\sharp\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$

such that the induced stalk maps

$$f^\sharp_y \mathscr{O}_{Y,y} \to (f_*\mathscr{O}_X)_y$$

at each $y \in f(X) \subset Y$ are local homomorphisms, i.e. the preimage of the maximal ideal of $(f_*\mathscr{O}_X)_y$ is the maximal ideal of $\mathscr{O}_{Y,y}$.

**Example 1.23.** Let $M, N$ be smooth manifolds with $\mathscr{O}_M$ and $\mathscr{O}_N$ the respective smooth function sheaves. Suppose $f\colon M \to N$ is a smooth map. Recall that, on each open set $V \subset N$,

$$(f_*\mathscr{O}_M)(U) = \mathscr{O}_M(f^{-1}U)) = \mathscr{C}^\infty(f^{-1}U)).$$

Define $f^\sharp\colon \mathscr{O}_N \to f_*\mathscr{O}_M$ by

$$f^\sharp_V(g) = g \circ f \in \mathscr{C}^\infty(f^{-1}U)).$$

One can check that $f^\sharp$ commutes properly with restrictions and is indeed a morphism of sheaves. The fact that $f^\sharp_{f(x)}$ is a local homomorphism follows from the fact that $g$ is 0 at $f(x)$ if, and only if, $g \circ f$ is 0 at $x$.

We can finally state the main theorem of this talk.

**Theorem 1.24.** $\mathbf{Man}^\infty$ *embeds fully faithfully in* **LRS**.

**Definition 1.25.** Recall that a functor $F\colon \mathscr{C} \to \mathscr{D}$ is *faithful* (resp. *full*) if $F$ is injective (resp. surjective) on morphisms, i.e. $\phi \in \mathscr{C}(A, B)$ maps to $F\phi \in \mathscr{C}(FA, FB)$ is injective (resp. surjective). We say $F$ is an *embedding* if it is injective on objects and faithful.

*Remark* 1.26. The main theorem can thus be interpreted as saying: "Manifolds are a subclass of locally ringed spaces, and morphisms of these locally ringed spaces are precisely the same data as smooth maps."

*Proof.* Define a functor $\Phi\colon \mathbf{Man}^\infty \to \mathbf{LRS}$ by the description of the previous example. Explicitly, if $M$ is a manifold, then

$$\Phi(M) = (M, \mathcal{C}_M)$$

and if $f\colon M \to N$ is a smooth map, then $\Phi(f) = (f, f^\sharp)$, where $f^\sharp\colon \mathcal{C}_N \to f_*\mathcal{C}_M$ is defined by

$$f^\sharp_V(g) = g \circ f \in \mathcal{C}_M(f^{-1}V)).$$

Injectivity of $\Phi$ on objects is clear. Furthermore, $f \mapsto (f, f^\sharp)$ is clearly injective, so $\Phi$ is faithful. It remains to show $\Phi$ is full.

Suppose $(f, F)\colon (M, \mathcal{C}_M) \to (N, \mathcal{C}_N)$ is a morphism, so

(A) $f\colon M \to N$ is continuous (not a priori smooth) and

(B) $F\colon \mathcal{C}_N \to f_*\mathcal{C}_M$ is a morphism of sheaves which is a local homomorphism on stalks.

We claim $F = f^\sharp$ and $f$ is smooth. Suppose $g \in \mathcal{C}_N(V)$ for $V \subset N$ open. It suffices to show that $(g \circ f)(p) = F_V(g)(p)$ for all $p \in U = f^{-1}V$). Let $p \in U$ and $q = f(p)$. Because the induced map on stalks

$$F_q\colon \mathcal{C}_{N,q} \to (f_*\mathcal{C}_M)_q$$

is a local homomorphism, if $F_V(g)(p) = 0$ then $g(q) = 0$. Suppose $F_V(g)(p) = c \in \mathbb{R}$. Then, $F_V(g-c)(p) = 0$, so $g(q) - c = 0$, so $g(q) = c$. Thus, $F_V(g)(p) = (g \circ f)(p)$, as desired.

It is not hard to see that $f$ is smooth: for any $g\colon V \to \mathbb{R}$ smooth, $g \circ f\colon f^{-1}V) \to \mathbb{R}$ is smooth. Taking $g$ to be a chart post-composed with a coordinate projection, one can show $f$ is smooth. □

## References


[Har] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics.

[Lee] John Lee, *Introduction of smooth manifolds*, Graduate Texts in Mathematics.

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