Morphisms

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In this section, Hartshorne, in true category-pilled fashion, introduces the notion of morphisms in order to make our objects into a category.

In what follows, we let Y be a quasi-affine variety in $\mathbb{A}^n$ (open subset of an affine variety). The easiest case is when the codomain is $k = \mathbb{A}^1$.

Definition 1. *A function $f : Y \to k$ is* ***regular*** *at a point $P \in Y$ if there is an open neighborhood $P \in U \subset Y$ and polynomials $g, h \in A = k[x_1, \dots, x_n]$ such that h does not vanish on U and $f = g/h$ on U. We say f is regular on Y if it is regular at each point.*

In other words, a regular function $Y \to k$ is one that is locally a rational function.

Lemma 2. *A regular function is Zariski-continuous as a map $Y \to k = \mathbb{A}^1$.*

Proof. We will show that f^{-1} maps closed sets to closed sets. A closed set of $\mathbb{A}^1$ is a finite set of points, so we need only show that level sets $f^{-1}(a)$ are closed. Closedness is locally checkable, in the sense that a subset of Y is closed in Y if is closed with respect to an open cover.

In this case, we take an open cover of Y on which f is a rational function. If U is one of these sets, then

$$f^{-1}(a) \cap U = \left\{ P \in U : \frac{g(P)}{h(P)} = a \right\} = Z(g - ah) \cap U,$$

which is certainly closed. □

We can similarly define regular maps on quasi-projective varieties $Y \subset \mathbb{P}^n$.

Definition 3. *A function $f : Y \to k$ is* ***regular*** *at a point $P \in Y$ if there is an open neighborhood $P \in U \subset Y$ and homogeneous polynomials $g, h \in S = k[x_0, \dots, x_n]$ of the same degree such that h does not vanish on U and $f = g/h$ on U.*

(It is critical that g and h have the same degree in order for g/h to be well-defined on the projective set Y.)

One can similarly show that a regular function on a quasi-projective variety is continuous. (k has the same topology as before.) On a variety open sets are dense, so if two regular functions $f, g : Y \to k$ are equal on a non-empty open subset of Y, then they are equal on all of Y, sense the set where they are equal will be closed (preimage of 0) and dense.

Definition 4. *Let k be an algebraically closed field. A **variety** over k is an affine, quasi-affine, projective, or quasi-projective variety. If X, Y are two varieties, a **morphism** is a continuous map $\varphi : X \to Y$ such that for every open set $V \subset Y$ and regular function $f : V \to k$, $f \circ \varphi : \varphi^{-1}(V) \to k$ is regular.*

It's easy to see that this forms a category. Isomorphisms in this category are homeomorphisms, but the converse is not true.

Definition 5. *If Y is a variety, then $\mathcal{O}(Y)$ is the ring of all regular functions on Y. (They form a ring since rational functions form a ring.) For a point $P \in Y$, the **local ring** $\mathcal{O}_{P,Y}$ of P on Y is the ring of germs of regular functions on Y near P, in the form of pairs $\langle U, f \rangle$.*

As one would hope, the local ring of P is actually a local ring. Just like for manifolds, the maximal ideal of $\mathcal{O}_P$ is the set of germs of regular functions that vanish at P, as everything else is locally invertible. The residue field $\mathcal{O}_P/\mathfrak{m}$ is isomorphic to k, the isomorphism being the value of the germ at P.

Definition 6. *If Y is a variety, then the set $K(Y)$ of all germs of regular functions on Y (equivalence relation: equal on the intersection of the open sets) is calle dthe **function field** of Y, with its elements being the **rational functions** on Y.*

Unlike manifolds, since Y is irreducible, any two non-empty open sets have non-trivial intersection. This lets us define addition and multiplication for any two germs on Y, making $K(Y)$ a ring. To show it is a field, note that if $\langle U, f \rangle \in K(Y)$ is nonzero, then f does not vanish on $V = U \setminus Z(f)$, $Z(f)$ being closed when f is regular. Thsu $1/f$ is regular on V, giving us an inverse $\langle V, 1/f \rangle \in K(Y)$.

The ring of regular functions $\mathcal{O}(Y)$ maps into each $\mathcal{O}_P$, which in turn maps into $K(Y)$. The first map is injective by our previous remark about equality of regular functions, and the second map is just an inclusion. We thus identify $\mathcal{O}(Y)$ and $\mathcal{O}_P$ with subrings of $K(Y)$. Furthermore, an isomorphism of varieties induces an isomorphism of each of these rings.

Theorem 7. *Let $Y \subset \mathbb{A}^n$ be an affine variety with affine coordinate ring $A(Y)$. Then:*

(a) for each $P \in Y$, if $\mathfrak{m}_P$ be the ideal of functions in $A(Y)$ vanishing at P, then $P \mapsto \mathfrak{m}_P$ is a bijection from Y to the maximal ideals of $A(Y)$;

(b) for each P, $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$ and $\dim \mathcal{O}_P = \dim Y$;

(c) $K(Y)$ is isomorphic to the field of fractions of $A(Y)$. Thus $K(Y)$ is a finitely generated algebra over k of transcendence degree $\dim Y$.

(d) $\mathcal{O}(Y) \cong A(Y)$; in particular $A(Y)$ is an isomorphism invariant of Y;

Proof. Every polynomial $f \in A = k[x_1, \ldots, x_n]$ defines a regular function on Y, giving a homomorphism $A \to \mathcal{O}(Y)$. Its kernel is $I(Y)$, giving us an injective homomorphism $\alpha : A(Y) \to \mathcal{O}(Y)$.

(a) The minimal algebraic subsets of Y are just the points $P \in Y$, which means the points of Y correspond 1-1 with the maximal ideals of A containing $I(Y)$, which in turn correspond to the maximal ideals of $A(Y)$. Tracing this bijection shows that it takes a point P to the ideal

$$\mathfrak{m}_P = \{f \in A(Y) : f(P) = 0\}.$$

(b) For each P, the map $\alpha : A(Y) \to \mathcal{O}(Y)$ composed with the natural map $\mathcal{O}(Y) \to \mathcal{O}_P$ takes every function in $A(Y)$ that vanishes at P, i.e., the elements of $\mathfrak{m}_P$, to a unit in $\mathcal{O}_P$. Universal property of localization then implies we get a map $A(Y)_{\mathfrak{m}_P} \to \mathcal{O}_P$. The original map is injective, so this map is also injective. Since $A(Y)_{\mathfrak{m}_P}$ is just quotients of polynomials by polynomials that don't vanish at P, this map is also surjective onto the germs of regular functions at P. Thus $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$.

The primes of $A(Y)_{\mathfrak{m}_P}$ correspond to the primes of $A(Y)$ contained in $\mathfrak{m}_P$, which implies that $\dim \mathcal{O}_P$ is equal to the height of $\mathfrak{m}_P$. Since $A(Y)/\mathfrak{m}_P \cong k$,

$$\operatorname{height}(\mathfrak{m}_P) = \dim A(Y) - \dim A(Y)/\mathfrak{m}_P = \dim Y - \dim k = \dim Y.$$

(c) The field of fractions of $A(Y)$ is equal to the field of fractions of each $A(Y)_{\mathfrak{m}_P}$, which from the last part can be identified with $\mathcal{O}_P$. The field of fractions of $\mathcal{O}_P$ is just $K(Y)$, as the localization takes care of the polynomials that vanish at P. This establishes the isomorphism.

For the second part, we note that $A(Y)$ is a finitely generated k-algebra (as it is a quotient of A), which implies the same is true of its field of fractions $K(Y)$. An earlier fact states that the transcendence degree of the field of fractions of an integral domain (finitely generated) algebra over k is the dimension of the algebra. In this case that is $\dim A(Y) = \dim Y$.

(d) Note that $\mathcal{O}(Y) \subset \cap_{P \in Y} \mathcal{O}_P$, where all of these rings are embedded into $K(Y)$. Therefore, working in the quotient field of $A(Y)$,

$$A(Y) \subset \mathcal{O}(Y) \subset \bigcap_{P \in Y} \mathcal{O}_P \subset \bigcap_{\mathfrak{m}} A(Y)_{\mathfrak{m}};$$

the first inclusion is the map induced by the natural inclusion $A \to \mathcal{O}_Y$, while the final inclusion is a consequence of (b). However $\bigcap_{\mathfrak{m}} A(Y)_{\mathfrak{m}} = A(Y)$ since $A(Y)$ is an integral domain, thus proving the claim.

□