Morphisms, cont.

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Here's a basic isomorphism:

Proposition 1. *Let $U_i \subset \mathbb{P}^n$ be the open set defined by $x_i \neq 0$. Then the mapping*

$$\varphi_i : U_i \to \mathbb{A}^n \qquad \varphi_i(a_0, \ldots, a_n) = \left(\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i} \right)$$

is an isomorphism of varieties.

Proof. □

For the next theorem, we need to introduce some notation. We start with a graded ring S, the main relevant example for this purpose being a projective coordinate ring $S(Y)$. Given a *homogeneous* prime ideal $\mathfrak{p}$ of S, let T be the set of *homogeneous* elements of S that do not lie in $\mathfrak{p}$ form a multiplicative subset of S (since $\mathfrak{p}$ is prime). We can therefore consider its localization $T^{-1}S$, and this ring inherits a grading from S: if f is homogeneous and $g \in T$, then we define

$$\deg(f/g) = \deg f - \deg g.$$

The subset of $T^{-1}S$ consisting of the elements of degree 0 form a subring of $T^{-1}S$, which we call $S_{(\mathfrak{p})}$. This is presumably the projective analogue of the localization $S_{\mathfrak{p}}$. Just as with $S_{\mathfrak{p}}$ (and for the same reason, namely considering the units), it is a local ring whose maximal ideal is $(\mathfrak{p} \cdot T^{-1}S) \cap S_{(\mathfrak{p})}$, the elements of $S_{(\mathfrak{p})}$ that are multiples of elements of $\mathfrak{p}$. Similarly, if $f \in S$ is a homogeneous element, then $S_{(f)}$ is the subring of degree 0 elements in the localization S_f.

Theorem 2. *Let $Y \subset \mathbb{P}^n$ be a projective variety with homogeneous coordinate ring $S(Y)$. Then*

(a) for any $P \in Y$, if $\mathfrak{m}_P$ is the ideal of $S(Y)$ generated by the homogeneous $f \in S(Y)$ such that $f(P) = 0$, then $\mathcal{O}_P = S(Y)_{(\mathfrak{m}_P)}$ (the ring of germs of regular functions at P);

(b) *$K(Y) \cong S(Y)_{((0))}$ (field of all germs of regular functions); as occurs when we localize at 0 usually, this is a field, essentially the projective version of the field of fractions;*

(c) *$\mathcal{O}(Y) = k$ (the ring of regular functions);*

For comparison, let's recall the corresponding results for an affine variety Y:

(a) $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$ (here $\mathfrak{m}_P$ is the ideal of functions in $A(Y)$ that vanish at P);

(b) $K(Y) \cong \operatorname{Frac}(A(Y))$;

(c) $\mathcal{O}(Y) \cong A(Y)$;

We can see from this that they are fairly similar (besides the first one), once we replace the classical localization with this projective localization.

Proof. We start with some preliminaries. As with the previous proposition, let $U_i \subset \mathbb{P}^n$ be the open set $x_i \neq 0$, and let $Y_i = Y \cap U_i$. Since U_i is isomorphic to $\mathbb{A}^n$, the same isomorphism φ_i allows us to view each Y_i as an affine variety. Furthermore, $A(Y_i)$ is naturally isomorphic to $S(Y)_{(x_i)}$. To prove this, we start with the map

$$\begin{aligned} k[y_1, \ldots, y_n] &\to k[x_0, \ldots, x_n]_{(x_i)} = S_{(x_i)} \\ f(y_1, \ldots, y_n) &\mapsto f\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right), \end{aligned}$$

which is an isomorphism. Of course $A(Y_i)$ is the quotient of the left side by $I(Y_i)$, so we just need to examine the quotient of $S_{(x_i)}$ by the image of $I(Y_i)$. It was shown in an exercise that this image is $I(Y)S_{(x_i)}$, which means the quotient is indeed $S(Y)_{(x_i)}$. We will label the isomorphism of the quotients as φ_i^*.

(a) Let i be such that $P \in Y_i$, which is always possible since we're in projective space. If $\mathfrak{m}'_P$ is the maximal ideal of functions in $A(Y_i)$ vanishing at P, then viewed from the affine perspective $\mathcal{O}_P \cong A(Y_i)_{\mathfrak{m}'_P}$. The image of $\mathfrak{m}'_P$ under φ_i^* is $\mathfrak{m}_P \cdot S(Y)_{(x_i)}$. By our choice of i, $x_i \notin \mathfrak{m}_P$ (the projective one), and we can thus localize by it:

$$\mathcal{O}_P \cong A(Y_i)_{\mathfrak{m}'_P} \cong (S(Y)_{(x_i)})_{\mathfrak{m}_P \cdot S(Y)_{(x_i)}} = S(Y)_{(\mathfrak{m}_P)}.$$

(b) Again, we start by examining it from the affine lens. $K(Y_i)$ is the same as $K(Y)$, since Y_i is just the intersection of Y with an open set, and germs are determined up to open sets. The affine theorem showed that $K(Y_i)$ is isomorphic to the fraction field of $A(Y_i)$. $A(Y_i)$ is isomorphic to $S(Y)_{(x_i)}$, and the fraction field of this is just the maximal homogeneous localization, which is $S(Y)_{((0))}$.

(c) Let $f \in \mathcal{O}(Y)$, that is a regular function on Y. For each i, f is therefore also in $\mathcal{O}(Y_i)$. Recall from the previous lecture that we can identify $\mathcal{O}$ with coordinate rings A when we embed both in K. From this view, $f \in A(Y_i) \cong S(Y)_{(x_i)}$ for all i. Therefore $f = g_i/x_i^{N_i}$ for some homogeneous $g_i \in S(Y)$ of degree N_i (since the element must have degree 0).

We can take these identifications further and view $\mathcal{O}(Y)$, $K(Y)$, and $S(Y)$ as all being subrings of the quotient field L of $S(Y)$. ($K(Y)$ is due to (b), $\mathcal{O}(Y)$ is using its natural inclusion into $K(Y)$.) What we have just showed is that there exist N_i such that $x_i^{N_i} f \in S(Y)_{N_i}$ (the elements of degree N_i).

Let $N \geq \sum N_i$. Then $S(Y)_N$ is spanned over k by monomials of degree N in $x_0, \ldots, x_n$. Furthermore, due to the size of N, each of these monomials has a variable x_i that occurs to a power $\geq N_i$. Thus $S(Y)_N \cdot f \subset S(Y)_N$. Iterating gives us

$$S(Y)_N \cdot f^2 = (S(Y)_N \cdot f) \cdot f \subset S(Y)_N \cdot f \subset S(Y)_N,$$

and we can do this for all f^q, $q \geq 1$. In particular, $x_0^N f^q \in S(Y)$ for all $q \geq 1$. Consequently, the subring $S(Y)[f]$ of L is contained in $x_0^{-N} S(Y)$, which is a finitely generated $S(Y)$-module. A theorem of commutative algebra (AM Proposition 5.1) then implies f is integral over $S(Y)$. (Hartshorne also brings up that $S(Y)$ is Noetherian but I don't think this is necessary?) Thus

$$f^m + a_1 f^{m-1} + \cdots + a_m = 0,$$

$a_i \in S(Y)$. We described at the top that $f \in S(Y)_{(x_i)}$, which means it has degree 0, which means this equation is valid if we replace the a_i by their homogeneous components of degree 0. (This is just taking the degree 0 parts of both sides.) That is to say, we replace them by their constant terms. This shows that f is actually algebraic over k. The algebraic closure of k implies $f \in k$.

$\square$

Proposition 3. *Let X be a variety and Y an affine variety. Then there is a natural bijection*

$$\alpha : \mathrm{Hom}_{Var}(X, Y) \to \mathrm{Hom}_{Alg}(A(Y), \mathcal{O}(X)).$$

Proof. Let $\varphi : X \to Y$ be a morphism of varieties. Then φ carries regular functions on Y to regular functions on X by definition, giving us a map $\mathcal{O}(Y) \to \mathcal{O}(X)$, which is a k-algebra homomorphism. We now apply the isomorphism $\mathcal{O}(Y) \cong A(Y)$ for affine varieties. This gives us the map α.

Conversely, let $h : A(Y) \to \mathcal{O}(X)$ be an algebra homomorphism. Define $\overline{x}_i$ to be the image of the coordinate function x_i in the quotient $A(Y)$. We can then define a map $\psi : X \to \mathbb{A}^n$ by

$$\psi(P) = (h(\overline{x}_1)(P), \ldots, h(\overline{x}_n)(P)).$$

Each $h(\overline{x}_i)$ is in $\mathcal{O}(X)$, so this makes sense (maps X to $\mathbb{A}^1$.) Goal: ψ maps into Y. Since $Y = Z(I(Y))$, we can do this by showing that for any $P \in X$ and $f \in I(Y)$, $f(\psi(P)) = 0$. However, since f is a polynomial and h is a k-algebra homomorphism,

$$f \circ \psi = f(h(\overline{x}_1), \ldots, h(\overline{x}_n)) = h(f(\overline{x}_1, \ldots, \overline{x}_n)),$$

where f should be interpreted as the corresponding polynomial on the quotient variables. Now, f is in $I(Y)$: it's a polynomial that when you pass the usual variables in, it vanishes on Y. If you do this to the variables in $A/I(Y)$, it the result is 0, as desired. Thus ψ is a map $X \to Y$ that induces h. What remains to show is that it is a morphism, which follows from the following lemma: $\square$

Lemma 4. *Let X be a variety and $Y \subset \mathbb{A}^n$ an affine variety. A function $\psi : X \to Y$ is a morphism iff $x_i \circ \psi$ is regular for each i.*

In our example, $x_i \circ \psi = h(\overline{x}_i) \in \mathcal{O}(X)$.

Proof. If ψ is a morphism, $x_i \circ \psi$ is a morphism since morphisms are closed under composition. Conversely, suppose $x_i \circ \psi$ are regular. Then for any polynomial f on $\mathbb{A}^n$, $f \circ \psi$ is regular on X.

A closed set of Y is the zero set of some polynomial f. Thus $\psi^{-1}(K) = (f \circ \psi)^{-1}(0)$ for some f, and the latter is closed since $f \circ \psi$ is regular. Thus ψ is continuous.

Finally, let g be a regular function on an open subset of Y. This is locally a quotient of polynomials, so locally $g \circ \psi$ is a quotient of polynomials composed with ψ. These are regular, and thus a quotient of regulars. But the local structure implies these are in turn quotients of polynomials. Thus $g \circ \psi$ is regular, making ψ a morphism. □

Corollary 5. *If X and Y are two affine varieties, then X and Y are isomorphic if and only if $A(X)$ and $A(Y)$ are isomorphic k-algebras.*

Proof. Follows from the naturality. □

The category way of stating this is that $X \mapsto A(X)$ is a functor that induces an arrow-reversing equivalence between the category of affine varieties and teh category of finitely-generated integral domains over k

There's also a commutative algebra result, but it's proved in another book and only used in the exercises.