Algebraic Geometry Learning Seminar Talk Notes

Kishen Narayan

April 8, 2024

1 Nonsingular Varieties

The idea of a manifold coincides with that of a nonsingular variety. Specifically, both are motivated by the notion of *smoothness*. Let $J = (\frac{\partial f_i}{\partial x_j}(P))$ be the *Jacobian matrix* at P. Note that differentiation here refers to a *formal derivative* so we are not taking any limits.

Definition. (Nonsingular Variety) Let $Y \subset \mathbf{A}^n$ be an affine variety, and let $f_1, \ldots, f_t \in A = k[x_1, \ldots, x_n]$ be a set of generators for the ideal of Y. Y is *nonsingular at a point* $P \in Y$ if the rank of the matrix J is $n - r$ where r is the dimension of Y. Y is *nonsingular* if it is nonsingular at every point.

Remark. The definition of nonsingularity is independent of the set of generators of the ideal of Y chosen. To see this, take any h that is a A-linear combination of the f_i and observe for $a_i \in A$ we have

$$\begin{aligned}\frac{\partial}{\partial x_j} h &= \frac{\partial}{\partial x_j}\left(\sum a_i f_i\right) \\ &= \sum \frac{\partial}{\partial x_j}(a_i f_i) \\ &= \sum a_i \frac{\partial f_i}{\partial x_j} + f_i \frac{\partial a_i}{\partial x_j} \\ &= \sum a_i \frac{\partial f_i}{\partial x_j} + \sum f_i \frac{\partial a_i}{\partial x_j}\end{aligned}$$

Note that $\sum a_i \frac{\partial f_i}{\partial x_j}$ is an A-linear combination of the $\frac{\partial f_i}{\partial x_j}$ and as $\sum f_i \frac{\partial a_i}{\partial x_j}(P) = 0$ as $f_i(P) = 0$, we see that if h is in the ideal of Y and we add it to the set of f_i's, the rank of J remains unchanged. So we see that if $\{f_i\}$ and $\{f_i'\}$ are two sets of generators for the ideal of Y, then we can add each f_i' as above without changing the rank of J and through row reductions we will have that the rank of J with respect to $\{f_i'\}$ remains the same.

Examples.

- Let $X = \mathbf{A}^n$. Then $I(X) = (0)$. For any $P \in X$ we see that J is the zero matrix so its rank is 0. Hence, X is nonsingular as the dimension of $\mathbf{A}^n$ is n.
- Let $X = \{P\} \subset \mathbf{A}^n$ where $P = (a_1, \ldots, a_n)$. Note that it can be shown that $I(X) = (x_1 - a_1, \ldots, x_n - a_n)$ from which it immediately follows that $J = I_n$ and so as its rank is n we have that X is nonsingular as its dimension of a single point is 0.

Let's start trying to generalize this definition so that we are not reliant on $Y \subset \mathbf{A}^n$.

Definition. (Regular Local Ring) Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. A is a *regular local ring* if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

Theorem. *Let $Y \subset \mathbf{A}^n$ be an affine variety. Then Y is nonsingular at P a point if and only if $\mathcal{O}_P$ is a regular local ring.*

Proof. $\Rightarrow$ Let $P = (a_1, \ldots, a_n) \in \mathbf{A}^n$, and let $\mathfrak{a} = (x_1 - a_1, \ldots, x_n - a_n)$ be the corresponding maximal ideal in $A = k[x_1, \ldots, x_n]$. We define a linear map $\varphi : A \to k^n$ given by $f \mapsto \left(\frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P)\right)$. Note that

$\varphi(x_i - a_i) = (0, \ldots, 1, \ldots 0)$ where 1 is in the i^{th} coordinate. So these $x_i - a_i$ for $i = 1, \ldots, n$ form a basis of k^n. Furthermore, observe that $\varphi((x_i - a_i)(x_j - a_j)) = (0, \ldots, x_j - a_j, \ldots, x_i - a_i, \ldots, 0)(P)$ where $x_j - a_j$ is in the i^{th} coordinate and $x_i - a_i$ is in the j^{th} coordinate. Evaluating this at P is equivalent to letting $x_i = a_i, x_j = a_j$ hence, $\varphi((x_i - a_i)(x_j - a_j)) = 0$. So then we see that $\varphi(\mathfrak{a}^2) = 0$ and φ induces an vector space isomorphism σ, $\mathfrak{a}/\mathfrak{a}^2 \cong k^n$.

Now let $\mathfrak{b} = (f_1, \ldots, f_t)$ be the ideal of Y in A generated by the f_i. By our construction of φ, we see that the rank of J at P is equivalent to $\dim_k \varphi(\mathfrak{b})$ where $\varphi(\mathfrak{b})$ is viewed as a subspace of k^n. Observe that using the fact that σ is an isomorphism we have that

$$\sigma^{-1}\varphi(\mathfrak{b}) = (\mathfrak{b} + \mathfrak{a}^2)/\mathfrak{a}^2$$

hence $\dim_k \varphi(\mathfrak{b})$ is equivalent to the dimension of $(\mathfrak{b} + \mathfrak{a}^2)/\mathfrak{a}^2$ as a subspace of $\mathfrak{a}/\mathfrak{a}^2$. Note that the local ring of P on Y is simply $\mathcal{O}_{\mathcal{P}} = (A/\mathfrak{b})_{\mathfrak{a}}$ and its maximal ideal is $\mathfrak{m} = (\mathfrak{b} + \mathfrak{a})/\mathfrak{b}$. So its clear that $\mathfrak{m}^2 = (\mathfrak{b} + \mathfrak{a}^2)/\mathfrak{b}$, hence $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{a}/(\mathfrak{b} + \mathfrak{a}^2)$. So, counting the dimensions of vector spaces, gives us that $\dim_k \mathfrak{m}/\mathfrak{m}^2 + \text{rank } J = n$.

$\Leftarrow$ The dimension of the local ring $\mathcal{O}_P$ as a ring is the dimension of Y as a variety. So, $\mathcal{O}_p$ is regular if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim Y = r$. But this is equivalent to saying that rank $J = n - r$ which in turn shows that Y is nonsingular at P. □

This theorem allows us to extend our definition of nonsingularity as we just showed that nonsingularity was not tied to the affinity of Y!

Definition. (Nonsingularity) Let Y be any variety. Y is *nonsingular* at a point $P \in Y$ if the local ring $\mathcal{O}_{P,Y}$ is a regular local ring. Y is *nonsingular* if it is nonsingular at every point. Y is *singular* if it is not nonsingular.

In fact, most points of a variety are nonsingular. To see this however, we will need a fact from commutative algebra.

Proposition. *If A is a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, then $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$.*

Theorem. *Let Y be a variety. Then the set Sing Y of singular points of Y is a proper closed subset of Y.*

Proof. The general strategy will be to prove it for the affine case. Any variety can be covered by affine open subsets, that is for a variety Y, and open cover $\cup Y_i$, if we show that Sing Y_i is closed for each i then we are done. So assume Y is affine and $\dim Y = r$. By the first theorem, we see that any singular point of Y are those with rank $J < n - r$. Consider $M = \left(\frac{\partial f_i}{\partial x_j}\right)$. The only way we could not achieve full rank is if and only if one of M's $(n-r) \times (n-r)$ sub-matrices has a determinant of 0. So Sing Y is $I(Y) \cup \{$determinants of the $(n-r) \times (n-r)$ sub-matrices of $M\}$ which is closed.

Lastly, we show that Sing $Y \neq Y$. Note that Y being a variety must be birational to a hypersurface in $\mathbf{P}^n$. The open subsets of birational varieties are isomorphic, so it suffices to show the case for a hypersurface, and moreover, it is enough to check any open affine subset of Y. So assume that Y is a hypersurface in $\mathbf{A}^n$ defined by $f(x_1, \ldots, x_n) = 0$ irreducible. If Sing $Y = Y$, then $\frac{\partial f}{\partial x_i}$ are 0 on $Y \implies \frac{\partial f}{\partial x_i} \in I(Y)$ for each i. As $I(Y) = (f)$ and $\deg(\frac{\partial f}{\partial x_i}) \leq \deg f - 1$, we have that $\frac{\partial f}{\partial x_i} = 0$ for each i. This is impossible in characteristic 0 as if x_i appears as a term in f, then $\frac{\partial f}{\partial x_i} \neq 0$, so char $k = p > 0$ for p a prime. But then $\frac{\partial f}{\partial x_i} = 0 \implies f$ is a polynomial in x_i^p. As k is algebraically closed, the p^{th} roots of the coefficients of f are in k, so $f = g^p$ for some polynomial $g(x_1, \ldots, x_n)$. But this is absurd as we took f to be irreducible. □

To finish off section 5, we look at *completions.* Let A be a local ring with maximal ideal $\mathfrak{m}$. We will look at $\mathfrak{m}-$adic topology on A.

Definition. (Completion) We define the *completion* of A, $\hat{A}$, with the $\mathfrak{m}$-adic topology as the inverse limit $\hat{A} = \varprojlim A/\mathfrak{m}^n$.

Examples.

- Let $A = \mathbf{Z}$ and $\mathfrak{m} = (p)$ for p a prime. The completion of $\mathbf{Z}$ is the ring of p-adic integers $\mathbf{Z}_p$.

An important exercise mentioned in this section is 4.7, which states that for varieties X, Y with $P \in X$ and $Q \in Y$, if $\mathcal{O}_P \cong \mathcal{O}_Q$, then P and Q have isomorphic neighborhoods which shows that X and Y are birational. So $\mathcal{O}_P$ contains information about all of X in this sense. It is in this sense that we try to understand what information $\widehat{\mathcal{O}_P}$ contains. To do so, we look at some results on completions from commutative algebra, and then study some examples.

Theorem. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$, and let $\hat{A}$ be its completion.*

(a) *$\hat{A}$ is a local ring, with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A}$, and there is a natural injective homomorphism $A \to \hat{A}$.*

(b) *If M is a finitely generated A-module, its completion $\hat{M}$ with respect to its $\mathfrak{m}$-adic topology is isomorphic to $M \otimes_A \hat{A}$.*

(c) $\dim A = \dim \hat{A}$.

(d) *A is regular if and only if $\hat{A}$ is regular.*

Theorem. *(Cohen Structure Theorem) If A is a complete regular local ring of dimension n containing some field, then $A \cong k[[x_1, \ldots, x_n]]$, the ring of formal power series over the residue field k of A.*

Definition. We say two points $P \in X$ and $Q \in Y$ are *analytically isomorphic* if there is an isomorphism $\widehat{\mathcal{O}_P} \cong \widehat{\mathcal{O}_Q}$ as k-algebras.

Examples.

- If $P \in X$ and $Q \in Y$ are analytically isomorphic, then $\dim X = \dim Y$. To obtain this result, use the theorem before the Cohen Structure Theorem and exercise 3.12 which states that any local ring of a point on a variety has the same dimension as the variety.

- If $P \in X$ and $Q \in Y$ are nonsingular points on varieties of the same dimension, then P and Q are analytically isomorphic. To obtain this result, use the preceding two theorems. This example is algebraic analogous to the result that any two manifolds of the same dimension are locally isomorphic.

The last result in this section is used in an exercise.

Theorem. *(Elimination Theory) Let $f_1, \ldots, f_r$ be homogeneous polynomials in $x_0, \ldots, x_n$, having indeterminate coefficients a_{ij}. Then there is a set $g_1, \ldots, g_t$ of polynomials in the a_{ij}, with integer coefficients, which are homogeneous in the coefficients of each f_i separately, with the following property: for any field k, and for any set of special values of the $a_{ij} \in k$, a necessary and sufficient condition for the f_i to have a common zero different from $(0, \ldots, 0)$ is that the a_{ij} are a common zero of the polynomials g_j.*

References

[1] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977.

[2] KReiser (https://math.stackexchange.com/users/21412/kreiser). The rank of jacobian matrix at a point of affine variety is independent of choice of generators. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/2561081 (version: 2017-12-11).

[3] Adam Block. Nonsingular varieties. `https://www.columbia.edu/~abb2190/Nonsingular.pdf`, July 2017.

[4] Ravi Vakil. Introduction to algebraic geometry, class 15. `https://virtualmath1.stanford.edu/~vakil/725/class15.pdf`, May 2000.

[5] Ziyu Zhang. Non-singularity of varieties. `https://ziyuzhang.github.io/ma40188/Lecture14.pdf`, November 2015.