Algebraic Geometry Learning Seminar Talk Notes

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1 Nonsingular Varieties

The idea of a manifold coincides with that of a nonsingular variety. Specifically, both are motivated by the notion of *smoothness*. Let $J = (\frac{\partial f_i}{\partial x_j}(P))$ be the *Jacobian matrix* at P. Note that differentiation here refers to a *formal derivative* so we are not taking any limits.

Definition. (Nonsingular Variety) Let $Y \subset \mathbf{A}^n$ be an affine variety, and let $f_1, \ldots, f_t \in A = k[x_1, \ldots, x_n]$ be a set of generators for the ideal of Y. Y is *nonsingular at a point* $P \in Y$ if the rank of the matrix J is $n - r$ where r is the dimension of Y. Y is *nonsingular* if it is nonsingular at every point.

Remark. The definition of nonsingularity is independent of the set of generators of the ideal of Y chosen. To see this, take any h that is a A-linear combination of the f_i and observe for $a_i \in A$ we have

$$\begin{aligned} \frac{\partial}{\partial x_j} g &= \frac{\partial}{\partial x_j}\left(\sum a_i f_i\right) \\ &= \sum \frac{\partial}{\partial x_j}(a_i f_i) \\ &= \sum a_i \frac{\partial f_i}{\partial x_j} + f_i \frac{\partial a_i}{\partial x_j} \\ &= \sum a_i \frac{\partial f_i}{\partial x_j} + \sum f_i \frac{\partial a_i}{\partial x_j} \end{aligned}$$

Note that $\sum a_i \frac{\partial f_i}{\partial x_j}$ is an A-linear combination of the $\frac{\partial f_i}{\partial x_j}$ and as $\sum f_i \frac{\partial a_i}{\partial x_j}(P) = 0$ as $f_i(P) = 0$, we see that if h is in the ideal of Y and we add it to the set of f_i's, the rank of J remains unchanged. So we see that if $\{f_i\}$ and $\{f_i'\}$ are two sets of generators for the ideal of Y, then we can add each f_i' as above without changing the rank of J and through row reductions we will have that the rank of J with respect to $\{f_i'\}$ remains the same.

Examples.

- Let $X = \mathbf{A}^n$. Then $I(X) = (0)$. For any $P \in X$ we see that J is the zero matrix so its rank is 0. Hence, X is nonsingular as the dimension of $\mathbf{A}^n$ is n.
- Let $X = \{P\} \subset \mathbf{A}^n$ where $P = (a_1, \ldots, a_n)$. Note that it can be shown that $I(X) = (x_1 - a_1, \ldots, x_n - a_n)$ from which it immediately follows that $J = I_n$ and so as its rank is n we have that X is nonsingular as its dimension of a single point is 0.

Let's start trying to generalize this definition so that we are not reliant on $Y \subset \mathbf{A}^n$.

Definition. (Regular Local Ring) Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. A is a *regular local ring* if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

Theorem. *Let $Y \subset \mathbf{A}^n$ be an affine variety. Then Y is nonsingular at P a point if and only if $\mathcal{O}_P$ is a regular local ring.*

Proof. $\Rightarrow$ Let $P = (a_1, \ldots, a_n) \in \mathbf{A}^n$, and let $\mathfrak{a} = (x_1 - a_1, \ldots, x_n - a_n)$ be the corresponding maximal ideal in $A = k[x_1, \ldots, x_n]$. We define a linear map $\varphi : A \to k^n$ given by $f \mapsto \left(\frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P)\right)$. Note that

$\varphi(x_i - a_i) = (0, \ldots, 1, \ldots 0)$ where 1 is in the i^{th} coordinate. So these $x_i - a_i$ for $i = 1, \ldots, n$ form a basis of k^n. Furthermore, observe that $\varphi((x_i - a_i)(x_j - a_j)) = (0, \ldots, x_j - a_j, \ldots, x_i - a_i, \ldots, 0)(P)$ where $x_j - a_j$ is in the i^{th} coordinate and $x_i - a_i$ is in the j^{th} coordinate. Evaluating this at P is equivalent to letting $x_i = a_i, x_j = a_j$ hence, $\varphi((x_i - a_i)(x_j - a_j)) = 0$. So then we see that $\varphi(\mathfrak{a}^2) = 0$ and φ induces an vector space isomorphism σ, $\mathfrak{a}/\mathfrak{a}^2 \cong k^n$.

Now let $\mathfrak{b} = (f_1, \ldots, f_t)$ be the ideal of Y in A generated by the f_i. By our construction of φ, we see that the rank of J at P is equivalent to $\dim_k \varphi(\mathfrak{b})$ where $\varphi(\mathfrak{b})$ is viewed as a subspace of k^n. Observe that using the fact that σ is an isomorphism we have that

$$\sigma^{-1}\varphi(\mathfrak{b}) = (\mathfrak{b} + \mathfrak{a})/\mathfrak{a}^2$$

hence $\dim_k \varphi(\mathfrak{b})$ is equivalent to the dimension of $(\mathfrak{b} + \mathfrak{a})/\mathfrak{a}^2$ as a subspace of $\mathfrak{a}/\mathfrak{a}^2$. Note that the local ring of P on Y is simply $\mathcal{O}_{\mathcal{P}} = (A/\mathfrak{b})_{\mathfrak{a}}$ and its maximal ideal is $\mathfrak{m} = (\mathfrak{b} + \mathfrak{a})/\mathfrak{b}$. So its clear that $\mathfrak{m}^2 = (\mathfrak{b} + \mathfrak{a}^2)/\mathfrak{b}$, hence $\mathfrak{m}/\mathfrak{m}^2 = \mathfrak{a}/(\mathfrak{b} + \mathfrak{a}^2)$. So, counting the dimensions of vector spaces, gives us that $\dim_k \mathfrak{m}/\mathfrak{m}^2 + \text{rank } J = n$.

$\Leftarrow$ The dimension of the local ring $\mathcal{O}_P$ as a ring is the dimension of Y as a variety. So, $\mathcal{O}_p$ is regular if and only if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim\, Y = r$. But this is equivalent to saying that rank $J = n - r$ which in turn shows that Y is nonsingular at P. □

This theorem allows us to extend our definition of nonsingularity as we just showed that nonsingularity was not tied to the affinity of Y!

Definition. (Nonsingularity) Let Y be any variety. Y is *nonsingular* at a point $P \in Y$ if the local ring $\mathcal{O}_{P,Y}$ is a regular local ring. Y is *nonsingular* if it is nonsingular at every point. Y is *singular* if it is not nonsingular.

In fact, most points of a variety are nonsingular. To see this however, we will need a fact from commutative algebra.

Proposition. *If A is a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, then $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$.*

Theorem. *Let Y be a variety. Then the set Sing Y of singular points of Y is a proper closed subset of Y.*

Proof. The general strategy will be to prove it for the affine case. Any variety can be covered by affine open subsets, that is for a variety Y, and open cover $\cup Y_i$, if we show that Sing Y_i is closed for each i then we are done. So assume Y is affine and $\dim\, Y = r$. By the first theorem, we see that any singular point of Y are those with rank $J < n - r$. Consider $M = \left(\frac{\partial f_i}{\partial x_j}\right)$. The only way we could not achieve full rank is if and only if one of M's $(n - r) \times (n - r)$ sub-matrices has a determinant of 0. So Sing Y is $I(Y) \cup \{$determinants of the $(n - r) \times (n - r)$ sub-matrices of $M\}$ which is closed.

Lastly, we show that Sing $Y \neq Y$. Note that Y being a variety must be birational to a hypersurface in $\mathbf{P}^n$. The open subsets of birational varieties are isomorphic, so it suffices to show the case for a hypersurface, and moreover, it is enough to check any open affine subset of Y. So assume that Y is a hypersurface in $\mathbf{A}^n$ defined by $f(x_1, \ldots, x_n) = 0$ irreducible. If Sing $Y = Y$, then $\frac{\partial f}{\partial x_i}$ are 0 on $Y \implies \frac{\partial f}{\partial x_i} \in I(Y)$ for each i. As $I(Y) = (f)$ and $\deg(\frac{\partial f}{\partial x_i}) \leq \deg f - 1$, we have that $\frac{\partial f}{\partial x_i} = 0$ for each i. This is impossible in characteristic 0 as if x_i appears as a term in f, then $\frac{\partial f}{\partial x_i} \neq 0$, so char $k = p > 0$ for p a prime. But then $\frac{\partial f}{\partial x_i} = 0 \implies f$ is a polynomial in x_i^p. As k is algebraically closed, the p^{th} roots of the coefficients of f are in k, so $f = g^p$ for some polynomial $g(x_1, \ldots, x_n)$. But this is absurd as we took f to be irreducible. □

References

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