I.6. Nonsingular Curves:

Classification Questions

min. model program — (a). Classify Varieties up to birational Equivalence

(b). Find a non-singular proj. var in each eq. class

(c). Classify non-singular proj. var. in a given class.

We will answer these for curves!

Plan: introduce abstract curves, prove q.proj curves $\subseteq$ abstract.

Motivating Example:

$X = \mathbb{P}^1 = \{[x:y] \mid x, y \in k\}$. $\quad k[t] \quad k[t]$

$K(X) = \mathrm{Frac}\, k[t] = k(t)$. $\quad \mathbb{A}^1 \subset \mathbb{P}^1 \supset \mathbb{A}^1$

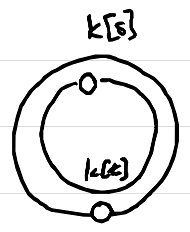
$P = [a:1] \in \mathbb{P}^1, \quad \mathcal{O}_P = k[t]_{(t-a)} = \left\{\frac{f}{g} : g(a) \neq 0\right\}$.

E.g. $\mathrm{ord}_P\left(\frac{(t-a)^2(t-b)}{\lambda(t-c)}\right) = 2, \quad \mathrm{ord}_P\left(\frac{t-b}{(t-a)^2}\right) = -2.$

$\mathrm{ord}_P\left(\frac{f}{g}\right) := \mathrm{ord}_a(f)$.

Q: What about $Q = [1:0]$?

$s = t^{-1} \quad \mathcal{O}_Q = k[s]_{(s)} = \left\{\frac{h}{k} : k(0) \neq 0\right\}$



$\frac{f}{g} = \frac{(t-1)(t+3)}{t} = \frac{(1-s)(1+3s)}{s}$

$\mathrm{ord}_Q\left(\frac{f}{g}\right) = -1 \qquad \mathrm{ord}_Q\left(\frac{f(t)}{g(t)}\right) = \deg g - \deg(f).$

Rmk: All the points have a valuation! on $K(X)$

Commutative Algebra Preliminaries

Def: A discrete valuation on K is

$\nu : K - \{0\} \longrightarrow \mathbb{Z}$. s.t.

(1). $\nu(xy) = \nu(x) + \nu(y)$

(2). $\nu(x+y) \geq \min(\nu(x), \nu(y))$.

Keep in mind: $\mathrm{ord}_0\left(\frac{f(t)}{g(t)}\right) \curvearrowright K = k(t)$.

Out loud: • Valuation ring $K \supset R = \{x \in K : \nu(x) \geq 0\}$ $\quad k[t]_{(t)}$

• $\nu(k) = 0$ then ν : val. of K/k.

Def: $(A, \mathfrak{m}_A), (B, \mathfrak{m}_B) \subset K$ local rings.

B dominates A: $A < B$ if • $A \subseteq B$

• $\mathfrak{m}_B \cap A = \mathfrak{m}_A$.

E.g. $k[t^2]_{(t^2)} < k[t]_{(t)}$

$2 \; \nu(t^2) = ?$

Theorem: $R \subset K$ is a valuation ring iff it is maximal w.r.t. domination.

Thm: $(A, \mathfrak{m})$ Noeth. Local domain of dimension 1:

TFAE:

- A: a DVR $\quad$ poles & zeros!
- $\Uparrow$ A: integrally closed (Normal) $\quad$ Formal properties.
- A: a regular local ring $\quad$ smoothness!
- $\mathfrak{m}$ is principal. $\quad$ Think about it

Def: A Dedekind domain is an integrally closed Noeth. domain of dim. $= 1$.

E.g. $k[t]$: dedekind.

Affine charts on a smooth curve!

Thm:
$$\begin{array}{ccccc} \text{int. clos. Dedekind} \Rightarrow & \bar{R} & \subset & L & \\ & | & & | & < \infty \\ \text{Dedekind} \Rightarrow & R & \subset & K & \end{array}$$

Motivation:

$P \in Y$ = smooth curve $K = K(Y)$.

$\mathcal{O}_P \subset K(Y)$

↳ regular dim $1 \Rightarrow$ DVR

$\therefore\ \exists$ map of sets:

$$Y \longrightarrow C_K := \{R \subset K(Y) : R \text{ is a DVR}\}$$

⚠ : INJectivity

Lem 6.4: Y: quasi-projective, $P, Q \in Y$ & $\mathcal{O}_P \subset \mathcal{O}_Q$ $\Rightarrow P = Q$.

Pf Sketch: $P, Q \in Y \cap (\mathbb{P}^n \setminus H_0)$ is affine.

$\mathcal{O}_P = A_{\mathfrak{m}}, \mathcal{O}_Q = A_{\mathfrak{n}}$, $\mathcal{O}_P \subset \mathcal{O}_Q \Rightarrow \mathfrak{n} \subseteq \mathfrak{m}$

$\Rightarrow \mathfrak{n} = \mathfrak{m}$

$\Rightarrow P = Q$ ■

Building up to defining C_K as an abstract variety

Lemma 6.5: K = fn field of dim $1/k$, $x \in K$.
then $\{R \in C_K \mid x \notin R\}$ is finite.

Rmk: finitely many poles!

Pf Sketch: same as finitely many zeros:

$x \notin R \iff y = \frac{1}{x} \notin \mathfrak{m}_R$.

Ded Dom! $B \subset K$

$$\begin{array}{ccc} B & \subset & K \\ | & & | \\ k[y] & \subset & k(y) \end{array}$$

Ded. Domain – $k[y]$ (UFD)

$\because R$: int. closed

$y \in R \subset K$, then $B \subseteq R$

$\mathfrak{n} = \mathfrak{m}_R \cap B$, then $\mathfrak{n}$: max'l in B $\because$ prime + dim 1

$\Rightarrow B_{\mathfrak{n}} < R \Rightarrow B_{\mathfrak{n}} = R$ by maximality.

DVR

Maximality + dim. $1 \Rightarrow$ if $y \in R$ then $R = B_{\mathfrak{n}}$ for some $\mathfrak{n}$.

i.e. $y \in R \iff y \in \mathfrak{n} \triangleleft B$.

but B is affine. & $y \in \mathfrak{n} \iff y$ vanishes as a reg fn. on Y

$|V(y)| < \infty \Rightarrow \#\{R \subset K : y \in \mathfrak{m}_R\} < \infty$. ■

Rmk: Integral closure was critical!

"SURJECTIVITY"

Cor: Any DVR $R \subset K/k$ is iso. to $\mathcal{O}_P$ for some $P \in Y$ a non-sing. affine curve.

Pf: Use Construction. ■

Construction:

K/k trans. deg $=1$, The <u>abstract curve</u> C_K is:

As a <u>set</u>:

$$C_K = \{R \subset K \mid R : DVR\}$$

These are points!

<u>Topologically</u>:

$U \subset C_K$ open $\iff$ cofinite.

<u>Regular fns</u>:

$$\mathcal{O}(U) = \bigcap_{P \in U} R_P$$

$$\left(\begin{array}{l} f \in \mathcal{O}(U) : U \longrightarrow k \\ \qquad\qquad P \longmapsto \bar{f} \in R_P/\mathfrak{m}_P \overset{\text{(Lemma 6.6)}}{\cong} k \end{array} \right)$$

Q: Maybe $g, f \in \mathcal{O}(U)$ take same values?

A: then $g - f \in \mathfrak{m}_R$ for ∞ly many R

Lemma 6.5 $\Rightarrow g - f = 0$!

Note: $f \in K$ is in $\mathcal{O}(U)$ for some U, by Lem. 6.5!

Remark: Just think in terms of $\mathbb{P}^1$.

Def: $U \subseteq C_K$ is an <u>abstract nonsingular curve</u>

Def: $\varphi : X \longrightarrow Y$ ANSC or Varieties is:

- cts
- $U \subset Y$, $\forall f \in \mathcal{O}(U)$, $f \circ \varphi \in \mathcal{O}(\varphi^{-1}(U))$

Aim: Show: $\{\text{non-sing. q-proj curve}\} \subseteq \{ANSC\}$
and $\supseteq$ too!

Prop: $\subseteq$

pf: Y: nonsing. q. proj. curve. $K := K(Y)$

$\varphi: Y \longrightarrow C_K$
$\quad P \longmapsto \mathcal{O}_P \subset K$
$\qquad$ DVR $\because$ $P \in Y$ smooth.

Injective: by lemma 6.4.

Let $\operatorname{im}(\varphi) = U \subseteq C_K$.

We need to show:

(1). U is open

(2). φ is a morphism.

(1). STP U contains an open subset. So assume Y: affine.

Let $Y = \operatorname{Spec} A$.
$\qquad$ f.g. k-alg.

Now $K = \operatorname{Frac} A$ and:

$$U = \{R \subset K : DVR \,\&\, A \subset R\}$$

Let $x_1, \dots, x_n$ gen. A/k.

then $A \subset R \iff x_1, \dots, x_n \in R$

$\therefore U_i := \{R \in C_K \mid x_i \in R\}$ is open

$\because \{R \in C_K \mid x_i \notin R\}$ is finite (Lemma 6.5)

$\therefore U = \bigcap_{i=1}^{n} U_i$ is open.

(2). Recall, for $V \subset Y$.

$\mathcal{O}(V) = \bigcap_{P \in V} \mathcal{O}_{Y,P}$. $\therefore \varphi$ is a morphism.

$\blacksquare$

END LECTURE 1

Now Want: $\{ANSC\} \subseteq \{q\text{-proj. nonsing. curves}\}$.

Plan: $C_K \to Y \subset \mathbb{P}^N$
$\qquad\qquad\quad$ ⊂ projective.

Prop 6.8: X: abstract nonsingular curve, $P \in X$, Y: proj. var. $\varphi: X \setminus \{P\} \to Y$ a morphism. Then $\exists! \ \bar{\varphi}: X \to Y$ extending φ.

Note: rational maps into projectives extend.

E.g. $\mathbb{A}^1 \setminus (0) \subseteq \mathbb{P}^1 \longrightarrow \mathbb{P}^2$

$[t:1] \longmapsto [t : t^2 : t^3]$

$[1:0] \longmapsto [t^{-1} : t^{-2} : t^{-3}] = [t^2 : t : 1]$.

$[0:1] \longmapsto [1:0:0] \qquad = [0:0:1]$

(sketch).

Pf: Let $Y \subseteq \mathbb{P}^n$. STP for $Y = \mathbb{P}^n$ $\because \operatorname{im} \bar{\varphi} \subseteq Y$.

(Let $U = D(x_0) \cap D(x_1) \cap \dots \cap D(x_n)$
suppose $\varphi(X \setminus \{P\}) \cap U \neq \emptyset$.
if it is then by induction done.)

$$X \setminus \{P\} \xrightarrow{\varphi} U$$

$$f_{ij} = \frac{x_i}{x_j} \circ \varphi \qquad \frac{x_i}{x_j}$$

$\quad \cap$
$K \leftarrow$ fn field.

Let ν = valuation for $P \in X$:

$$r_i := \nu(f_{i0})$$

Since $\frac{x_i}{x_j} = \frac{x_i}{x_0} \cdot \frac{x_0}{x_j}$ we have

$$\nu(f_{ij}) = r_i - r_j$$

Choose $r_k = \min\{r_0, \dots, r_n\}$

then $\nu(f_{ik}) \geq 0 \ \forall i$

$\therefore f_{0k}, \dots, f_{nk} \in R_P \subset K$
$\qquad\qquad\qquad\qquad$ ↑ DVR for $P \in X$.

Define: $\bar{\varphi}: X \to Y$

$$P \mapsto (f_{0k}(P), \dots, f_{nk}(P))$$
$$Q \mapsto \varphi(Q), \ Q \neq P.$$

Check: (a) is a morphism

(b) uniqueness: "clear" from construction

or fns agree on open set agree.

(a).

Need to show: fns in a nbd of $\bar{\varphi}(P)$ pull back to fns on X:

$$\bar{\varphi}(P) \subset U_k \quad \because f_{kk}(P) = 1$$

co-ord ring:

$$K \longleftarrow k[x_0/x_k, \dots, x_n/x_k]$$

$$f_{ik} \longleftarrow\!\shortmid \ x_i/x_k$$

$\cap$
R_P by construction.

$\therefore \ \forall \bar{\varphi}(P) \in V \subset U_k$ pullback to reg. fns ∎

Thm 6.9. K/k: fn field of dim 1. Then C_K is isomorphic to a non-singular projective curve.

Idea:
Pf: $C = C_K$.

Plan:

Cover $C = \bigcup_i U_i$ affines.

$Y_i = \overline{U_i} \subset \mathbb{P}^n$.

$C \hookrightarrow \prod Y_i$ is the iso.

$P \in C$, Cor 6.6 $\Rightarrow \mathcal{O}_{C,P} \simeq R_Q$ where $V = \operatorname{Spec} R$ & $Q \in V$.

$\therefore k(V) = K$

$\therefore V \cong$ open subset of $C = C_K$. (Lemma 6.8)

Cover C by finitely many $U_i \cong V_i \leftarrow$ affine var.

$$\begin{array}{ccc} V_i & \subset & \mathbb{A}^{n_i} \\ \cap & & \cap \\ Y_i := \overline{V_i} & \subset & \mathbb{P}^{n_i} \end{array}$$

Then Y_i: projective & $\varphi_i : U_i \to Y_i$ iso onto image.

Prop 6.8. extend $\overline{\varphi_i} : C \to Y_i$

Then $\varphi : C \to \prod Y_i \leftarrow$ Projective!

Let $Y = \overline{\operatorname{im}(\varphi)}$.

Then $\varphi : C \to Y$ has dense image

($\therefore Y$ is a curve).

Now to show φ is an iso. STP:

①. $\mathcal{O}_{Y,\varphi(P)} \to \mathcal{O}_{C,P}$ an iso.

②. Bijectivity

①.

$$\begin{tikzcd} C \arrow[r, "\varphi"] & Y \arrow[d, "\pi"] \\ U_i \arrow[u, hook] \arrow[r, "\varphi_i"] & V_i \subset Y_i \end{tikzcd}$$

$\therefore$ iso.

$$\begin{tikzcd} \mathcal{O}_{Y_i, \varphi_i(P)} \arrow[r, hook] \arrow[rr, bend right] & \mathcal{O}_{Y, \varphi(P)} \arrow[r, hook] & \mathcal{O}_{C,P} \end{tikzcd}$$

iso $\because \varphi_i : U_i \to V_i$ an iso.

② . surjectivity: $Q \in Y$, $\quad U_i \xrightarrow{\sim} V_i \hookrightarrow Y$ open subset.

$\therefore k(Y) = K$.

$\therefore \mathcal{O}_Q \subset K$ local ring. $\therefore$

$\mathcal{O}_Q < R \subset K$
$\quad\ \ \subset$ DVR

But $R = R_P$ for some $P \in C_K$

& $\mathcal{O}_{\varphi(P)} \cong R$.

$\therefore \mathcal{O}_Q \subset \mathcal{O}_{\varphi(P)} \overset{\text{lem 6.4 (inj)}}{\Longrightarrow} \underline{Q = \varphi(P)}$.

injectivity:

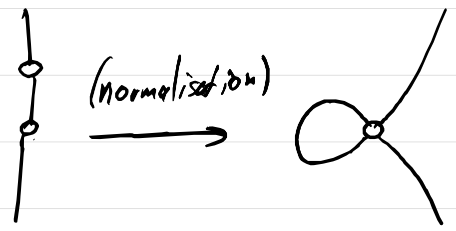
$P \neq P' \in C_K \quad \therefore \mathcal{O}_{\varphi(P)} \neq \mathcal{O}_{\varphi(P')} \Rightarrow \varphi(P) \neq \varphi(P')$.

$R_P \neq R_{P'}$

Cor 6.10: $\{ANSC\} \subseteq \{q.\text{proj curve}\}$

Every non-singular quasiprojective curve is isomorphic to an open subset of a nonsingular proj. curve.

E.g.



(b). Cor 6.11: Every curve is birationally equivalent to a non-singular projective curve

(c). Cor 6.12: Three categories are equivalent.

(i). non-sing. proj. curves w/ dom. morphisms

(ii). q. proj curves & dom. rational maps

(iii). fn fields of dim $1/k$ & k-homomorphisms.

Pf: Functors: (i) $\to$ (ii) inclusion

(ii) $\to$ (iii) equiv. cats. by $\underline{\text{Rat}} \leftrightarrow$ Fields. Laney.

Need (iii) $\to$ (i).

$K_2 \to K_1$ fields. C_{K_1} C_{K_2} proj. curves

(ii) $\simeq$ (iii) $\Rightarrow$ $\varphi: U \to C_{K_1}$, since C_{K_1} is $\underline{\text{projective}}$

$\exists!$ complete $\bar{\varphi}: C_{K_2} \to C_{K_1}$

Functor?

$$K_3 \to K_2 \to K_1$$

uniqueness of completions $\Rightarrow$

$$\begin{tikzcd} K_1 \arrow[rr] \arrow[dr] & \arrow[d, phantom, "G"] & K_3 \\ & K_2 \arrow[ur] & \end{tikzcd}$$

$\therefore K \mapsto C_K$ (iii) $\mapsto$ (i). a functor

(iii) $\to$ (i) $\longrightarrow$ (iii)

$K \mapsto C_K \longmapsto K$ id. on objects & morphisms.

(i) $\longrightarrow$ (iii) $\longrightarrow$ (i)

$V \longmapsto k(V) \mapsto C_{k(V)} \simeq V$ $\because V$: proj. + birational.

$$\begin{tikzcd} X \arrow[d] & & C_{k(X)} \arrow[d] & U \arrow[l] \arrow[r, phantom, "\subset"] & X \arrow[d] \arrow[ll, bend right, "\simeq"'] \\ Y & \longmapsto & C_{k(Y)} & V \arrow[l] \arrow[r, phantom, "\subset"] & Y \arrow[ll, bend left, "\simeq"] \end{tikzcd}$$

G

by uniqueness of completion.