I.7. ~~Intersections in $\mathbb{P}^n$~~
The Hilbert Polynomial

Thm: Y, Z Varieties of dim r, s in $\mathbb{A}^n$ (or $\mathbb{P}^n$)
$W \subset Y \cap Z$ irred comp. has $\dim(W) \geq r+s-n$
($Y \cap Z \neq \emptyset$ in $\mathbb{P}^n$ if $r+s-n \geq 0$).
Slogan: "codimensions add"

Pf ideas: 1) Prove for hypersurfaces (Krull Hauptidealsatz)
2) then via $(X \times Y) \cap \Delta \subset \mathbb{A}^n \times \mathbb{A}^n$ is a hypersurfaces
3). Pass to cones $C(Y), C(Z) \subseteq \mathbb{A}^{n+1}$ then dimension count.

Polynomials Galore:

Def: A numerical polynomial is $P(z) \in \mathbb{Q}[z]$ s.t. $P(n) \in \mathbb{Z}$ for $n \in \mathbb{Z}$, $n >> 0$.

Prop.

(a). If $P(z) \in \mathbb{Q}[z]$ is a numerical Polynomial.

Then

$$P(z) = c_0 \binom{z}{r} + c_1 \binom{z}{r-1} + \cdots + c_r$$

with $c_0, \ldots, c_r \in \boxed{\mathbb{Z}\;!}$

((b). $f : \mathbb{Z} \to \mathbb{Z}$ a fn. s.t. $\Delta f(n) := f(n+1) - f(n)$ is eventually a numerical polynomial then $f(n) = P(n)$, $n >> 0$, $P \in \mathbb{Q}[z]$ num. poly.)

Pf ideas:

(a). Induction:

$+ \Delta \binom{z}{r} = \binom{z}{r-1}$. for integrality.

(b). similar ∎

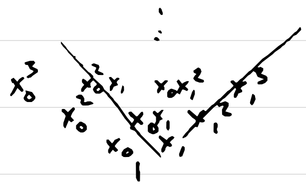
Def: M: f.g. graded module over $k[x_0,\dots,x_n]=S$.
The Hilbert fn of M is
$$\varphi_M:\mathbb{Z}\longrightarrow\mathbb{Z},\ \varphi_M(\ell)=\dim M_\ell$$

Thm: M: f.g. graded $S=k[x_0,\dots,x_n]$-Module.
then φ_M is eventually polynomial.
Furthermore $\deg P_M(z)=\dim Z(\mathrm{Ann}\,M)$

Def: $Y\subset\mathbb{P}^n$ alg. set of dim $=r$.
then The Hilbert polynomial P_Y is the unique polynomial $\varphi_{S(Y)}$ for:
$$S(Y)=k[x_0,\dots,x_n]/I(Y).$$
Note: $\mathrm{degree}(P_Y)=\dim(Y)$

Def: $\deg Y:=r!\cdot(\text{Leading coefficient of } P_Y)$.

Examples:

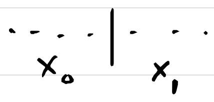
$\mathbb{P}^1$: $Y=\{pt\}$:
$$S(pt)=k[x_0,x_1]/(x_0)$$


$$\varphi_{pt}=1$$
$$\therefore\ P_{pt}(z)=1$$
$$\boxed{\deg(pt)=1}$$

$Y=\mathbb{P}^1$:
$$\varphi_Y(n)=\dim k[x_0,x_1]_n=n+1$$
$$\boxed{\deg(Y)=1}$$

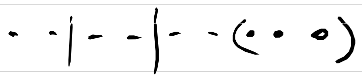
$\mathbb{P}^2$: • $P_{pt}(n)=\left(k[x_0,x_1,x_2]/(x_0,x_2)\right)_n=1.$
$\deg(pt)=1.$

• $Y=V(x_2)$: $\left(k[x_0,x_1,x_2]/x_2\right)_n=n+1$

• $Y=V(x_1x_2)$
$$\left(k[x_0,x_1,x_2]/(x_1x_2)\right)_n=2\cdot\binom{n+1}{1}=2n+2$$


$\deg(Y)=2.$

$$0\to S(-2)\to S\to S(Y)\to 0$$
$$1\longmapsto x_1x_2$$
$$\varphi_Y(\ell)=\binom{\ell+2}{2}-\binom{\ell}{2}=\frac{(\ell+2)(\ell+1)}{2}-\frac{\ell(\ell-1)}{2}$$
$$=2\ell+2$$

• $Y=V(x_0x_1^2-x_3^3)$
$$\left(k[x_0,x_1,x_2]/(x_0x_1^2-x_3^3)\right)_n=\binom{n+2}{2}-\binom{n-1}{2}$$
$$=\frac{(n+2)(n+1)}{2}-\frac{(n-1)(n-2)}{2}$$

$$=\frac{6n}{2}=3n.$$
n^2+3n+2
n^2-3n+2
$$\boxed{\deg Y=3}$$

• $Y=\mathbb{P}^n$: $k[x_0,\dots,x_n]_p=\binom{n+r-1}{r}=\frac{n(n-1)\cdots(n-(r-1))}{r!}$
$$\boxed{\deg=\frac{r!}{r!}=1.}$$

Prop 7.6.

(a). $\emptyset \neq Y \subset \mathbb{P}^n$, $\deg(Y) \geq 0$

(b). $Y = Y_1 \cup Y_2$, $\dim Y_1 = \dim Y_2 = r$ and $\dim Y_1 \cap Y_2 < r$. Then $\deg Y = \deg Y_1 + \deg Y_2$

(c). $\deg \mathbb{P}^n = 1$

(d). $H \subseteq \mathbb{P}^n$ hypersurface, whose homog. ideal is gen. by f of degree d. then $\deg H = d$

Pf:

(a) P_Y is a numerical polynomial, $\therefore$ by lemma on numerical poly's $\deg Y = c_0$. pos $\because$ bounded below.

(c). done

(b). $Y_i := V(I_i)$, $i = 1, 2$.

then $Y = V(\underbrace{I_1 \cap I_2}_{I})$. $\therefore$

$$0 \to S/I \xrightarrow{\Delta} S/I_1 \oplus S/I_2 \xrightarrow{+} S/(I_1 + I_2) \to 0$$

$Z(I_1 + I_2) = Y_1 \cap Y_2$ has dim $< r$.

$\therefore$ $P_{S/(I_1+I_2)}$ has degree $< r$.

$$P_{S/I_1} + P_{S/I_2} = P_{S/I} + P_{S/I_1+I_2}$$

$$P_{Y_1} + P_{Y_2} = P_Y + P_{Y_1 \cap Y_2}$$

compare leading terms.

(d). $f \in S = k[x_0, \ldots, x_n]$ homog. degree d.

$$0 \to S(-d) \to S \to S/(f) \to 0, \qquad 1 \longmapsto f, \qquad S/(f) = S(Y)$$

$$\therefore \quad P_Y(z) = P_{\mathbb{P}^n}(z) - P_{\mathbb{P}^n}(z-d)$$

$$= \binom{z+n}{n} - \binom{z+n-d}{n}$$

$$= \frac{d}{(n-1)!} z^{n-1} + \cdots$$

$\therefore$ degree $(H) = d$. ∎

Applications:

Genus of a Curve:

$Y \subset \mathbb{P}^r$ a dim 1 variety. Define the (arithmetic) genus:

$$p_a(Y) := 1 - P_Y(0)$$

Recall: • $\mathbb{P}^1 \subset \mathbb{P}^1$, $P_{\mathbb{P}^1}(n) = n + \boxed{1}$ $\boxed{g=0}$

• $\mathbb{P}^1 \subset \mathbb{P}^2$, $P_{\mathbb{P}^2}(n) = n + \boxed{1}$ $\boxed{g=0}$

• $\mathbb{P}^1 = V(xz - y^2) \subset \mathbb{P}^2$

$$P_Y(n) = \binom{n+2}{2} - \binom{n}{2} = \frac{(n+2)(n+1)}{2} - \frac{n(n-1)}{2}$$
$$= 2n + \boxed{1}$$

• $V(y^2z - x^3 + xz^2) \subset \mathbb{P}^2$ $\quad P_Y(n) = \binom{n+1}{2} - \binom{n-1}{2}$
$$= 3n$$

$\boxed{g=1}$

• $V(f) \subset \mathbb{P}^2$, $\deg(f) = d$.

$$P_Y(n) = \binom{n+2}{2} - \binom{n-d+2}{2}$$
$$= \frac{(n+2)(n+1)}{2} - \frac{(n-d+2)(n-d+1)}{2}$$

$2 \Leftrightarrow n-d$.

$$g = 1 - P_Y(0) = 1 - \frac{2}{2} - \frac{(-d+2)(-d+1)}{2}$$
$$= \frac{(d-1)(d-2)}{2}.$$

Rmk: This is independent of the choice of embedding in $\mathbb{P}^n$!

Proof? Use sheaf Cohomology!

Thm: $Y \subseteq \mathbb{P}^n$ variety $\dim Y \geq 1$.

H: hypersurface not containing Y.

$Y \cap H = Z_1 \cup \cdots \cup Z_s$. Then

$$\sum_{j=1}^{s} i(Y, H; Z_j) \cdot \deg Z_j = (\deg Y)(\deg H)$$

$i(Y, H; Z_j)$ def. by filtration of $M = S/(I_Y + I_H)$

$$0 = M^0 \subseteq M^1 \subseteq M^2 \subseteq \cdots \subseteq M^r = M.$$
$$M^i / M^{i-1} \cong (S/\mathfrak{p}_j)(\ell_i)$$
$$\mathfrak{p}_j \leftrightarrow Z_j.$$