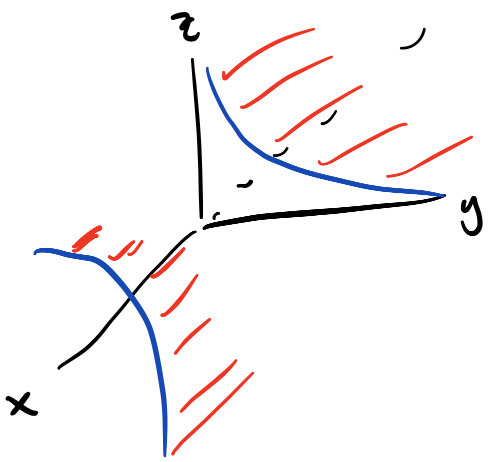
Goal: Rat is equivalent to f.g. field extns

category of varieties w/ dominated rat'l maps

Lemma: $Y \subset \mathbb{A}^n \Rightarrow \mathbb{A}^n - Y \cong$ hypersurface $\subseteq \mathbb{A}^{n+1}$

hypersurface



$yz - 1 = 0$

Pf: $f = 0$ defines Y

$x_{n+1} f = 1$ defines H

$\varphi : H \to \mathbb{A}^n$

$\varphi(a_1, \ldots, a_{n+1})$

$= (a_1, \ldots, a_n) \notin Y$

since $a_{n+1} \underbrace{f(a_1, \ldots, a_n)}_{\neq 0} = 1$

$$\varphi^{-1}(a_1,\dots,a_n) = \left(a_1,\dots,a_n, \frac{1}{f(a_1,\dots,a_n)}\right)$$

Lemma: On variety Y, $\exists$ base for topol. of open affine sets.

Pf: $p \in U \subseteq Y$. WTS $\exists V$ s.t. $p \in V \subseteq U$

U quasi-proj., $U = Y$

Take Y quasi-affine

i.e. $Y = \overline{Y} - Z$ — $Z = \overline{Y} - Y$ closed in $\mathbb{A}^n$

$I = I(Z)$, $p \notin Z$

$\exists f \in I$ s.t. $f(p) \neq 0$

H def'd by $f = 0$, $Z \subseteq H$, $p \notin H$

$p \in Y - Y \cap H$ is open in Y (show this is V we want!)

$$Y - Y \cap H = Y \cap (\mathbb{A}^n - H)$$

$$= (\bar{Y} - Z) \cap (\mathbb{A}^n - H) \quad (Z \subseteq H)$$

$$= \bar{Y} \cap (\mathbb{A}^n - H)$$ is closed in $\mathbb{A}^n - H$

which is affine by prev. lemma

$\square$

Proof of Goal:

(A.a) Y : (affine) variety $\subseteq \mathbb{A}^n$

$K(Y)$ is f.g. field extn of k

(A.b) K f.g. field extn of k

$B = k(y_1, \dots, y_n) = k[y_1, \dots, y_n] / I(Y)$

integral domain

B.a $\varphi : X \to Y$

/

$\langle U, \varphi \rangle$

$f \in K(Y)$, $\langle V, f \rangle$, f is regular on V

$\varphi(U) \subseteq Y$ dense

$\varphi^{-1}(V)$ nonempty, open

$f \circ \varphi$ is regular func. on $\varphi^{-1}(V)$

so have $K(Y) \to \mathcal{O}(\varphi^{-1}(V)) \hookrightarrow K(X)$

B.b $\Theta : K(Y) \to K(X)$

Y : affine $\qquad K(Y) \cong K(U_i)$

$y_1, \dots, y_n$ gens for $A(Y)$

$\theta(y_1), \dots, \theta(y_n) \in K(X)$

$\exists\, U \subseteq X$ s.t. $\theta(y_i)$ regular $\forall i$

$\theta : A(Y) \to \mathcal{O}(U)$

since $\mathrm{Hom}(A(Y), \mathcal{O}(U)) \xrightarrow{\sim} \mathrm{Hom}(U, Y)$

Corollary: X, Y varieties, TFAE

1. X, Y birational
2. $U \subset X$, $V \subset Y$, $U \cong V$
3. $K(X) \cong K(Y)$

Prop: Any var. of dim r is birational to a hypersurface $\mathbb{P}^{r+1}$

(Check Dummit & Foote on separable field extns)