Projective Varieties

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Fix an algebraically closed field k.

Definition 1. Define *projective n-space*, denoted $\mathbb{P}^n$, to be $(k^{n+1} - 0)/\sim$, where

$$(a_0, \dots, a_{n+1}) \sim (b_0, \dots, b_{n+1}) \iff \exists \lambda \in k, b_i = \lambda a_i, \forall i.$$

An equivalence class $P \in \mathbb{P}^n$ (a point in projective space) is often denoted $[a_0 : \cdots : a_n]$.

Let $A = k[x_1, \dots, x_n]$ and $S = k[x_0, \dots, x_n]$.

Proposition 2. *The decomposition of S into the direct sum of S_d, where S_d contains all monomials of degree d, gives S the strucutre of a graded ring, i.e.*

$$S = \bigoplus_{d \geq 0} S_d$$

and $S_d S_e \subset S_{d+e}$.

In the affine setting, we have a correspondence between closed sets in $\mathbb{A}^n$ and ideals in A. In the projective setting, we will define a topology on $\mathbb{P}^n$ which gives a correspondence between closed sets in $\mathbb{P}^n$ and *homogeneous* ideals in S.

Definition 3. Let $\mathfrak{a} \subset S$ be an ideal in S (or more generally in any graded ring). TFAE:

i. $\mathfrak{a}$ is generated by homogeneous elements, i.e. elements in S_d for some d,

ii. $\mathfrak{a} = \bigoplus_{d \geq 0} (\mathfrak{a} \cap S_d)$.

In the case that $\mathfrak{a}$ satisfies (i) and (ii), we say $\mathfrak{a}$ is *homogeneous*.

Proposition 4. *The collection of homogeneous ideals in S is closed under sum, product, intersection, and radical.*

We are now ready to define the closed sets of $\mathbb{P}^n$.

Definition 5. Given a collection T of homogeneous elements in S, we define the *zero set* (or *vanishing set*) of T to be

$$Z^+(T) = \{[a_0 : \cdots : a_n] \in \mathbb{P}^n \mid f(a_0, \ldots, a_n) = 0, \forall f \in T\}.$$

Because all $f \in T$ are homogeneous, $Z^+(T)$ is well-defined.

We call the sets $Z^+(T)$ *algebraic sets* in $\mathbb{P}^n$.

Definition 6. The complements of algebraic sets in $\mathbb{P}^n$ are the open sets of the *Zariski topology* on $\mathbb{P}^n$.

Definition 7. If $X \subset \mathbb{P}^n$ is closed and irreducible, we say it is a *projective algebraic variety.* We say an open subset of a projective algebraic variety is a *quasiprojective algebraic variety.*

Given $Y \subset \mathbb{P}^n$, define

$$I^+(Y) = \{f \in S_d \mid d \geq 0, f(a_0, \ldots, a_n) = 0, \forall [a_0 : \cdots : a_n] \in Y\}.$$

Note $I^+(Y)$ is well-defined because the f are required to be homogeneous. We call $I^+(Y)$ the homogeneous ideal of (associated to) Y.

Remark 8. Let $q\colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the quotient map. If $Y \subset \mathbb{P}^n$, then $I^+(Y) = I(q^{-1}Y)$. If $T \subset S$ is homogeneous, then $Z^+(T) = q(Z(T) \setminus \{0\})$.

Proposition 9. *We have the following:*

a. $T_1 \subset T_2$ *homogeneous subsets of* S*, then* $Z^+(T_1) \supset Z^+(T_2)$,

b. $Y_1 \subset Y_2$ *subsets of* $\mathbb{P}^n$*, then* $I^+(Y_1) \supset I^+(Y_2)$,

c. $Y_1, Y_2 \subset \mathbb{P}^n$*, then* $I^+(Y_1 \cup Y_2) = I^+(Y_1) \cap I^+(Y_2)$,

d. $\mathfrak{a}$ *homogeneous ideal with* $Z^+(\mathfrak{a}) \neq \emptyset$*, then* $I^+(Z^+(\mathfrak{a})) = r(\mathfrak{a})$,

e. $Y \subset \mathbb{P}^n$ *implies* $Z^+(I^+(Y)) = \overline{Y}$.

Property (d) is the only one that looks slightly strange. In the affine case, $I(Z(\mathfrak{b}))$ holds for any ideal $\mathfrak{b} \subset A$. However, $Z^+(\mathfrak{a}) = \emptyset$ does not imply $\mathfrak{a} = (1)$ in the projective setting. In fact, we have the following proposition.

Proposition 10. *TFAE:*

i. $Z^+(\mathfrak{a}) = \emptyset$,

ii. $\mathfrak{a} \supset S_d$ *for some* $d > 0$,

iii. $r(\mathfrak{a})$ *is either* S *or* $S_+ = \bigoplus_{d>0} S_d$.

Proof. (i) $\implies$ (iii): Because $Z^+(\mathfrak{a}) = \emptyset$,

$$Z(\mathfrak{a}) - 0 = \{(a_0, \ldots, a_n) \neq 0 \mid f(a_0, \ldots, a_n) = 0, \forall f \in \mathfrak{a}\} = \emptyset.$$

Thus, $Z(\mathfrak{a}) = \emptyset$ or $Z(\mathfrak{a}) = \{0\}$. If $Z(\mathfrak{a}) = \emptyset$, $\mathfrak{a} = S$, so $r(\mathfrak{a}) = S$. If $Z(\mathfrak{a}) = \{0\}$, then there exist $d_0, \ldots, d_n$ such that $x_i^{d_i} \in \mathfrak{a}$. Thus, $r(\mathfrak{a}) \supset S_+$. Every element of S_0 is a unit, so $r(\mathfrak{a}) = S_+$.

(iii) $\implies$ (ii): If $r(\mathfrak{a}) = S$ or S^+, then $x_0^{d_0}, \ldots, x_n^{d_n} \in \mathfrak{a}$ for some $d_0, \ldots, d_n$. Thus, for $d = \max\{d_i\}$, $x_i^d \in \mathfrak{a}$ for all i. Thus, $S_d \subset \mathfrak{a}$.

(ii) $\implies$ (i): We see $Z^+(\mathfrak{a}) \subset Z^+(x_0^d, \ldots, x_n^d) = \emptyset$. □

The ideal S_+ is sometimes called the *irrelevant* maximal ideal of S, because there is no algebraic set whose associated ideal is S_+.

Remark 11. Homogeneous radical ideals correspond to algebraic sets, and homogeneous prime ideals correspond to projective varieties.

Remark 12. $\mathbb{P}^n$ is irreducible and noetherian.

Remark 13. $\mathbb{P}^n$ is locally homeomorphic to $\mathbb{A}^n$: for

$$U_i = \{[a_0 : \ldots, a_n] \in \mathbb{P}^n \mid a_i \neq 0\},$$

we have a homeomorphism $\phi_i \colon U_i \to \mathbb{A}^n$ given by

$$[a_0 : \cdots : a_n] \mapsto (a_0/a_i, \ldots, a_{i-1}/a_i, a_{i+1}/a_i, \ldots, a_n/a_i).$$

Recall the following notions of dimension.

Definition 14. For X a topological space, we define $\dim X$ to be the maximum n such that

$$X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_n \subset X$$

where each X_i is closed, irreducible.

For R a ring, we define $\dim R$ to be the maximum n such that

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subset X$$

where each $\mathfrak{p}_i$ is prime.

In the affine case we have the following result.

Proposition 15. *Suppose $X \subset \mathbb{A}^n$ is an affine variety. Then,*

$$\dim X = \dim A(X),$$

where $A(X) = A/I(X)$ is the affine coordinate ring *of X.*

In the projective case, we have a natural analogue.

Proposition 16. *Suppose $Y \subset \mathbb{P}^n$ is a projective variety. Then,*

$$\dim Y = \dim S(Y) - 1,$$

where $S(Y) = S/I^+(Y)$ is the projective coordinate ring *of Y.*

Example 17. *To illustrate the need for subtracting 1, observe that*

$$\dim S/I^+(\mathbb{P}^n) = \dim k[x_0, \dots, x_n]/(0) = \dim k[x_0, \dots, x_n] = n + 1.$$

Thus, the result implies $\dim \mathbb{P}^n = n + 1 - 1 = n$. This makes sense intuitively, because $\mathbb{P}^n$ is locally homeomorphic to $\mathbb{A}^n$.

To prove this proposition, we require the following theorem and proposition.

Theorem 18 (Thm 1.8A)**.** *Suppose B is an integral domain which is a finitely generated k-algebra. Then, $\dim B$ is the transcendence degree of $K(B) := \mathrm{Frac}(B)$ over k.*

Proposition 19 (Exer 1.10b)**.** *If X is a space covered by a family of open sets $\{U_i\}$, then*

$$\dim X = \sup \dim U_i.$$

We now prove the proposition.

Proof. Let $Y \subset \mathbb{P}^n$ be a projective variety. Consider $\phi_0 \colon U_0 \to \mathbb{A}^n$, the homeomorphism given above, and let $Y_i = U_i \cap Y$.

Without loss of generality consider the case $i = 0$. We identify $A(Y_0)$ with the $\deg 0$ elements of $S(Y)_{x_0}$ via the injection

$$p(x_1, \dots, x_n) \mapsto \bar{p} = p(x_1/x_0, \dots, x_n/x_0).$$

Extend this map to an isomorphism $A(Y_0)[x_0, x_0^{-1}] \to S(Y)_{x_0}$ by mapping

$$q_{-n}x_0^{-n} + \cdots + q_m x_0^m \mapsto \bar{q}_{-n}x_0^{-n} + \cdots + \bar{q}_m x_0^m.$$

Thus, we have

$$\dim S(Y)_{x_0} = \dim A(Y_0)[x_0, x_0^{-1}].$$

Because $I^+(Y)$ is prime, $S(Y)$ is an integral domain (hence so is $A(Y_0)$). By Theorem 1.8A, we can pass to the field of fractions and look at transcendence degree. We see

$$\dim A(Y_0)[x_0, x_0^{-1}] = \operatorname{tr.deg}_k K(A(Y_0)[x_0, x_0^{-1}]) = \operatorname{tr.deg}_k K(A(Y_0)[x_0]) = \operatorname{tr.deg}_k K(A(Y_0)) + 1 = \dim A(Y_0$$

and

$$\dim S(Y)_{x_0} = \operatorname{tr.deg}_k K(S(Y)_{x_0}) = \operatorname{tr.deg}_k K(S(Y)) = \dim S(Y).$$

Thus,

$$\dim S(Y) = \dim A(Y_0) + 1.$$

We can do the same argument with any other $i = 0, \dots, n$. By Exercise 1.10b,

$$\dim Y = \sup \dim Y_i = \sup \dim A(Y_i) = \sup(\dim S(Y) - 1) = \dim S(Y) - 1.$$

Thus,

$$\dim S(Y) = \dim Y + 1.$$

$\square$