Projective Varieties 2

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0.1 Dimension

Recall the following facts about affine space. Fix an algebraically closed field k, $A = k[x_1, \dots, x_n]$, $S = k[x_0, \dots, x_n]$.

Proposition 1. *If $Y \subset \mathbb{A}^n$ is a quasi-affine variety, then $\dim Y = \dim \overline{Y}$.*

Proof. If $Z_0 \subset \dots \subset Z_m = Y$ is a sequence of distinct closed irreducible subsets of Y, $\overline{Z}_0 \subset \dots \subset \overline{Z}_m \subset \overline{Y}$ remain closed, irreducible, and distinct. Thus, $\dim Y \leq \dim \overline{Y}$.

Suppose $\dim Y = m$, so there exists a maximal chain $Z_0 \subset \dots \subset Z_m = Y$. It is clear from maximality of Z_i that Z_0 is a point P.

We claim $P = \overline{Z}_0 \subset \dots \subset \overline{Z}_m = \overline{Y}$ is a chain in $\overline{Y}$ maximal in the sense that there are no missing intermediate elements. Suppose there exists $X \subset \overline{Y}$ closed irreducible such that $\overline{Z}_i \subset X \subset \overline{Z}_{i+1}$. Then, $Z_i \subset X \cap Y \subset Z_{i+1}$, where $X \cap Y$ is closed and irreducible. Hence, $X \cap Y = Z_i$ or Z_{i+1}. Thus, $X = \overline{X \cap Y} = \overline{Z}_i$ or $\overline{Z}_{i+1}$. Thus, $\overline{Z}_i$ is a maximal chain.

The closed irreducible $\overline{Z}_i$ correspond to a chain $0 = \mathfrak{p}_m \subset \mathfrak{p}_{m-1} \subset \dots \subset \mathfrak{p}_0 \subset A(\overline{Y})$ of prime ideals. Because $\overline{Z}_0$ is a point, $\mathfrak{p}_0$ is maximal. Because $A(\overline{Y})$ is catenary, height $\mathfrak{p}_0 = m$. Thus, by Thm 1.8A,

$$\dim A(\overline{Y}) = \operatorname{height} \mathfrak{p}_0 + \dim A(\overline{Y})/\mathfrak{p}_0 = \operatorname{height} \mathfrak{p}_0 + \dim k = m = \dim Y.$$

□

Proposition 2. *An affine variety has dimension $n-1$ if, and only if, it is the zero set $Z(f)$ of a single nonconstant irreducible polynomial in $A = k[x_1, \dots, x_n]$.*

Proof. See Hartshorne Proposition 1.13. □

Recall the following theorem.

Theorem 3 (Exer 2.6)**.** *If Y is a projective variety, $\dim S(Y) = \dim Y + 1$.*

Using this theorem, we can generalize the two propositions to projective space.

Proposition 4 (Exer 2.7)**.** *If $Y \subset \mathbb{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \overline{Y}$.*

Proof. For some i, $\overline{Y} \cap U_i$ is nonempty, so $\dim \overline{Y} = \dim(\overline{Y} \cap U_i)$. Because $\overline{Y} \cap U_i$ is affine and $\overline{Y} \cap Y_i = \overline{Y \cap U_i}$ in U_i,

$$\dim(\overline{Y} \cap U_i) = \dim(Y \cap U_i).$$

Thus,

$$\dim \overline{Y} = \sup_{\overline{Y} \cap U_i \neq \emptyset} \dim(\overline{Y} \cap U_i) = \sup \dim(Y \cap U_i) = \dim Y.$$

□

Proposition 5 (Exer 2.8)**.** *A projective variety $Y \subset \mathbb{P}^n$ has dimension $n-1$ if, and only if, $Y = Z(f)$ for $f \in S$ nonconstant, homogeneous, and irreducible.*

Proof. If $Y = Z(f)$, then $\dim S(Y) - 1 = \dim S/(f) - 1 = n - 1$.

Suppose $\dim Y = n - 1$. There exists U_0 such that $Y \cap U_0 \neq \emptyset$. Note that $Y \cap U_0$ remains irreducible, as shown last time. Then, $Y \cap U_0 = Z(g)$ for $g \in k[x_1, \ldots, x_n]$ irreducible, non-constant. Let $f \in S$ be defined by

$$f(x_0, \ldots, x_n) = x_0^{\deg g} \cdot g(x_1/x_0, \ldots, x_n/x_0).$$

Then, f is homogeneous and non-constant with $\deg f = \deg g$.

We claim $Y = Z(f)$. It is clear that $Y \cap U_0 = Z(f) \cap U_0$. Because $Z(f)$ and Y are closed, U_0 is dense, we have $Y = Z(f)$. Because Y is a variety, f is irreducible. □

Definition 6. Let $q\colon \mathbb{A}^{n+1} - 0 \to \mathbb{P}^n$ be the quotient map. For $Y \subset \mathbb{P}^n$ a nonempty algebraic set, we define the *affine cone* over Y to be

$$C(Y) = q^{-1}(Y) \cup \{(0, \ldots, 0)\}.$$

Proposition 7 (Exer 2.8)**.** *The following hold:*

a. $C(Y)$ *is an algebraic set with ideal* $I(Y)$*,*

b. $C(Y)$ *is irreducible if, and only if,* Y *is irreducible,*

c. $\dim C(Y) = \dim Y + 1$.

Proof. Suppose $Y = Z(\mathfrak{a})$.

(a): $C(Y) = \{0\} \cup \{x \in \mathbb{A}^{n+1} \mid q(x) \in Y\} = \{0\} \cup \{x \in \mathbb{A}^{n+1} \mid f(x) = 0 \forall f \in \mathfrak{a}\} = Z(\mathfrak{a})$.

(b): $C(Y)$ irreducible $\iff$ $\mathfrak{a}$ is prime $\iff$ Y irreducible.

(c): $\dim Y = \dim S(Y) - 1 = \dim C(Y) - 1$. □

0.2 Linear Varieties

Definition 8 (Exer 2.11)**.** A *hyperplane* (in $\mathbb{P}^n$) is a hypersurface $Z(f)$ defined by a linear polynomial f.

Proposition 9. *If $Y \subset \mathbb{P}^n$ is a variety, then Y is an intersection of hyperplanes if, and only if, $I(Y)$ is generated by linear polynomials.*

Proof. $Y = Z(f_1) \cap \cdots \cap Z(f_n) \iff I(Y) = (f_1, \ldots, f_n)$ when the f_i are irreducible (S is a UFD). □

Proposition 10. *Let $Y \subset \mathbb{P}^n$ be a linear variety of dimension r. Then, $I(Y)$ is minimally generated by $n - r$ homogeneous linear polynomials.*

Proof. It suffices to show Y is the intersection of at least $n - r$ homogeneous linear polynomials.

Suppose
$$Y = Z(T_1, \ldots, T_m) = Z(T_1) \cap \cdots \cap Z(T_m),$$
where m is minimal. Because m is minimal, $T_1, \ldots, T_m$ are linearly independent. Thus, by a linear change of variables, it suffices to consider the case

$$Y = Z(X_n, \ldots, X_{n-m}).$$

In this case, it is clear that $I(Y) = (X_n, \ldots, X_{n-m})$, and $\dim S/I(Y) = n + 1 - m$. Thus, since
$$r = \dim Y = \dim S/I(Y) - 1 = n - m,$$
we conclude $m = n - r$. □

Proposition 11. *Suppose $Y, Z \subset \mathbb{P}^n$ are linear varieties of dimension r, s. Then,*

$$r + s - n \geq 0 \implies Y \cap Z \neq \emptyset.$$

Proof. Observe that $\dim_k C(Y) = r + 1$, $\dim_k C(Z) = s + 1$. Then,

$$\dim_k C(Y) \cap C(Z) \geq r + 1 + s + 1 - n - 1 = r + s - n + 1 \geq 1.$$

□

0.3 Two Canonical Embeddings

Definition 12 (The Segre Embedding)**.** Let $r, s \in \mathbb{N}$ and $N = rs + r + s$. Define $\psi \colon \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ to be the map

$$[a_0, \ldots, a_r] \times [b_0, \ldots, b_s] \mapsto [a_0 b_0, a_0 b_1, \ldots, a_0 b_s, a_1 b_0, \ldots, a_r b_s].$$

Proposition 13. *The Segre embedding is well-defined and injective.*

Proof. Well-definedness is clear by homogeneity of each component and the fact that some $a_i b_j \neq 0$. For injectivity, suppose $\psi(a, b) = \psi(c, d)$. Without loss of generality $a_0 b_0 = c_0 d_0 = 1$. Then,
$$\frac{a_i}{a_0} = \frac{a_i b_0}{a_0 b_0} = \frac{c_i d_0}{c_0 d_0} = \frac{c_i}{c_0}$$
and similarly $\frac{b_j}{b_0} = \frac{d_j}{d_0}$. Thus, $a = c$ and $b = d$. □

Proposition 14. *The image of the Segre embedding is a subvariety of* $\mathbb{P}^N$.

Proof. Consider the map $\Gamma\colon k[z_{ij}] \to k[x_0, \ldots, x_r, y_0, \ldots, y_s]$ which maps $z_{ij} \mapsto x_i y_j$. Observe that $\mathfrak{a} = \ker \Gamma$ is a homogeneous ideal. We claim $\operatorname{Im}(\psi) = Z(\mathfrak{a})$.

For the easy direction ($\subset$), consider $c = \psi(a, b) = [a_0 b_0 : \cdots : a_r b_s] \in \operatorname{Im}(\psi)$. Let $f \in \mathfrak{a}$, so $\Gamma(f) = 0$. We see

$$f(c) = f(\psi(a, b)) = \Gamma(f)(a, b) = 0.$$

Thus, $c \in Z(\mathfrak{a})$.

For the hard direction ($\supset$), suppose $c = [c_{00} : \cdots : c_{rs}] \in Z(\mathfrak{a})$. Without loss of generality suppose $c_{00} \neq 0$. Observe that $\Gamma(z_{i0} z_{0j} - z_{00} z_{ij}) = 0$ for all i, j. Define $a = [a_0 : a_1 : \cdots : a_r] \in \mathbb{P}^r$, $b = [b_0 : b_1 : \cdots : b_s] \in \mathbb{P}^s$ by

$$a_i = \frac{c_{i0}}{c_{00}}, \quad b_j = \frac{c_{0j}}{c_{00}}.$$

Because $c_{00} \neq 0$, a and b are well-defined. We see

$$\begin{aligned} \psi(a, b) &= \left[\frac{c_{00}^2}{c_{00}^2} : \frac{c_{10} c_{00}}{c_{00}^2} : \cdots : \frac{c_{r0} c_{0s}}{c_{00}^2}\right] \\ &= [c_{00} : \cdots : c_{rs}]. \end{aligned}$$

Thus, $c \in \operatorname{Im} \psi$.

Irreducibility of $Z(\mathfrak{a})$ follows from the fact that 0 is prime in $k[x_0, \ldots, x_r, y_0, \ldots, y_s]$. Thus, $\operatorname{Im} \psi = Z(\mathfrak{a})$ is a subvariety of $\mathbb{P}^N$. □

Definition 15 (The d-Uple Embedding)**.** Fix $n, d > 0$ and let $N = \binom{n+d}{n} - 1$. Denote by $M_0, \ldots, M_N$ the monomials of degree d in $n + 1$ variables $x_0, \ldots, x_n$. Let $\rho_d\colon \mathbb{P}^n \to \mathbb{P}^N$ be defined, for $a = [a_0 : \cdots : a_n] \in \mathbb{P}^n$, by

$$\rho_d(a) = [M_0(a) : \cdots : M_N(a)].$$

Proposition 16. *The d-Uple embedding is well-defined and injective.*

Proof. Well-definedness is clear because M_i are all homogeneous of the same degree and at least one of $a_0^d, \ldots, a_n^d$ is necessarily nonzero. For injectivity, suppose $[M_0(a) : \cdots : M_N(a)] = [M_0(b) : \cdots : M_N(b)]$. Without loss of generality suppose $a_0^d = b_0^d \neq 0$. Then,

$$\frac{a_i}{a_0} = \frac{a_0^{d-1} a_i}{a_0^d} = \frac{b_0^{d-1} b_i}{b_0^d} = \frac{b_i}{b_0}$$

for all i, so $a = b$. □

Proposition 17. *The image of the d-Uple embedding is a subvariety of* $\mathbb{P}^N$.

Proof. Consider $\Theta\colon k[y_0,\dots,y_N] \to k[x_0,\dots,x_n]$ which maps y_i to M_i. Let $\mathfrak{b} = \ker\Theta$, a homogeneous ideal. We claim $\operatorname{Im}(\rho_d) = Z(\mathfrak{b})$.

For the easy direction ($\subset$), let $c = \rho_d(a) = [M_0(a),\dots,M_N(a)]$. Let $f \in \mathfrak{b}$. Then,

$$f(c) = f(\rho_d(a)) = \Theta(f)(a) = 0.$$

Thus, $c \in Z(\mathfrak{b})$.

For the hard direction ($\supset$), let $c \in Z(\mathfrak{b})$. Denote by $c[x_0^{i_0}\dots x_n^{i_n}] = c[x^I]$ the component of c corresponding to the monomial x^I, where $I = (i_0,\dots,i_n)$, $i_0 + \dots + i_n = d$.

We claim $c[x_k^d] \neq 0$ for some k. For any I, the polynomial $p \in k[y_0,\dots,y_N]$ defined by

$$p(y) = y[x^I]^d - y[x_0^d]^{i_0} \cdot \ldots y[x_n^d]^{i_n}.$$

We see

$$\Theta(p) = x_0^{i_0 d} \dots x_n^{i_n d} - x_0^{d i_0} \dots x_n^{d i_n} = 0.$$

Thus,

$$c[x^I]^d = c[x_0^d]^{i_0} \cdot \ldots c[x_n^d]^{i_n}.$$

Because $c \in \mathbb{P}^N$, there exists $J = (j_0,\dots,j_n)$ such that $c[x^J] \neq 0$. Because $j_k \neq 0$ for some k, there exists k such that $c[x_k^d] \neq 0$.

Without loss of generality let $k = 0$. Define $a \in \mathbb{P}^n$ by

$$a_i = c[x_0^{d-1} x_i].$$

It is clear that $a_0 \neq 0$ so $a \in \mathbb{P}^n$.

Let $b = \rho_d(a)$. We claim $b = c$. Indeed, we see

$$\frac{b[x_i^d]}{b[x_0^d]} = \frac{a_i^d}{a_0^d} = \frac{c[x_0^{d-1}x_i]^d}{c[x_0^d]^d} = \frac{c[x_0^d]^{d-1} c[x_i^d]}{c[x_0^d]^d} = \frac{c[x_i^d]}{c[x_0^d]}.$$

Further checks require a few more calculations which we leave as an exercise for a lazy sunday afternoon. □