(Affine)

Nullstellensatz (Zero Place Thm)

Ring, Ideal

Radical of Ideal

Z - zero set
I - ideal

} Inclusion reversing

Z(I) vs I(Z)

'Nullstellensatz

'Multidimensional Version of Fundamental Thm of Algebra'

ideal gen'd by single poly radical iff poly has no repeated roots

$$\begin{tikzcd} & (x-c)^n \arrow[dl, "Z"'] \arrow[d] \\ c \arrow[r, leftrightarrow, "I"'] & (x-c) \end{tikzcd}$$

in $\mathbb{C}[z]$

knowing zero set of f is enough to get radical

radical ideal $I \subset \mathbb{C}[z_1, \dots, z_n]$ is det'd by $Z(I)$

Pf of Null (Allcock w/help of K. Conrad)

essentially same as Zariski's

Jacobson / Hilbert rings are rings where this proof works

Examples of k-algebras (Vector space over a field)

complex numbers, VS is $\mathbb{R}^2$, bilinear product is $(a+bi)(c+di)$ $i^2=-1$

cross product of 3-vectors, VS is $\mathbb{R}^3$, $\vec{a} \times \vec{b}$

polys $\mathbb{R}[x]$ is VS, product is poly mult.

General algebra:

$f: A \to B$ ring homom

$a \in A$, $b \in B$: $a \cdot b = f(a)\, b$

$\downarrow$

gives A-module str to B

B called an A-algebra

i.e. B an A-algebra $\Rightarrow$ B a ring + ring hom $f: A \to B$

any ring is a $\mathbb{Z}$-algebra

finite if B is fin'ly gen'd

finite type if $\exists$ A-alg homom from $A[t_1, \dots, t_n] \to B$

k alg closed field (e.g. $\mathbb{C}$)

affine n-space over k $\mathbb{A}^n_k$ ($\mathbb{A}^n$)

set of all n-tuples of elts of k (i.e. k^n)

$P \in \mathbb{A}^n$ is a pt

$= (a_1, \dots, a_n)$, $a_i \in k$

coords of P

$A = k[x_1, \dots, x_n]$ is fncs $f : \mathbb{A}^n_k \to k$

$f(P) = f(a_1, \dots, a_n)$

$Z(f) = \{P \in \mathbb{A}^n : f(P) = 0\}$

zeroes of f

for $T \subset A$, zero set of T $Z(T) = \{P \in \mathbb{A}^n : f(P) \ \underline{\forall} f \in T\}$

α ideal gen'd by $T \Rightarrow Z(T) = Z(\alpha)$

A noetherian $\Rightarrow \alpha = (f_1, \dots, f_r)$ (α is what matters, not the f_i)

so $Z(T) = \{P \in \mathbb{A}^n : f_1(P) = \dots = f_r(P) = 0\}$

$Y \subset \mathbb{A}^n$ is algebraic if $\exists T \subseteq A$ s.t.

$Y = Z(T)$

Prop 1.1 Union of 2 alg sets is an alg set

Intersection of any family of alg sets

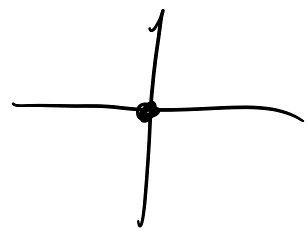
is an alg set. $\emptyset$ & whole space are alg sets

$$\emptyset = Z(1) \qquad \text{whole space} = Z(0)$$

Zariski Topology (open sets are complements of alg sets)

Not Hausdorff!

Union : $V(x) \cup V(y) = V(xy)$



Intersection : $V(x) \cap V(y)$

$$\bigcap Y_\alpha = \bigcap Z(T_\alpha) = Z\left(\bigcup T_\alpha\right)$$

let $P \in \bigcap Y_\alpha$

so $f_{\alpha_i}(P) = 0 \quad \forall f_{\alpha_i}$

$$Z\left(\bigcup T_\alpha\right) \neq \bigcup Z(T_\alpha) = \{P : f(P) = 0 \text{ for } f \text{ in some } T_\alpha\}$$

$\bigcup T_\alpha$ = union of all f in each T_α

= think of this as a collection of polys, a T'

$Y \subseteq X$ irreducible if it can't be expressed as union of 2 proper subsets,

$Y = Y_1 \cup Y_2$, each closed in Y

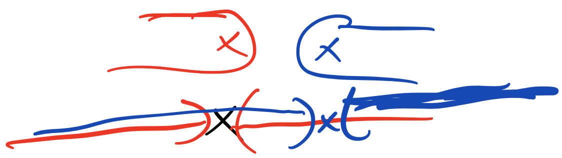
Eg. $\mathbb{A}^1$, $k[x]$ is PID so $Y = \underset{Z(f)}{\underset{\shortparallel}{Z(\alpha)}}$, $\alpha = (f)$

$f = c(x-a_1)\cdots(x-a_n)$ so

$Z(f) = \{a_1, \dots, a_n\}$

alg. sets are finite sets & $\mathbb{A}^1$ $\leftarrow f=0$

cofinite sets & $\emptyset$ are open (not Hausdorff)



$\mathbb{A}^1$ is irred (only proper closed subsets are finite)

Affine alg var. an irred closed subset of $\mathbb{A}^n$

open subset of affine is quasi-affine variety

For $Y \subset \mathbb{A}^n$, ideal of Y in $A = k[x_1, \dots, x_n]$

in $I(Y) = \{ f \in A \mid f(P) = 0 \ \forall P \in Y \}$

$$\left\{ \begin{array}{c} \text{subsets of} \\ A \end{array} \right\} \xrightarrow{Z} \{\text{alg sets}\}$$

$$\{\text{ideals}\} \xleftarrow{I} \left\{\begin{array}{c}\text{subsets of} \\ A^n\end{array}\right\}$$

Might expect inclusion-reversing correspondence

Null : This is between alg sets &
radical ideals

if $T_1 \subset T_2 \Rightarrow Z(T_1) \supseteq Z(T_2)$

if $Y_1 \subset Y_2 \Rightarrow I(Y_1) \supseteq I(Y_2)$

$$I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$$

$$Z\,I(Y) = \overline{Y}$$

$$I\,Z(\alpha) = \sqrt{\alpha}$$ Null

Pf starts here

Pf of Null :

Thm 1 $\longrightarrow$ Wk Null $\longrightarrow$ Null

(Mumford uses Noether Norm Lemma & Going-Up thm of Cohen-Seidenberg)

Thm 1: k a field, K field ext

fin. gen'd as k-alg. $\Rightarrow$ K is

algebraic over k (

every $\alpha \in K$ is root of some nonzero poly w/ coefficients in k

simple transcendental extension

idea of Pf: if k infinite & $K = k(x)$ ← not a root of any univariate poly w/ coeffs in k

then if $f_1, \dots, f_n \in K$, then k-Alg they gen is smaller than K (A)

Choose a c not one of the poles of the f_i,

then $\frac{1}{x-c} \in K \setminus A$ ✓

<u>Actual Pf:</u> Assume K is transcendental over k & fin gen'd as a k-alg

$\Rightarrow \neg$ ⤴ $\#$

Case 1: K has transcendence degree 1

i.e. contains subfield $k(x)$ a copy of rat'l function field,

K is alg. over $k(x)$

so since K is fin. gen'd, it has finite dim as a $k(x)$-VS.

Choose basis $e_1, \dots, e_\ell$,

$$\text{Then } e_i e_j = \sum_k \frac{a_{ij_k}(x)}{b_{ij_k}(x)} e_k$$

where a's, b's $\in k[x]$

Show for any $f_1, \dots, f_m \in K$, k-alg A they gen is smaller than K.

Add 1 as a generator
$\overset{\shortparallel}{f_0}$

Express f_i in terms of e_i:

$$f_i = \sum_j \frac{c_{ij}(x)}{d_{ij}(x)} e_j \quad , c\text{'s}, d\text{'s} \in k[x]$$

So $a \in A$ is k-lin comb. of $f_0 = 1$ & products of f_i's

expand in terms of basis so that a is k-lin comb of products of e_i w/ denoms ~~~~ involve only d's

mult rule for $e_i e_j$ now has denoms only involve b's & d's and have a $k(x)$-linear comb of e_i

i.e. a expressed in lowest terms as a

~~the~~ means all coeff's denom's irred factors

~~are~~ come from irred factors of b's & d's

Pick some other irred factor α

then $\frac{1}{\alpha} \notin A \Rightarrow A$ smaller than K

(Need infinitely many irred polys in $k[x]$

for k infinite use polys $(x-c)$, $c \in k$

else mimic Euclid's pf of infinitude of primes)

Case 2: Suppose K has trdeg > 1 over k.

Choose a subext k' where K has trdeg$=1$ over k'

From Case 1: K not fin gen'd as a k'-alg,

so can't be fin gen'd as a k-alg.

(Can build k' explicitly)

choose k-alg gens $x_1, \ldots, x_n$ for K over k,

$k = k(x_1, \ldots, x_{\ell-1})$

Last of x_i's transcendental over field gen'd by predecessors

Wk Nullstellensatz:

k alg. closed. Then every max'l ideal in poly ring $R = k[x_1, \ldots, x_n]$ has form $(x - a_1, \ldots, x - a_n)$ for some $a_1, \ldots, a_n \in k$.

Thus a family of poly functions on k^n w/ no common zeros generates unit ideal of R.

Pf: If m is max'l ideal of R

$\Rightarrow R/m$ is field finitely gen'd as a k-alg. (R is noetherian)

By prev. thm, R/m is alg over k.

$\Rightarrow R/m = k$ (alg. closed)

$\Rightarrow$ each $\underset{R}{x_i \in} \mapsto$ some $a_i \in k$ under $R \to R/m = k$

$\Rightarrow m \supset (x_1 - a_1, \ldots, x_n - a_n)$

this is maximal so $(x_1 - a_1, \ldots, x_n - a_n) = m$

Part 2: Consider ideal I gen'd by

some poly fncs w/ no common zeros

Then $I \not\subset (x_1 - a_1, \ldots, x_n - a_n)$

for some $(a_1, \ldots, a_n) \in k^n$

$\Rightarrow I = R = (1)$.

Nullstellensatz:

k alg. closed, $g, f_1, \ldots, f_m \in R = k[x_1, \ldots, x_n]$

regarded as poly fncs on k^n.

If g vanishes on common zero locus of f_i's

then $g \in \sqrt{(f_1, \ldots, f_m)}$ (i.e. $g^n \in (f_1, \ldots, f_m)$ for some $n > 0$)

$g \in I(Z((f_1, \ldots, f_m)))$

Pf: <u>Rabinowitsch trick.</u>

Extend to k^{n+1} & look at $f_1, \ldots, f_m, x_{n+1} g - 1$

These have no common zeros in k^{n+1}.

Wk Null : $\overset{\star}{1} = p_1 f_1 + \cdots + p_m f_m + p_{m+1}(x_{n+1} g - 1)$

p_i are polys in $x_1, \ldots, x_{n+1}$

$p_i(x_1, \ldots, x_{n+1})$

Take image of $\star$ under homom

$$k[x_1, \ldots, x_{n+1}] \to k(x_1, \ldots, x_n)$$
$$x_{n+1} \longmapsto 1/g$$

$\star$ $1 = p_1(x_1, \ldots, x_n, 1/g) f_1 + \cdots + p_m(x_1, \ldots, x_n, 1/g) f_m$

Multiply out by necessary power of g to clear out denoms

$g^n = b_1 f_1 + \cdots + b_m f_m$ ✓