Linear Systems Examples

Example 1 *(Basic Example - Single Solution)*

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 31 \\ 4x_1 + x_2 - 5x_3 &= -18 \\ x_1 - x_2 + x_3 &= 3 \end{aligned}$$

*The **Augmented Matrix** is given by*

$$\left[\begin{array}{rrr|r} 2 & 3 & 4 & 31 \\ 4 & 1 & -5 & -18 \\ 1 & -1 & 1 & 3 \end{array}\right]$$

We can use our three admissible operations to produce equivalent matrices.

$$\textbf{\textit{interchange row1 and row 3}} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 4 & 1 & -5 & -18 \\ 2 & 3 & 4 & 31 \end{array}\right]$$

$$\textbf{\textit{Row 2 subtract (4}} \times \textbf{\textit{row 1)}} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 0 & 5 & -9 & -30 \\ 2 & 3 & 4 & 31 \end{array}\right]$$

$$\textbf{\textit{Row 3 subtract (2}} \times \textbf{\textit{row 1)}} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 0 & 5 & -9 & -30 \\ 0 & 5 & 2 & 25 \end{array}\right]$$

We've now placed zeros below the first non-zero element in row 1. We call the 1 *in the* a_{11} *position a **pivot element** (a pivot is the first nonzero element in a row and requires that the leading entries in the rows below are located in columns to the right of this element). Our goal is to clear out all the element below each pivot. the next pivot is the* 5 *in row 2. Proceeding*

$$\textbf{\textit{Row 3 subtract row 2}} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 3 \\ 0 & 5 & -9 & -30 \\ 0 & 0 & 11 & 55 \end{array}\right]$$

Finally, our last pivot -11 in the third row has nothing below it, so we are now ready to back-solve. Rewriting back in equation form, we see the third row of our last matrix gives

$$11x_3 = 55 \ \textit{or} \ x = 5.$$

Using our second row yields

$$5x_2 - 9x_3 = -30, \ \textit{or} \ 5x_2 - 9(5) = -30, \ \textit{or} \ 5x_2 = 15 \Rightarrow x_2 = 3.$$

Finally, substituting into row 1 gives

$$x_1 - (3) + (5) = 3 \ \textit{or} \ x_1 = 1.$$

This gives the solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

Example 2 *(Important Example - Infinite Number of Solutions).*

$$\begin{aligned} 1x_1 + 2x_2 + 3x_3 + 4x_4 &= 10 \\ 2x_1 + 4x_2 + 6x_3 + 9x_4 &= 22 \\ 3x_1 + 6x_2 + 8x_3 + 11x_4 &= 29 \\ 4x_1 + 8x_2 + 11x_3 + 15x_4 &= 39 \end{aligned}$$

or equivalently in matrix notation

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 6 & 8 & 11 \\ 4 & 8 & 11 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 22 \\ 29 \\ 39 \end{bmatrix}.$$

The Augmented Matrix is given by

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 2 & 4 & 6 & 9 & 22 \\ 3 & 6 & 8 & 11 & 29 \\ 4 & 8 & 11 & 15 & 39 \end{array}\right].$$

*Row reducing gives (**see if you can get this**)*

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

*We have pivots in the first, third, and fourth columns. When the matrix is in **row echelon form (**has zeros below each pivot**),** we call the variables that correspond to columns that contain pivot elements **basic variables**. The variables corresponding to columns without pivots are called **free variables**. In the example above, x_1, x_3, x_4 are basic variables and $x_2$is a free variable. We find in the above that*

$$\begin{aligned} x_4 &= 2 \\ x_3 &= -1 \\ x_2 &= \textit{free} \\ x_1 &= 5 - 2x_2 \end{aligned}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 - 2x_2 \\ x_2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

*Note that since we are **free** to let x_3 take on any value (hence free variable), we have an infinite number of solutions.*

Example 3 *(Another important example - the Homogenous Case)*

$$\begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & 7 & 3 & 0 \\ 3 & 1 & -5 & 0 \\ 6 & 5 & -7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It is not necessary to use an augmented matrix as the last column will remain zero forever. However feel free to include it if you like. Using row reduction gives us (again., see if you can get this)

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This has one free variable, x_3*. We can write the solution as*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Being able to write are solutions in the form of the previous 2 examples is very important. It will be essential to do this in order to find ***eigenvectors****, which we will learn more about later.*