Solve the equation

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \vec{x}$$

using the eigenvectors

$$\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 8-\sqrt{2}i \\ -1-4\sqrt{2}i \\ 9+3\sqrt{2}i \end{bmatrix}$$

- Using

$$\vec{v} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

we get

$$A\vec{v} = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

and the solution is $\lambda = -2$, giving us our first solution

$$\vec{x}^{\,1}(t) = e^{\lambda t}\vec{v}_\lambda = e^{-2t} \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

- Using

$$\vec{v} = \begin{bmatrix} 8-\sqrt{2}i \\ -1-4\sqrt{2}i \\ 9+3\sqrt{2}i \end{bmatrix}$$

we get

$$A\vec{v} = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 8-\sqrt{2}i \\ -1-4\sqrt{2}i \\ 9+3\sqrt{2}i \end{bmatrix} = \begin{bmatrix} -6+9\sqrt{2}i \\ 9+3\sqrt{2}i \\ -15+6\sqrt{2}i \end{bmatrix} = \lambda \begin{bmatrix} 8-\sqrt{2}i \\ -1-4\sqrt{2}i \\ 9+3\sqrt{2}i \end{bmatrix}$$

and the solution is $\lambda = -1+\sqrt{2}i$, giving us our complex solution

$$\vec{z}(t) = e^{\left(-1+\sqrt{2}i\right)t} \begin{bmatrix} 8-\sqrt{2}i \\ -1-4\sqrt{2}i \\ 9+3\sqrt{2}i \end{bmatrix} = e^{-t}\left(\cos(\sqrt{2}\,t) + i\sin(\sqrt{2}\,t)\right)\left(\begin{bmatrix} 8 \\ -1 \\ 9 \end{bmatrix} + i\begin{bmatrix} -\sqrt{2} \\ -4\sqrt{2} \\ 3\sqrt{2} \end{bmatrix}\right)$$

$$= e^{-t} \begin{bmatrix} 8\cos(\sqrt{2}\,t) + \sqrt{2}\sin(\sqrt{2}\,t) \\ -\cos(\sqrt{2}\,t) + 4\sqrt{2}\sin(\sqrt{2}\,t) \\ 9\cos(\sqrt{2}\,t) - 3\sqrt{2}\sin(\sqrt{2}\,t) \end{bmatrix} + ie^{-t} \begin{bmatrix} -\sqrt{2}\cos(\sqrt{2}\,t) + 8\sin(\sqrt{2}\,t) \\ -4\sqrt{2}\cos(\sqrt{2}\,t) - \sin(\sqrt{2}\,t) \\ 3\sqrt{2}\cos(\sqrt{2}\,t) + 9\sin(\sqrt{2}\,t) \end{bmatrix}$$

giving us the two real solutions

$$\vec{x}^{\,2}(t) = e^{-t} \begin{bmatrix} 8\cos(\sqrt{2}\,t) + \sqrt{2}\sin(\sqrt{2}\,t) \\ -\cos(\sqrt{2}\,t) + 4\sqrt{2}\sin(\sqrt{2}\,t) \\ 9\cos(\sqrt{2}\,t) - 3\sqrt{2}\sin(\sqrt{2}\,t) \end{bmatrix} \quad \text{and} \quad \vec{x}^{\,3}(t) = e^{-t} \begin{bmatrix} -\sqrt{2}\cos(\sqrt{2}\,t) + 8\sin(\sqrt{2}\,t) \\ -4\sqrt{2}\cos(\sqrt{2}\,t) - \sin(\sqrt{2}\,t) \\ 3\sqrt{2}\cos(\sqrt{2}\,t) + 9\sin(\sqrt{2}\,t) \end{bmatrix}$$

- The general solution is

$$\vec{x}(t) = c_1 e^{-2t} \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} + c_2\, e^{-t} \begin{bmatrix} 8\cos(\sqrt{2}\,t) + \sqrt{2}\sin(\sqrt{2}\,t) \\ -\cos(\sqrt{2}\,t) + 4\sqrt{2}\sin(\sqrt{2}\,t) \\ 9\cos(\sqrt{2}\,t) - 3\sqrt{2}\sin(\sqrt{2}\,t) \end{bmatrix} + c_3\, e^{-t} \begin{bmatrix} -\sqrt{2}\cos(\sqrt{2}\,t) + 8\sin(\sqrt{2}\,t) \\ -4\sqrt{2}\cos(\sqrt{2}\,t) - \sin(\sqrt{2}\,t) \\ 3\sqrt{2}\cos(\sqrt{2}\,t) + 9\sin(\sqrt{2}\,t) \end{bmatrix}$$