Info for Exam 3

- Let $A \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. If there is a **NONZERO** vector $\vec{v} \in \mathbb{R}^n$ such that

$$A\vec{v} = \lambda\vec{v}$$

  then we say λ is an eigenvalue for A, and $\vec{v}$ is an eigenvector for A with eigenvalue λ. We also write $\vec{v} = \vec{v}_\lambda$.

- If $A \in M_n(\mathbb{R})$ has eigenvalue λ and eigenvector $\vec{v}_\lambda$, then

$$\vec{x}(t) = e^{\lambda t}\vec{v}_\lambda$$

  is a solution to the system

$$\frac{d}{dt}\vec{x} = A\vec{x}.$$

- If $A \in M_n(\mathbb{R})$, the characteristic polynomial of A is

$$p(\lambda) = \det(A - \lambda I_n).$$

  The roots of $p(\lambda)$ are the eigenvalues of A. The multiplicity of a root λ is denoted m_λ. For example, if A has characteristic polynomial

$$p(\lambda) = (\lambda - 2)^3(\lambda + 1)(\lambda + 5)^2$$

  then the eigenvalues of A are

$$\lambda = 2, -1, \text{ and } -5$$

  with multiplicities

$$m_2 = 3, m_{-1} = 1, \text{ and } m_{-5} = 2.$$

- If $A \in M_n(\mathbb{R})$ has eigenvalue λ, then the set of eigenvectors (together with the zero vector) is a subspace of $\mathbb{R}^n$ denoted E_λ:

$$E_\lambda = \text{Nul}(A - \lambda I)$$

  The standard basis of E_λ is the standard basis of $\text{Nul}(A - \lambda I)$.

- The real eigenvalue/eigenvector method in Braun, Section 3.8, for solving a system of equations

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

  involves finding the eigenvalues of A, then finding the standard basis for E_λ:

$$\{\vec{v}_\lambda^1, \vec{v}_\lambda^2, \ldots, \vec{v}_\lambda^r\}$$

  and creating the linear independent solutions

$$\vec{x}^1 = e^{\lambda t}\vec{v}_\lambda^1, \vec{x}^2 = e^{\lambda t}\vec{v}_\lambda^2, \ldots, \vec{x}^r = e^{\lambda t}\vec{v}_\lambda^r.$$

- If A has a complex eigenvalue, then the conjugate is also an eigenvalue, thus A has eigenvalues $\alpha \pm \beta i$. In this case, just use

$$\lambda = \alpha + \beta i \quad (\text{assuming } \beta > 0,)$$

  find the complex eigenvector $\vec{v}_{\alpha+\beta i}$ by row reducing $A - (\alpha + \beta i)I$ (there will only be one for us) and compute the complex solution

$$\vec{z}(t) = e^{(\alpha+\beta i)t}\vec{v}_{\alpha+\beta i} = \vec{x}^1(t) + i\vec{x}^2(t).$$

  Then the two real solutions to the system are $\vec{x}^1(t)$ and $\vec{x}^2(t)$.

- If $A \in M_n(\mathbb{R})$, we define the matrix exponential of A to be

$$e^{At} = I_n + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots$$

If λ is any scalar in $\mathbb{R}$, we can "center" e^{At} at λ:

$$e^{At} = e^{\lambda t}\left(I_n + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \frac{t^3}{3!}(A - \lambda I)^3 + \cdots\right)$$

In this class, we only do this when λ is an eigenvalue of A.

- If $A \in M_n(\mathbb{R})$ has eigenvalue λ with multiplicity $m_\lambda > 1$, and

$$\dim(E_\lambda) < m_\lambda$$

then more linearly independent solutions to the system

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

can be found by taking bases of $\text{Nul}(A - \lambda I)^k$ for $k > 1$.

1. λ is the **ONLY** eigenvalue of A.
   In this case, some power of $(A - \lambda I)$ is the zero matrix, and so e^{At} can be computed directly from the power series

$$e^{At} = e^{\lambda t}\left(I_n + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \frac{t^3}{3!}(A - \lambda I)^3 + \cdots\right)$$

   Then the general solution to

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

   is the linear combination of the columns of e^{At}:

$$\vec{x}(t) = e^{At} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

2. $\dim(E_\lambda) = 1$.
   In this case we find a Jordan Cycle. First, put the solution to

$$(A - \lambda I)\vec{J}_1 = \vec{v}_\lambda$$

   in parametric form and find $\vec{J}_1$ by setting the free variable to zero. (There will be only one free variable since $\dim(E_\lambda) = 1$.) Then

$$\vec{x}^2(t) = e^{\lambda t}\left(I_n + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \frac{t^3}{3!}(A - \lambda I)^3 + \cdots\right)\vec{J}_1.$$

   Further solutions can be found similarly be setting the one free variable equal to zero in the solutions to

$$(A - \lambda I)\vec{J}_{i+1} = \vec{J}_i$$

   and forming

$$\vec{x}^{i+2}(t) = e^{\lambda t}\left(I_n + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \frac{t^3}{3!}(A - \lambda I)^3 + \cdots\right)\vec{J}_{i+1}.$$

3. $\dim(E_\lambda) > 1$.
   For all other cases, see the handout on Generalized Eigenvectors and Arrow Diagrams.

- Having found n solutions, $\{\vec{x}^1, \vec{x}^2, \ldots, \vec{x}^n\}$, to

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

we want to make sure they give us the general solution, i.e. we want to make sure $\{\vec{x}^1, \vec{x}^2, \ldots, \vec{x}^n\}$ is a basis for V_A. To do this, we put the vectors in a matrix

$$\chi(t) = \begin{bmatrix} \vec{x}^1(t) & \vec{x}^2(t) & \ldots & \vec{x}^n(t) \end{bmatrix}$$

and check that

$$\det(\chi(0)) \neq 0.$$

Then $\{\vec{x}^1, \vec{x}^2, \ldots, \vec{x}^n\}$ is a basis for V_A and the general solution to the system is

$$\vec{x} = c_1\vec{x}^1 + c_2\vec{x}^2 + \cdots + c_n\vec{x}^n = \chi(t) \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

In this case, we say $\chi(t)$ is a fundamental matrix solution.

- An equivalent definition of a fundamental matrix solution is an $n \times n$ matrix $\chi(t)$ of functions such that

$$\frac{d}{dt}\chi(t) = A \cdot \chi(t) \quad \text{and} \quad \det(\chi(0)) \neq 0.$$

- The matrix exponential, e^{At}, is a fundamental matrix solution, since

$$\frac{d}{dt}e^{At} = A \cdot e^{At}$$

and

$$e^{A \cdot 0} = I_n \quad \Rightarrow \quad \det(e^{A \cdot 0}) = \det(I_n) = 1 \neq 0.$$

In fact, it is the case that any fundamental matrix $\chi(t)$ such that $\chi(0) = I_n$ must be the matrix exponential:

$$\frac{d}{dt}\chi(t) = A \cdot \chi(t) \quad \text{and} \quad \chi(0) = I_n \quad \Rightarrow \quad \chi(t) = e^{At}.$$

- This gives us the following method for computing e^{At} for any $A \in M_n(\mathbb{R})$:

  1. Find a basis $\{\vec{x}^1, \vec{x}^2, \ldots, \vec{x}^n\}$ for V_A.
  2. Construct the fundamental matrix solution

$$\chi(t) = \begin{bmatrix} \vec{x}^1(t) & \vec{x}^2(t) & \ldots & \vec{x}^n(t) \end{bmatrix}$$

  3. Compute $(\chi(0))^{-1}$, the inverse.
  4. Finally,

$$e^{At} = \chi(t) \cdot (\chi(0))^{-1}$$

- The columns of e^{At} are the standard basis of V_A.
- To solve an I.V.P.

$$\frac{d}{dt} e^{At} = A \cdot e^{At} \qquad \vec{x}(0) = \vec{b}$$

first solve the O.D.E. to get a fundamental matrix solution $\chi(t)$, then row reduce the augmented matrix

$$\left[\begin{array}{c|c} \chi(0) & \vec{b} \end{array} \right] \to \cdots \to \left[\begin{array}{c|c} I_n & \vec{c} \end{array} \right].$$

The vector

$$\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the vector of coefficients and the solution is

$$\vec{x} = c_1 \vec{x}^1 + c_2 \vec{x}^2 + \cdots + c_n \vec{x}^n = \chi(t) \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

Note that if $\chi(t) = e^{At}$, then

$$\left[\begin{array}{c|c} \chi(0) & \vec{b} \end{array} \right] = \left[\begin{array}{c|c} I_n & \vec{c} \end{array} \right]$$

thus $\vec{b} = \vec{c}$ and the solution is

$$\vec{x} = b_1 \vec{x}^1 + b_2 \vec{x}^2 + \cdots + b_n \vec{x}^n = \chi(t) \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

- If $A \in M_n(\mathbb{R})$, then we can recover A from e^{At} using the following formula:

$$\left(\frac{d}{dt} e^{At} \right) \Bigg|_{t=0} = A$$