**Linear Systems Application and Notes**

If you spray a crop with pesticide, the pesticide on the plant will seep into the surrounding dirt, and the pesticide on the dirt will respirate back up into the crop. Let $x_1(t)$ be the pesticide on the plant at time $t$, and $x_2(t)$ be the pesticide in the dirt at time $t$. If 1/2 the pesticide on the plant seeps into the dirt, and 1/2 the pesticide in the dirt respirates up to the plant, we get the following system:

$$\begin{aligned} x_1' &= -\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 \\ x_2' &= \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 \end{aligned}$$

which converts to the system

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \vec{x} = A\,\vec{x}$$

Let's look at some solutions:

- $$\frac{d}{dt}\vec{x} = A\,\vec{x}\,, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

  has solution

  $$\vec{x}^{\,1}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

  a constant solution.

- $$\frac{d}{dt}\vec{x} = A\,\vec{x}\,, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

  has solution

  $$\vec{x}^{\,2}(t) = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{-t} \\ \\ \frac{1}{2} - \frac{1}{2}e^{-t} \end{bmatrix}.$$

  Seen here: https://www.desmos.com/calculator/btpfymuu1e

- $$\frac{d}{dt}\vec{x} = A\,\vec{x}\,, \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  has solution

  $$\vec{x}^{\,3}(t) = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}e^{-t} \\ \\ \frac{1}{2} + \frac{1}{2}e^{-t} \end{bmatrix}.$$

  Seen here: https://www.desmos.com/calculator/aflmpbnymf

- Note that

  $$\vec{x}^{\,1}(t) = \vec{x}^{\,2}(t) + \vec{x}^{\,3}(t).$$

If we want to refine our model to include dissipation of the pesticide via evaporation into the air, then we will need to include a third quantity, $x_3(t)$, the quantity of pesticide in the air at time $t$. We will assume 1/3 of the pesticide in the ground evaporates into the air, and 1/3 respirates up into the plant. Similarly for the amount of pesticide on the plant.

$$\begin{aligned} x_1' &= -\tfrac{2}{3}x_1 + \tfrac{1}{3}x_2 \\ x_2' &= \tfrac{1}{3}x_1 - \tfrac{2}{3}x_2 \\ x_3' &= \tfrac{1}{3}x_1 + \tfrac{1}{3}x_2 - 2x_3 \end{aligned}$$

which converts to the system

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} -2/3 & 1/3 & 0 \\ 1/3 & -2/3 & 0 \\ 1/3 & 1/3 & -2 \end{bmatrix} \vec{x} = A\,\vec{x}$$

Let's look at some solutions:

- $$\frac{d}{dt}\vec{x} = A\,\vec{x}\ , \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  has solution

  $$\vec{x}^{\,1}(t) = \begin{bmatrix} 0 \\ 0 \\ e^{-2t} \end{bmatrix}.$$

- $$\frac{d}{dt}\vec{x} = A\,\vec{x}\ , \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

  has solution

  $$\vec{x}^{\,2}(t) = \begin{bmatrix} \frac{1}{2}e^{-1/3t} + \frac{1}{2}e^{-t} \\ \frac{1}{2}e^{-1/3t} - \frac{1}{2}e^{-t} \\ \frac{1}{5}e^{-1/3t} - \frac{1}{5}e^{-2t} \end{bmatrix}.$$

  Seen here: https://www.desmos.com/calculator/bkej1o5oqo

- $$\frac{d}{dt}\vec{x} = A\,\vec{x}\ , \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

  has solution

  $$\vec{x}^{\,3}(t) = \begin{bmatrix} \frac{1}{2}e^{-1/3t} - \frac{1}{2}e^{-t} \\ \frac{1}{2}e^{-1/3t} + \frac{1}{2}e^{-t} \\ \frac{1}{5}e^{-1/3t} - \frac{1}{5}e^{-2t} \end{bmatrix}.$$