Power Series Solutions
Solve

$$L[y] = y'' - 2ty' - 2y = 0$$

Solution:
Let

$$y(t) \quad = \quad \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots$$

then

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} a_n t^n &&= a_0 + a_1 t + a_2 t^2 + \cdots \\ y'(t) &= \sum_{n=1}^{\infty} n a_n t^{n-1} &&= a_1 + 2 \cdot a_2 t + 3 \cdot a_3 t^2 + \cdots \\ y''(t) &= \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} &&= 2 \cdot a_2 + 3 \cdot 2 \cdot a_3 t + 4 \cdot 3 \cdot 2 \cdot a_4 t^2 + \cdots \end{aligned}$$

Let's write these vertically, for fun:

$$y(t) = \begin{pmatrix} & \vdots \\ + & (a_n)\, t^n \\ & \vdots \\ + & (a_4)\, t^4 \\ + & (a_3)\, t^3 \\ + & (a_2)\, t^2 \\ + & (a_1)\, t \\ + & (a_0) \end{pmatrix}, \quad y'(t) = \begin{pmatrix} & \vdots \\ + & (n \cdot a_n)\, t^{n-1} \\ & \vdots \\ + & (4 \cdot a_4)\, t^3 \\ + & (3 \cdot a_3)\, t^2 \\ + & (2 \cdot a_2)\, t \\ + & (a_1) \\ & \end{pmatrix}, \quad y''(t) = \begin{pmatrix} & \vdots \\ + & (n \cdot (n-1) \cdot a_n)\, t^{n-2} \\ & \vdots \\ + & (4 \cdot 3 \cdot a_4)\, t^2 \\ + & (3 \cdot 2 \cdot a_3)\, t \\ + & (2 \cdot a_2) \\ & \\ & \end{pmatrix}$$

So

$$L[y] = y'' - 2ty' - 2y =$$

$$= \begin{pmatrix} & \vdots \\ + & (n \cdot (n-1) \cdot a_n)\, t^{n-2} \\ & \vdots \\ + & (4 \cdot 3 \cdot a_4)\, t^2 \\ + & (3 \cdot 2 \cdot a_3)\, t \\ + & (2 \cdot a_2) \\ & \\ & \end{pmatrix} - 2\,t \begin{pmatrix} & \vdots \\ + & (n \cdot a_n)\, t^{n-1} \\ & \vdots \\ + & (4 \cdot a_4)\, t^3 \\ + & (3 \cdot a_3)\, t^2 \\ + & (2 \cdot a_2)\, t \\ + & (a_1) \\ & \end{pmatrix} - 2 \begin{pmatrix} & \vdots \\ + & (a_n)\, t^n \\ & \vdots \\ + & (a_4)\, t^4 \\ + & (a_3)\, t^3 \\ + & (a_2)\, t^2 \\ + & (a_1)\, t \\ + & (a_0) \end{pmatrix}$$

$$= \left(\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} \right) \quad - 2\,t \quad \left(\sum_{n=1}^{\infty} n a_n t^{n-1} \right) \quad - 2 \quad \left(\sum_{n=0}^{\infty} a_n t^n \right)$$

$$= \begin{pmatrix} & \vdots \\ + & (n\cdot(n-1)\cdot a_n)\,t^{n-2} \\ & \vdots \\ + & (4\cdot 3\cdot a_4)\,t^2 \\ + & (3\cdot 2\cdot a_3)\,t \\ + & (2\cdot a_2) \\ & \end{pmatrix} + \begin{pmatrix} & \vdots \\ + & (-2\cdot n\cdot a_n)\,t^n \\ & \vdots \\ + & (-2\cdot 4\cdot a_4)\,t^4 \\ + & (-2\cdot 3\cdot a_3)\,t^3 \\ + & (-2\cdot 2\cdot a_2)\,t^2 \\ + & (-2\cdot a_1)\,t \\ & \end{pmatrix} + \begin{pmatrix} & \vdots \\ + & (-2\cdot a_n)\,t^n \\ & \vdots \\ + & (-2\cdot a_4)\,t^4 \\ + & (-2\cdot a_3)\,t^3 \\ + & (-2\cdot a_2)\,t^2 \\ + & (-2\cdot a_1)\,t \\ + & (-2\cdot a_0) \end{pmatrix}$$

$$= \left(\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}\right) + \left(\sum_{n=1}^{\infty}(-2)n a_n t^n\right) + \left(\sum_{n=0}^{\infty}(-2)a_n t^n\right)$$

$$= \begin{pmatrix} & \vdots \\ + & \textbf{?????} \\ & \vdots \\ + & (n\cdot(n-1)\cdot a_n)\,t^{n-2} \\ & \vdots \\ + & (4\cdot 3\cdot a_4)\,t^2 \\ + & (3\cdot 2\cdot a_3)\,t \\ + & (2\cdot a_2) \end{pmatrix} + \begin{pmatrix} & \vdots \\ + & (-2\cdot n\cdot a_n)\,t^n \\ & \vdots \\ + & (-2\cdot 4\cdot a_4)\,t^4 \\ + & (-2\cdot 3\cdot a_3)\,t^3 \\ + & (-2\cdot 2\cdot a_2)\,t^2 \\ + & (-2\cdot a_1)\,t \\ & \end{pmatrix} + \begin{pmatrix} & \vdots \\ + & (-2\cdot a_n)\,t^n \\ & \vdots \\ + & (-2\cdot a_4)\,t^4 \\ + & (-2\cdot a_3)\,t^3 \\ + & (-2\cdot a_2)\,t^2 \\ + & (-2\cdot a_1)\,t \\ + & (-2\cdot a_0) \end{pmatrix}$$

$$= \left(\sum_{n=0}^{\infty} \textbf{?????}\right) + \left(\sum_{n=1}^{\infty}(-2)n a_n t^n\right) + \left(\sum_{n=0}^{\infty}(-2)a_n t^n\right)$$

$$= \begin{pmatrix} & \vdots \\ + & ((n+2)(n+1)a_{n+2})\,t^n \\ & \vdots \\ + & (n\cdot(n-1)\cdot a_n)\,t^{n-2} \\ & \vdots \\ + & (4\cdot 3\cdot a_4)\,t^2 \\ + & (3\cdot 2\cdot a_3)\,t \\ + & (2\cdot a_2) \end{pmatrix} + \begin{pmatrix} & \vdots \\ + & (-2\cdot n\cdot a_n)\,t^n \\ & \vdots \\ + & (-2\cdot 4\cdot a_4)\,t^4 \\ + & (-2\cdot 3\cdot a_3)\,t^3 \\ + & (-2\cdot 2\cdot a_2)\,t^2 \\ + & (-2\cdot a_1)\,t \\ & \end{pmatrix} + \begin{pmatrix} & \vdots \\ + & (-2\cdot a_n)\,t^n \\ & \vdots \\ + & (-2\cdot a_4)\,t^4 \\ + & (-2\cdot a_3)\,t^3 \\ + & (-2\cdot a_2)\,t^2 \\ + & (-2\cdot a_1)\,t \\ + & (-2\cdot a_0) \end{pmatrix}$$

$$= \left(\sum_{n=0}^{\infty} ((n+2)(n+1)a_{n+2})\,t^n\right) + \left(\sum_{n=1}^{\infty}(-2)n a_n t^n\right) + \left(\sum_{n=0}^{\infty}(-2)a_n t^n\right)$$

$$
= \begin{pmatrix} & \vdots \\ + & ((n+2)(n+1)a_{n+2})\,t^n \\ & \vdots \\ + & (6\cdot 5\cdot a_6)\,t^4 \\ + & (5\cdot 4\cdot a_5)\,t^3 \\ + & (4\cdot 3\cdot a_4)\,t^2 \\ + & (3\cdot 2\cdot a_3)\,t \\ \cdots & \cdots\cdots\cdots \\ + & (2\cdot a_2) \end{pmatrix} + \begin{pmatrix} & \vdots \\ + & (-2\cdot n\cdot a_n)\,t^n \\ & \vdots \\ + & (-2\cdot 4\cdot a_4)\,t^4 \\ + & (-2\cdot 3\cdot a_3)\,t^3 \\ + & (-2\cdot 2\cdot a_2)\,t^2 \\ + & (-2\cdot a_1)\,t \\ \cdots & \cdots\cdots\cdots \\ & \end{pmatrix} + \begin{pmatrix} & \vdots \\ + & (-2\cdot a_n)\,t^n \\ & \vdots \\ + & (-2\cdot a_4)\,t^4 \\ + & (-2\cdot a_3)\,t^3 \\ + & (-2\cdot a_2)\,t^2 \\ + & (-2\cdot a_1)\,t \\ \cdots & \cdots\cdots\cdots \\ + & (-2\cdot a_0) \end{pmatrix}
$$

$$
= \begin{pmatrix} & \vdots \\ + & ((n+2)(n+1)a_{n+2})\,t^n \\ & \vdots \\ + & (6\cdot 5\cdot a_6)\,t^4 \\ + & (5\cdot 4\cdot a_5)\,t^3 \\ + & (4\cdot 3\cdot a_4)\,t^2 \\ + & (3\cdot 2\cdot a_3)\,t \end{pmatrix} + \begin{pmatrix} & \vdots \\ + & (-2\cdot n\cdot a_n)\,t^n \\ & \vdots \\ + & (-2\cdot 4\cdot a_4)\,t^4 \\ + & (-2\cdot 3\cdot a_3)\,t^3 \\ + & (-2\cdot 2\cdot a_2)\,t^2 \\ + & (-2\cdot a_1)\,t \end{pmatrix} + \begin{pmatrix} & \vdots \\ + & (-2\cdot a_n)\,t^n \\ & \vdots \\ + & (-2\cdot a_4)\,t^4 \\ + & (-2\cdot a_3)\,t^3 \\ + & (-2\cdot a_2)\,t^2 \\ + & (-2\cdot a_1)\,t \end{pmatrix}
$$

$$
+ \quad (2\cdot a_2) \quad + \quad (-2\cdot a_0)
$$

$$
= \left(\sum_{n=1}^{\infty}((n+2)(n+1)a_{n+2})\,t^n\right) + \left(\sum_{n=1}^{\infty}(-2)na_nt^n\right) + \left(\sum_{n=1}^{\infty}(-2)a_nt^n\right) + \left[2a_2 - 2a_0\right]
$$

$$
= \begin{pmatrix} & \vdots \\ + & \Big[(n+2)(n+1)a_{n+2} + (-2)na_n + (-2)a_n\Big]t^n \\ & \vdots \\ + & \Big[(5\cdot 4\cdot a_5) + (-2\cdot 3\cdot a_3) + (-2\cdot a_3)\Big]t^3 \\ + & \Big[(4\cdot 3\cdot a_4) + (-2\cdot 2\cdot a_2) + (-2\cdot a_2)\Big]t^2 \\ + & \Big[(3\cdot 2\cdot a_3) + (-2\cdot a_1) + (-2\cdot a_1)\Big]t \end{pmatrix} + \left[2a_2 - 2a_0\right]
$$

$$
= \left(\sum_{n=1}^{\infty}\Big[(n+2)(n+1)a_{n+2} + (-2)na_n + (-2)a_n\Big]t^n\right) + \left[2a_2 - 2a_0\right]
$$

$$
= 0
$$

$$\Rightarrow \qquad \begin{cases} (n+2)(n+1)a_{n+2} + (-2)na_n + (-2)a_n = 0, & \text{for } n = 1, 2, 3, ... \\ 2a_2 - 2a_0 = 0 \end{cases}$$

NOTE: The equation

$$(n+2)\,a_{n+2} - 2a_n = 0, \qquad \text{for } n = 1, 2, 3, ...$$

is called the *recurrence relation* or the *recursion relation*.

$$\Rightarrow \qquad \begin{cases} (n+2)\,a_{n+2} - 2a_n = 0, & \text{for } n = 1, 2, 3, ... \\ \\ 2a_2 - 2a_0 = 0 \end{cases}$$

So

$$\begin{aligned} a_2 &= a_0 \\ a_3 &= \frac{2}{3}a_1 \\ a_4 &= \frac{2}{4}a_2 = \frac{2}{4}a_0 = \frac{1}{2}a_0 \\ a_5 &= \frac{2}{5}a_3 = \frac{2}{5}\cdot\frac{2}{3}a_1 = \frac{4}{15}a_1 \\ a_6 &= \frac{2}{6}a_4 = \frac{2}{6}\cdot\frac{1}{2}a_0 = \frac{1}{6}a_0 \\ &\vdots \end{aligned}$$

gives

$$\begin{aligned} y(t) &= a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5 + a_6t^6 + \cdots \\ &= a_0 + a_1t + a_0t^2 + \frac{2}{3}a_1t^3 + \frac{1}{2}a_0t^4 + \frac{4}{15}a_1t^5 + \frac{1}{6}a_0t^6 + \cdots \\ &= a_0\left(1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \cdots\right) + a_1\left(t + \frac{2}{3}t^3 + \frac{4}{15}t^5 + \cdots\right) \end{aligned}$$

So

$$\begin{aligned} y_1(t) &= 1 + t^2 + \frac{1}{2}t^4 + \frac{1}{6}t^6 + \cdots = e^{t^2} \\ y_2(t) &= t + \frac{2}{3}t^3 + \frac{4}{15}t^5 + \cdots \end{aligned}$$