Solution to Braun, 3.8, Number 3
Solve the equation

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \vec{x}$$

- The characteristic polynomial of A is

$$p(\lambda) = -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = -(\lambda - 8)(\lambda + 1)^2$$

- The eigenvalues are $\lambda = 8$ (with multiplicity 1) and $\lambda = -1$ with multiplicity 2.
- For $\lambda = 8$, we find the eigenspace $E_\lambda = E_8 = \ker(A - 8I_3)$:

$$\text{RREF}(A - 8I_3) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

gives us equations

$$\begin{array}{rcl} x - z & = & 0 \\ y - \frac{1}{2}z & = & 0 \\ z \text{ is free} & & \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x & = & z \\ y & = & \frac{1}{2}z \\ z & = & z \end{array}$$

so the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 1/2z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$$

so

$$\vec{v}_\lambda = \vec{v}_8 = \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} \text{ and } E_8 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} \right\}$$

and

$$\vec{x}^1(t) = e^{\lambda t}\vec{v}_\lambda = e^{8t} \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$$

- For $\lambda = -1$, we find the eigenspace $E_\lambda = E_{-1} = \ker(A + I_3)$:

$$\text{RREF}(A + I_3) = \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

gives us equations

$$\begin{array}{rcl} x + \frac{1}{2}y + z & = & 0 \\ y \text{ is free} & & \\ z \text{ is free} & & \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x & = & -\frac{1}{2}y - z \\ y & = & y \\ z & = & z \end{array}$$

so the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/2y - z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

so

$$\vec{v}_{-1}^{\,1} = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_{-1}^{\,2} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and } E_{-1} = \text{span}\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$\vec{x}^{\,2}(t) = e^{-t} \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{x}^{\,3}(t) = e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- The general solution is

$$\vec{x}(t) = c_1 e^{8t} \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Suppose we also wanted to solve the I.V.P.

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

First, we would find the general solution, as above:

$$\vec{x}(t) = c_1 e^{8t} \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$\vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & -1 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

so we row reduce

$$\begin{bmatrix} 1 & -1/2 & -1 & 3 \\ 1/2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

to get

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

which tells us $c_1 = 2, c_2 = 0$, and $c_3 = -1$, thus the unique solution to the I.V.P. is

$$\vec{x}(t) = 2e^{8t} \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix} - e^{-t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$