Solution to Braun, 3.10, Example 2
Solve the equation

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

First, we will solve using a Jordan Cycle

- The characteristic polynomial of A is

$$p(\lambda) = -(\lambda - 2)^3$$

- The only eigenvalue is $\lambda = 2$ with multiplicity 3.
- For $\lambda = 2$, we find the eigenspace $E_\lambda = E_2 = \ker(A - 2I_3)$:

$$A - 2I_3 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \text{RREF}(A - 2I_3) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \vec{x}^{\,1}(t) = e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- To find more solutions, we must find generalized eigenvectors with eigenvalue 2. We do this by finding vectors in a Jordan Cycle.

  The next step is to find a generalized eigenvector, $\vec{J}_1$, such that

$$(A - 2I_3)\,\vec{J}_1 = \vec{v}_2$$

  So we are solving the non homogeneous system

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \vec{J}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

  We row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{to get} \quad \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

  Now write the equations

$$\begin{array}{rcl} x \text{ is free} & & \\ y & = & 1 \\ z & = & 0 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x & = & x \\ y & = & 1 \\ z & = & 0 \end{array}$$

  then write the solution in parametric form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and finally we obtain $\vec{J}_1$ by setting the free variable equal to 0:

$$\vec{J}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Now our second solution is

$$\vec{x}^{\,2}(t) = e^{A\,t}\vec{J}_1 = e^{2t}e^{(A-2I)t}\vec{J}_1 =$$

$$= e^{2t}\left(I_3 + t(A-2I_3) + \frac{t^2}{2}(A-2I_3)^2 + \cdots\right)\vec{J}_1$$

To distribute $\vec{J}_1$, we calculate

$$\begin{aligned} I_3\vec{J}_1 &= \vec{J}_1 \\ (A-2I_3)\vec{J}_1 &= \vec{v}_2 \quad \text{(that's how we found } \vec{J}_1\text{)} \\ (A-2I_3)^2\vec{J}_1 &= (A-2I_3)(A-2I_3)\vec{J}_1 = (A-2I_3)\vec{v}_2 = \vec{0} \\ (A-2I_3)^3\vec{J}_1 &= (A-2I_3)^2(A-2I_3)\vec{J}_1 = (A-2I_3)^2\vec{v}_2 = \vec{0} \\ (A-2I_3)^4\vec{J}_1 &= (A-2I_3)^3(A-2I_3)\vec{J}_1 = (A-2I_3)^3\vec{v}_2 = \vec{0} \\ &\vdots \end{aligned}$$

Thus

$$\vec{x}^{\,2}(t) = e^{2t}\left(\vec{J}_1 + t\,\vec{v}_2\right) = e^{2t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = e^{2t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$$

- The last step is to find a generalized eigenvector, $\vec{J}_2$, such that

$$(A-2I_3)\,\vec{J}_2 = \vec{J}_1$$

So we are solving the non homogeneous system

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\vec{J}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{to get} \quad \left[\begin{array}{ccc|c} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now write the equations

$$\begin{array}{rcl} x \text{ is free} & & \\ y &=& 3 \\ z &=& -1 \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x &=& x \\ y &=& 3 \\ z &=& -1 \end{array}$$

then write the solution in parametric form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} + x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and finally we obtain $\vec{J}_2$ by setting the free variable equal to 0:

$$\vec{J}_2 = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

Now our third solution is

$$\vec{x}^{\,3}(t) = e^{A\,t}\vec{J}_2 = e^{2t}e^{(A-2I)t}\vec{J}_2 =$$

$$= e^{2t}\left(I_3 + t(A-2I_3) + \frac{t^2}{2}(A-2I_3)^2 + \cdots\right)\vec{J}_2$$

To distribute $\vec{J}_2$, we calculate

$$\begin{array}{rcl}
I_3\vec{J}_2 & = & \vec{J}_2 \\
(A-2I_3)\vec{J}_2 & = & \vec{J}_1 \quad \text{(that's how we found } \vec{J}_2\text{)} \\
(A-2I_3)^2\vec{J}_2 & = & (A-2I_3)(A-2I_3)\vec{J}_2 = (A-2I_3)\vec{J}_1 = \vec{v}_2 \\
(A-2I_3)^3\vec{J}_2 & = & (A-2I_3)(A-2I_3)(A-2I_3)\vec{J}_2 = (A-2I_3)(A-2I_3)\vec{J}_1 \\
 & = & (A-2I_3)\vec{v}_2 = \vec{0} \\
(A-2I_3)^4\vec{J}_2 & = & (A-2I_3)^2(A-2I_3)(A-2I_3)\vec{J}_2 = (A-2I_3)^2(A-2I_3)\vec{J}_1 \\
 & = & (A-2I_3)^2\vec{v}_2 = \vec{0} \\
 & \vdots &
\end{array}$$

Thus

$$\begin{array}{rcl}
\vec{x}^{\,3}(t) & = & e^{2t}\left(\vec{J}_2 + t\,\vec{J}_1 + \tfrac{t^2}{2}\vec{v}_2\right) \\
 & = & e^{2t}\left(\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \tfrac{t^2}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) \\
 & = & e^{2t}\begin{bmatrix} t^2/2 \\ 3+t \\ -1 \end{bmatrix}
\end{array}$$

- The general solution is

$$\vec{x}(t) = c_1e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2e^{2t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3e^{2t}\begin{bmatrix} t^2/2 \\ 3+t \\ -1 \end{bmatrix}$$

Second, we will solve by calculating the matrix exponential

- We know by the Caley Hamilton Theorem that a matrix A satisfies its own characteristic polynomial. For our matrix A, the characteristic polynomial is

$$p(\lambda) = -(\lambda - 2)^3$$

thus we know

$$-\left(A - 2I_3\right)^3 = 0$$

so we know

$$\begin{aligned} e^{At} &= e^{2t}\left(I_3 + t\left(A - 2I_3\right) + \tfrac{t^2}{2}\left(A - 2I_3\right)^2 + \tfrac{t^2}{2}\left(A - 2I_3\right)^3 + \cdots\right) \\ &= e^{2t}\left(I_3 + t\left(A - 2I_3\right) + \tfrac{t^2}{2}\left(A - 2I_3\right)^2\right) \end{aligned}$$

- We compute the nonzero powers of $A - 2I_3$:

$$A - 2I_3 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - 2I_3)^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Now

$$\begin{aligned} e^{At} &= e^{2t}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} + \tfrac{t^2}{2}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \\ &= e^{2t}\begin{bmatrix} 1 & t & 3t - \frac{t^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

- The general solution is the span of the columns of e^{At}:

$$\vec{x}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 3t - \frac{t^2}{2} \\ -t \\ 1 \end{bmatrix}$$