Solution to Braun, 3.10, Exercise 5
Solve the equation

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ -2 & 1 & 3 \end{bmatrix} \vec{x}$$

First, we will solve using a Jordan Cycle

- The characteristic polynomial of A is
$$p(\lambda) = -(\lambda - 1)^3$$
- The only eigenvalue is $\lambda = 1$ with multiplicity 3.
- For $\lambda = 1$, we find the eigenspace $E_\lambda = E_1 = \ker(A - I_3)$:
$$A - I_3 = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \text{RREF}(A - I_3) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
so
$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \vec{x}^1(t) = e^t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
- To find more solutions, we must find generalized eigenvectors with eigenvalue 1. We do this by finding vectors in a Jordan Cycle.

  The next step is to find a generalized eigenvector, $\vec{J}_1$, such that
$$(A - I_3)\,\vec{J}_1 = \vec{v}_1$$
  So we are solving the non homogeneous system
$$\begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} \vec{J}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
  We row reduce the augmented matrix
$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 1 \\ -1 & 0 & 1 & 0 \\ -2 & 1 & 2 & 1 \end{array}\right] \quad \text{to get} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
  Now write the equations
$$\begin{array}{rcl} x - z & = & 0 \\ y & = & 1 \\ z \text{ is free} & & \end{array} \quad \Rightarrow \quad \begin{array}{rcl} x & = & z \\ y & = & 1 \\ z & = & z \end{array}$$
  then write the solution in parametric form
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 1 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and we obtain $\vec{J}_1$ by setting the free variable equal to 0:

$$\vec{J}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Now our second solution is

$$\begin{aligned} \vec{x}^{\,2}(t) &= e^{A\,t}\vec{J}_1 = e^t e^{(A-I)t}\,\vec{J}_1 \\ &= e^t\left(I_3 + t(A-I_3) + \tfrac{t^2}{2}(A-I_3)^2 + \cdots\right)\vec{J}_1 \\ &= e^t\left(\vec{J}_1 + t\,\vec{v}_1\right) \\ &= e^t\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = e^t\begin{bmatrix} t \\ 1 \\ t \end{bmatrix} \end{aligned}$$

- The last step is to find a generalized eigenvector, $\vec{J}_2$, such that

$$(A - I_3)\,\vec{J}_2 = \vec{J}_1$$

So we are solving the non homogeneous system

$$\begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}\vec{J}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We row reduce the augmented matrix

$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ -1 & 0 & 1 & 1 \\ -2 & 1 & 2 & 0 \end{array}\right] \quad \text{to get} \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and we obtain $\vec{J}_2$:

$$\vec{J}_2 = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

Now our third solution is

$$\begin{aligned} \vec{x}^{\,3}(t) &= e^{A\,t}\vec{J}_2 = e^t e^{(A-I)t}\,\vec{J}_2 \\ &= e^t\left(I_3 + t(A-I_3) + \tfrac{t^2}{2}(A-I_3)^2 + \cdots\right)\vec{J}_2 \\ &= e^t\left(\vec{J}_2 + t\,\vec{J}_1 + \tfrac{t^2}{2}\vec{v}_1\right) \\ &= e^t\left(\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \tfrac{t^2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = e^t\begin{bmatrix} -1+\frac{t^2}{2} \\ -2+t \\ \frac{t^2}{2} \end{bmatrix} \end{aligned}$$

- The general solution is

$$\vec{x}(t) = c_1\,e^t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2\,e^t\begin{bmatrix} t \\ 1 \\ t \end{bmatrix} + c_3\,e^t\begin{bmatrix} -1+\frac{t^2}{2} \\ -2+t \\ \frac{t^2}{2} \end{bmatrix}$$

Second, we will solve by calculating the matrix exponential

- We know by the Caley Hamilton Theorem that a matrix A satisfies its own characteristic polynomial. For our matrix A, the characteristic polynomial is
$$p(\lambda) = -(\lambda - 1)^3$$
thus we know
$$-\left(A - I_3\right)^3 = 0$$
so we know
$$\begin{aligned} e^{At} &= e^t \left(I_3 + t\,(A - I_3) + \tfrac{t^2}{2}\,(A - I_3)^2 + \tfrac{t^3}{3!}\,(A - I_3)^3 + \cdots \right) \\ &= e^t \left(I_3 + t\,(A - I_3) + \tfrac{t^2}{2}\,(A - I_3)^2 \right) \end{aligned}$$

- We compute the nonzero powers of $A - I_3$:
$$A - I_3 = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$
$$(A - I_3)^2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

- Now
$$\begin{aligned} e^{At} &= e^t \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} -2 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 2 \end{bmatrix} + \tfrac{t^2}{2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right) \\ &= e^t \begin{bmatrix} 1 - 2t - \frac{t^2}{2} & t & 2t + \frac{t^2}{2} \\ -t & 1 & t \\ -2t - \frac{t^2}{2} & t & 1 + 2t + \frac{t^2}{2} \end{bmatrix} \end{aligned}$$

- The general solution is the span of the columns of e^{At}:
$$\vec{x}(t) = c_1\, e^t \begin{bmatrix} 1 - 2t - \frac{t^2}{2} \\ -t \\ -2t - \frac{t^2}{2} \end{bmatrix} + c_2\, e^t \begin{bmatrix} t \\ 1 \\ t \end{bmatrix} + c_3\, e^t \begin{bmatrix} 2t + \frac{t^2}{2} \\ t \\ 1 + 2t + \frac{t^2}{2} \end{bmatrix}$$