Arrow Diagrams and Generalized Eigen Vectors

Arrow Diagrams

An arrow diagram is a table of arrows that represents the distribution of eigenvectors and generalized eigenvectors of a matrix $A \in M_n(\mathbb{R})$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of A. For each λ_i you will get a rectangular grid of arrows, for example

$$\lambda_i : \left\{ \begin{array}{cccc} & & & \leftarrow \\ & \leftarrow & \leftarrow & \leftarrow \\ \leftarrow & \leftarrow & \leftarrow & \leftarrow \end{array} \right.$$

with the following property: the number of arrows on the bottom row is the dimension of $E_\lambda = \ker(A - \lambda I)$, the number of arrows in the bottom two rows is the dimension of $\ker((A - \lambda I)^2)$, the number of arrows in the bottom three rows is the dimension of $\ker((A - \lambda I)^3)$, and so on. Continue until the total number of arrows equals the multiplicity of λ_i, m_{λ_i}. For example, if you "have all your eigenvectors" for an eigenvalue λ of multiplicity m_λ, that is to say $\dim(E_\lambda) = m_\lambda$, then the arrow diagram for that λ will be a single row of m_λ arrows:

$$\lambda : \{ \leftarrow \quad \leftarrow \quad \cdots \quad \leftarrow$$

The diagram always has one important feature: for any eigenvalue λ, the number of arrows in a row can not exceed the number of arrows in a lower row.

Thus, if you have an eigenvalue λ of multiplicity m_λ and you only have one eigenvector, that is to say $\dim(E_\lambda) = 1$, then the arrow diagram for that λ will be a single column of m_λ arrows:

$$\lambda : \left\{ \begin{array}{c} \leftarrow \\ \vdots \\ \leftarrow \\ \leftarrow \end{array} \right.$$

and this is exactly the situation in which we get a Jordan cycle.

If you have a 4×4 with eigenvalue -2 of multiplicity $m_{-2} = 1$ and eigenvalue -1 of multiplicity $m_{-1} = 3$, then the possible arrow diagrams are

(a)

$$-2 : \{ \leftarrow \qquad\qquad -1 : \{ \leftarrow \quad \leftarrow \quad \leftarrow$$

(b)

$$-2 : \{ \leftarrow \qquad\qquad -1 : \left\{ \begin{array}{cc} & \leftarrow \\ \leftarrow & \leftarrow \end{array} \right.$$

(c)

$$-2 : \{ \leftarrow \qquad\qquad -1 : \left\{ \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \right.$$

A 4×4 with a single eigenvalue, λ, has the following possible arrow diagrams:

$$\text{(a) } \lambda : \{ \leftarrow \quad \leftarrow \quad \leftarrow \quad \leftarrow \qquad \text{(b) } \lambda : \left\{ \begin{array}{ccc} & & \leftarrow \\ \leftarrow & \leftarrow & \leftarrow \end{array} \right. \qquad \text{(c) } \lambda : \left\{ \begin{array}{cc} \leftarrow & \leftarrow \\ \leftarrow & \leftarrow \end{array} \right. \qquad \text{(d) } \lambda : \left\{ \begin{array}{cc} & \leftarrow \\ & \leftarrow \\ \leftarrow & \leftarrow \end{array} \right. \qquad \text{(e) } \lambda : \left\{ \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \right.$$

Example 1
Consider

$$\frac{d}{dt}\vec{x} = A\vec{x} \text{ where } A = \begin{bmatrix} 0 & -3 & 3 \\ 1 & 4 & -1 \\ -2 & -2 & 5 \end{bmatrix} \text{ has eigenvalue } \lambda = 3, m_3 = 3.$$

1. What is the standard basis of E_3?
2. Give the solution(s) generated by the standard vector(s) $J \in \ker((A-\lambda I)^2) \setminus \ker(A - \lambda I)$. These are the vectors represented by arrows in the second row (from the bottom) of the arrow diagram.
3. What is the arrow diagram for the system?

Solution:

1. The standard basis of E_3:

$$(A-3I) = \begin{bmatrix} -3 & -3 & 3 \\ 1 & 1 & -1 \\ -2 & -2 & 2 \end{bmatrix} \quad \to \quad \text{RREF}(A-3I) = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the standard basis of E_3 is

$$\left\{ \vec{v}_3^1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3^2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since there are two eigenvectors, the bottom row of our arrow diagram has two arrows, and now we know the arrow diagram:

$$3 : \begin{cases} & \leftarrow \\ \leftarrow & \leftarrow \end{cases}$$

2. Solution(s) generated by generalized eigenvector(s) J in the second row:

We are looking for vectors $J \in \mathbb{R}^3$ such that $(A-3I)J$ is an eigenvector, that is

$$(A-3I)J \in E_3 = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} -3 & -3 & 3 \\ 1 & 1 & -1 \\ -2 & -2 & 2 \end{bmatrix} J = \alpha \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\alpha + \beta \\ \alpha \\ \beta \end{bmatrix}$$

So we row reduce

$$\left[\begin{array}{ccc|c} -3 & -3 & 3 & -\alpha+\beta \\ 1 & 1 & -1 & \alpha \\ -2 & -2 & 2 & \beta \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & \alpha \\ -3 & -3 & 3 & -\alpha+\beta \\ -2 & -2 & 2 & \beta \end{array}\right]$$

$$\underset{R_3 + 2R_1}{\overset{R_2 + 3R_1}{\longrightarrow}} \left[\begin{array}{ccc|c} 1 & 1 & -1 & \alpha \\ 0 & 0 & 0 & 2\alpha+\beta \\ 0 & 0 & 0 & 2\alpha+\beta \end{array}\right]$$

This system is consistent only when the entries in the augmented column of rows with all zeroes are also zero. This gives us a homogeneous system

$$\begin{aligned} 2\alpha + \beta &= 0 \\ 2\alpha + \beta &= 0 \end{aligned}$$

We solve this using parametric form and get

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \alpha = -\frac{1}{2}\beta \Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \beta \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

This gives us one linearly independent solution, $\alpha = -1/2, \beta = 1$, so we will get one vector J. We can find J by solving

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & \alpha \\ 0 & 0 & 0 & 2\alpha+\beta \\ 0 & 0 & 0 & 2\alpha+\beta \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad J = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

and

$$(A - 3I)J = \begin{bmatrix} -\alpha + \beta \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

This gives us a solution

$$\vec{x} = e^{3t}\left(J + t(A - 3I)J\right) = e^{3t}\left(\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right) = e^{3t} \begin{bmatrix} -\frac{1}{2} + \frac{3}{2}t \\ -\frac{1}{2}t \\ t \end{bmatrix}$$

3. Arrow diagram

$$3 : \left\{ \begin{array}{cc} & \leftarrow \\ \leftarrow & \leftarrow \end{array} \right.$$

Example 2
Consider

$$\frac{d}{dt}\vec{x} = A\vec{x} \text{ where } A = \begin{bmatrix} -1 & 0 & 2 & 0 \\ -2 & -3 & 1 & 2 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & 2 & 0 \end{bmatrix} \text{ has eigenvalue } \lambda = -1, m_{-1} = 4.$$

1. What is the standard basis of E_{-1}?
2. Give the solution(s) generated by the standard vector(s) $J \in \ker((A - \lambda I)^2) \setminus \ker(A - \lambda I)$. These are the vectors represented by arrows in the second row (from the bottom) of the arrow diagram.
3. What is the arrow diagram for the system?

Solution:

1. The standard basis of E_{-1}:

$$(A + I) = \begin{bmatrix} 0 & 0 & 2 & 0 \\ -2 & -2 & 1 & 2 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 2 & 1 \end{bmatrix} \quad \to \quad \text{RREF}(A + I) = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the standard basis of E_{-1} is

$$\left\{ \vec{v}^{\,1}_{-1} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}^{\,2}_{-1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since there are two eigenvectors, the bottom row of our arrow diagram has two arrows, and now we know only one of the following pictures is our possible arrow diagram

$$-1 : \left\{ \begin{matrix} \leftarrow & \leftarrow \\ \leftarrow & \leftarrow \end{matrix} \right. \qquad \text{or} \qquad -1 : \left\{ \begin{matrix} & \leftarrow \\ & \leftarrow \\ \leftarrow & \leftarrow \end{matrix} \right.$$

2. Solution(s) generated by generalized eigenvector(s) J in the second row:

We are looking for vectors $J \in \mathbb{R}^4$ such that $(A + I)J$ is an eigenvector, that is

$$(A + I)J \in E_{-1} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 0 & 0 & 2 & 0 \\ -2 & -2 & 1 & 2 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 2 & 1 \end{bmatrix} J = \alpha \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\alpha + \beta \\ \alpha \\ 0 \\ \beta \end{bmatrix}$$

So we row reduce

$$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 0 & -\alpha+\beta \\ -2 & -2 & 1 & 2 & \alpha \\ -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 2 & 1 & \beta \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_3 \\ -R_1 \\ \longrightarrow \\ \frac{1}{2}R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ -2 & -2 & 1 & 2 & \alpha \\ 0 & 0 & 1 & 0 & -\frac{1}{2}\alpha+\frac{1}{2}\beta \\ -1 & -1 & 2 & 1 & \beta \end{array}\right]$$

$$\begin{array}{c} R_2+2R_1 \\ -R_2 \\ \longrightarrow \\ R_4+R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -\alpha \\ 0 & 0 & 1 & 0 & -\frac{1}{2}\alpha+\frac{1}{2}\beta \\ 0 & 0 & 1 & 0 & \beta \end{array}\right] \begin{array}{c} R_1+R_2 \\ R_3-R_2 \\ \longrightarrow \\ R_4-R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -\alpha \\ 0 & 0 & 1 & 0 & -\alpha \\ 0 & 0 & 0 & 0 & \frac{1}{2}\alpha+\frac{1}{2}\beta \\ 0 & 0 & 0 & 0 & \alpha+\beta \end{array}\right]$$

This system is consistent only when the entries in the augmented column of rows with all zeroes are also zero. This gives us a homogeneous system

$$\begin{array}{rcl} \frac{1}{2}\alpha+\frac{1}{2}\beta & = & 0 \\ \alpha+\beta & = & 0 \end{array}$$

We solve this using parametric form and get

$$\begin{bmatrix} 1/2 & 1/2 \\ 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \alpha = -\beta \Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \beta \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

This gives us one linearly independent solution, $\alpha = -1, \beta = 1$, so we will get one vector J. Now we know our arrow diagram is

$$-1 : \left\{ \begin{array}{cc} & \leftarrow \\ & \leftarrow \\ \leftarrow & \leftarrow \end{array} \right.$$

We can find J by solving

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -\alpha \\ 0 & 0 & 1 & 0 & -\alpha \\ 0 & 0 & 0 & 0 & \frac{1}{2}\alpha+\frac{1}{2}\beta \\ 0 & 0 & 0 & 0 & \alpha+\beta \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad J = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and

$$(A+I)J = \begin{bmatrix} -\alpha+\beta \\ \alpha \\ 0 \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

This gives us a solution

$$\vec{x} = e^{-t}\left(J + t(A+I)J\right) = e^{-t}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right) = e^{-t} \begin{bmatrix} 1+2t \\ -t \\ 1 \\ t \end{bmatrix}$$

3. Arrow diagram

$$-1 : \left\{ \begin{array}{cc} & \leftarrow \\ & \leftarrow \\ \leftarrow & \leftarrow \end{array} \right.$$

Example 3
Consider

$$\frac{d}{dt}\vec{x} = A\vec{x} \text{ where } A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ -2 & -3 & 1 & 2 \\ 0 & 0 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{bmatrix} \text{ has eigenvalue } \lambda = -1, m_{-1} = 4.$$

1. What is the standard basis of E_{-1}?
2. Give the solution(s) generated by the standard vector(s) $J \in \ker((A - \lambda I)^2) \setminus \ker(A - \lambda I)$. These are the vectors represented by arrows in the second row (from the bottom) of the arrow diagram.
3. What is the arrow diagram for the system?

Solution:

1. The standard basis of E_{-1}:

$$(A + I) = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -2 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 1 \end{bmatrix} \quad \rightarrow \quad \text{RREF}(A + I) = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the standard basis of E_{-1} is

$$\left\{ \vec{v}^{\,1}_{-1} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}^{\,2}_{-1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since there are two eigenvectors, the bottom row of our arrow diagram has two arrows, and now we know only one of the following pictures is our possible arrow diagram

$$-1 : \left\{ \begin{array}{cc} \leftarrow & \leftarrow \\ \leftarrow & \leftarrow \end{array} \right. \qquad \text{or} \qquad -1 : \left\{ \begin{array}{cc} & \leftarrow \\ & \leftarrow \\ \leftarrow & \leftarrow \end{array} \right.$$

2. Solution(s) generated by generalized eigenvector(s) J in the second row:

We are looking for vectors $J \in \mathbb{R}^4$ such that $(A + I)J$ is an eigenvector, that is

$$(A + I)J \in E_{-1} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ -2 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 1 \end{bmatrix} J = \alpha \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\alpha + \beta \\ \alpha \\ 0 \\ \beta \end{bmatrix}$$

So we row reduce

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -\alpha+\beta \\ -2 & -2 & 1 & 2 & \alpha \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 1 & \beta \end{array}\right] \begin{array}{c} R_3 \leftrightarrow R_4 \\ R_2 + 2R_1 \\ \longrightarrow \\ R_3 + R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -\alpha+\beta \\ 0 & 0 & 3 & 0 & -\alpha+2\beta \\ 0 & 0 & 3 & 0 & -\alpha+2\beta \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} R_3 - R_2 \\ \frac{1}{3}R_2 \\ \longrightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & -\alpha+\beta \\ 0 & 0 & 1 & 0 & -\frac{1}{3}\alpha+\frac{2}{3}\beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ \longrightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -\frac{2}{3}\alpha+\frac{1}{3}\beta \\ 0 & 0 & 1 & 0 & -\frac{1}{3}\alpha+\frac{2}{3}\beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This system is consistent only when the entries in the augmented column of rows with all zeroes are also zero. Of course, these entries are already zero, which means α and β can be anything, or that they are free. We solve this using parametric form and get two solutions: $\alpha = 1, \beta = 0$, and $\alpha = 0, \beta = 1$. So we will get two linearly independent vectors, J^1 and J^2. Now we know our arrow diagram is

$$-1 : \begin{cases} \leftarrow & \leftarrow \\ \leftarrow & \leftarrow \end{cases}$$

(a) We can find J^1 using the solution $\alpha = 1, \beta = 0$.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -\frac{2}{3}\alpha+\frac{1}{3}\beta \\ 0 & 0 & 1 & 0 & -\frac{1}{3}\alpha+\frac{2}{3}\beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \quad J^1 = \begin{bmatrix} -\frac{2}{3} \\ 0 \\ -\frac{1}{3} \\ 0 \end{bmatrix}$$

and

$$(A+I)J^1 = \begin{bmatrix} -\alpha+\beta \\ \alpha \\ 0 \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

This gives us a solution

$$\vec{x} = e^{-t}\left(J^1 + t(A+I)J^1\right) = e^{-t}\left(\begin{bmatrix} -\frac{2}{3} \\ 0 \\ -\frac{1}{3} \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) = e^{-t}\begin{bmatrix} -\frac{2}{3} - t \\ t \\ -\frac{1}{3} \\ 0 \end{bmatrix}$$

(b) We can find J^2 using the solution $\alpha = 0, \beta = 1$.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & -\frac{2}{3}\alpha+\frac{1}{3}\beta \\ 0 & 0 & 1 & 0 & -\frac{1}{3}\alpha+\frac{2}{3}\beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & -1 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \quad J^2 = \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \\ 0 \end{bmatrix}$$

and

$$(A+I)J^2 = \begin{bmatrix} -\alpha+\beta \\ \alpha \\ 0 \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This gives us a solution

$$\vec{x} = e^{-t}\left(J^2 + t(A+I)J^2\right) = e^{-t}\left(\begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) = e^{-t}\begin{bmatrix} \frac{1}{3}+t \\ 0 \\ \frac{2}{3} \\ t \end{bmatrix}$$

So the answer to number 2 is

$$e^{At}J^1 = e^{-t}\begin{bmatrix} -\frac{2}{3}-t \\ t \\ -\frac{1}{3} \\ 0 \end{bmatrix} \quad \text{and} \quad e^{At}J^2 = e^{-t}\begin{bmatrix} \frac{1}{3}+t \\ 0 \\ \frac{2}{3} \\ t \end{bmatrix}$$

3. Arrow diagram

$$-1 : \left\{ \begin{matrix} \leftarrow & \leftarrow \\ \leftarrow & \leftarrow \end{matrix} \right.$$

Example 4
Consider

$$\frac{d}{dt}\vec{x} = A\vec{x} \text{ where } A = \begin{bmatrix} -4 & -3 & 3 & 3 \\ 1 & 0 & -1 & -1 \\ -2 & -2 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ has eigenvalue } \lambda = -1, m_{-1} = 4.$$

1. What is the standard basis of E_{-1}?
2. Give the solution(s) generated by the standard vector(s) $J \in \ker((A - \lambda I)^2) \setminus \ker(A - \lambda I)$. These are the vectors represented by arrows in the second row (from the bottom) of the arrow diagram.
3. What is the arrow diagram for the system?

Solution:

1. The standard basis of E_{-1}:

$$(A + I) = \begin{bmatrix} -3 & -3 & 3 & 3 \\ 1 & 1 & -1 & -1 \\ -2 & -2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \to \quad \text{RREF}(A + I) = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the standard basis of E_{-1} is

$$\left\{ \vec{v}^{\,1}_{-1} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}^{\,2}_{-1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}^{\,3}_{-1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since there are three eigenvectors, the bottom row of our arrow diagram has three arrows, and now we know the arrow diagram is

$$-1 : \begin{cases} & & \leftarrow \\ \leftarrow & \leftarrow & \leftarrow \end{cases}$$

2. Solution(s) generated by generalized eigenvector(s) J in the second row:

We are looking for a vector $J \in \mathbb{R}^4$ such that $(A + I)J$ is an eigenvector, that is

$$(A + I)J \in E_{-1} = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} -3 & -3 & 3 & 3 \\ 1 & 1 & -1 & -1 \\ -2 & -2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} J = \alpha \cdot \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\alpha + \beta + \gamma \\ \alpha \\ \beta \\ \gamma \end{bmatrix}$$

So we row reduce

$$\left[\begin{array}{cccc|c} -3 & -3 & 3 & 3 & -\alpha + \beta + \gamma \\ 1 & 1 & -1 & -1 & \alpha \\ -2 & -2 & 2 & 2 & \beta \\ 0 & 0 & 0 & 0 & \gamma \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ R_2 + 3R_1 \\ \longrightarrow \\ R_3 + 2R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & \alpha \\ 0 & 0 & 0 & 0 & 2\alpha + \beta + \gamma \\ 0 & 0 & 0 & 0 & 2\alpha + \beta \\ 0 & 0 & 0 & 0 & \gamma \end{array}\right]$$

This system is consistent only when the entries in the augmented column of rows with all zeroes are also zero. This gives us a homogeneous system

$$\begin{array}{rcl} 2\alpha + \beta + \gamma & = & 0 \\ 2\alpha + \beta & = & 0 \\ \gamma & = & 0 \end{array}$$

We solve this using parametric form and get

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{rcl} \alpha & = & -\frac{1}{2}\beta \\ \gamma & = & 0 \end{array} \Rightarrow \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \beta \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

This gives us one linearly independent solution, $\alpha = -1/2, \beta = 1, \gamma = 0$. We can find J by solving

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & \alpha \\ 0 & 0 & 0 & 0 & 2\alpha+\beta+\gamma \\ 0 & 0 & 0 & 0 & 2\alpha+\beta \\ 0 & 0 & 0 & 0 & \gamma \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & -1 & -1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad J = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$(A+I)J = \begin{bmatrix} -\alpha+\beta+\gamma \\ \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

This gives us a solution

$$\vec{x} = e^{-t}\left(J + t(A+I)J\right) = e^{-t}\left(\begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}\right) = e^{-t}\begin{bmatrix} -\frac{1}{2}+\frac{3}{2}t \\ -\frac{1}{2}t \\ t \\ 0 \end{bmatrix}$$

3. Arrow diagram

$$-1 : \left\{ \begin{array}{ccc} & & \leftarrow \\ \leftarrow & \leftarrow & \leftarrow \end{array} \right.$$