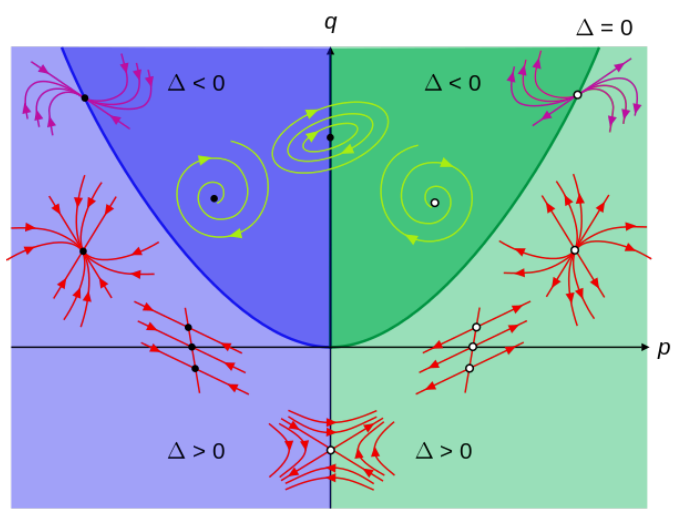
Analyzing Nonlinear Systems

We will focus on analyzing nonlinear 2×2 systems:

$$\begin{aligned} x'(t) &= f(x,y) \\ y'(t) &= g(x,y) \end{aligned}$$

For such a system

1. Find the **equilibrium points**. These are points (x_i, y_i) such that $f(x_i,y_i) = g(x_i,y_i) = 0$.
2. Compute the **Jacobian matrix** of the system. This is the matrix of partial derivatives
$$J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$
3. Compute the Jacobian at each equilibrium point:
$$J_i = J(x_i,y_i) = \begin{bmatrix} \frac{\partial f}{\partial x}(x_i,y_i) & \frac{\partial f}{\partial y}(x_i,y_i) \\ \\ \frac{\partial g}{\partial x}(x_i,y_i) & \frac{\partial g}{\partial y}(x_i,y_i) \end{bmatrix} \in M_2(\mathbb{R})$$
4. Analyze the phase plane at each equilibrium point. This is the phase plane of J_i shifted to the equilibrium point. This analysis should include the eigenvalues, any real eigenvectors, the class of phase plane (nodal sink, spiral source, etc.), and direction of motion.
5. Sketch the vector field given by the system by sketching the phase portrait of each J_i at the respective equilibrium point. Here are the kinds of phase portraits you will sketch:



$$\frac{dx}{dt} = Ax + By \qquad p = A + D$$
$$\frac{dy}{dt} = Cx + Dy \qquad q = AD - BC$$
$$\Delta = p^2 - 4q$$

Figure 1: Phase Sketch

This figure is located on wikipedia: Link to Phase Planes on Wikipedia

Example 1

Analyze the system

$$\begin{aligned} x' &= x^2 - y^2 \\ y' &= xy - 1 \end{aligned}$$

1. Find the **equilibrium points**:

   $$x^2 = y^2 \quad \Rightarrow \quad x = \pm y$$

   so the graph is the union of the lines $y = x$ and $y = -x$. Then

   $$xy = 1$$

   has graph the hyperbola $y = 1/x$. These graphs intersect at the equilibrium points

   $$(x_1, y_1) = (1, 1) \quad \text{and} \quad (x_2, y_2) = (-1, -1).$$

2. Compute the **Jacobian matrix** of the system:

   $$J(x, y) = \begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix}$$

3. Compute the Jacobian at each equilibrium point:

   $$J_1 = J(1, 1) = \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad J_2 = J(-1, -1) = \begin{bmatrix} -2 & 2 \\ -1 & -1 \end{bmatrix}$$

4. Analyze the phase plane at each equilibrium point:

   (1) At $(1, 1)$,

   $$J_1 \text{ has eigenvalues } \lambda = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

   which is a **spiral source**. Checking the path of a solution curve passing through $(1, 0)$ has tangent vector

   $$J_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

   we see the curve is moving to the right and upward as time passes. Thus the solutions are moving out and counterclockwise.

   (2) At $(-1, -1)$,

   $$J_2 \text{ has eigenvalues } \lambda = -\frac{3}{2} \pm i\frac{\sqrt{7}}{2}$$

   which is a **spiral sink**. Checking the path of a solution curve passing through $(1, 0)$ has tangent vector

   $$J_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

   we see the curve is moving to the left and downward as time passes. Thus the solutions are moving in and clockwise.

5. Sketch the vector field given by the system by sketching the phase portrait of each J_i at the respective equilibrium point.

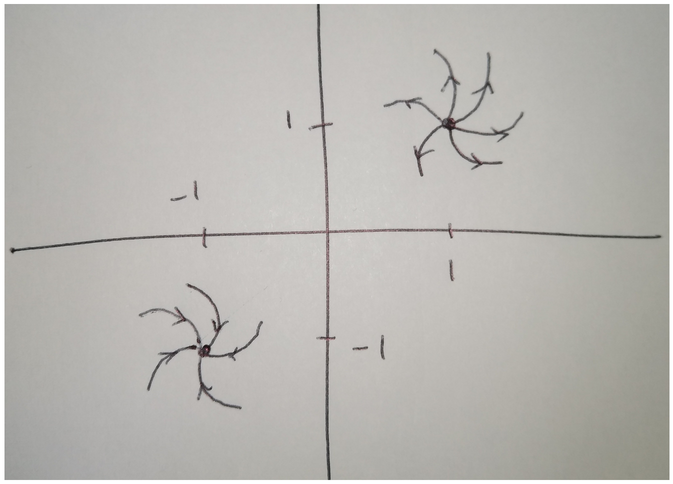
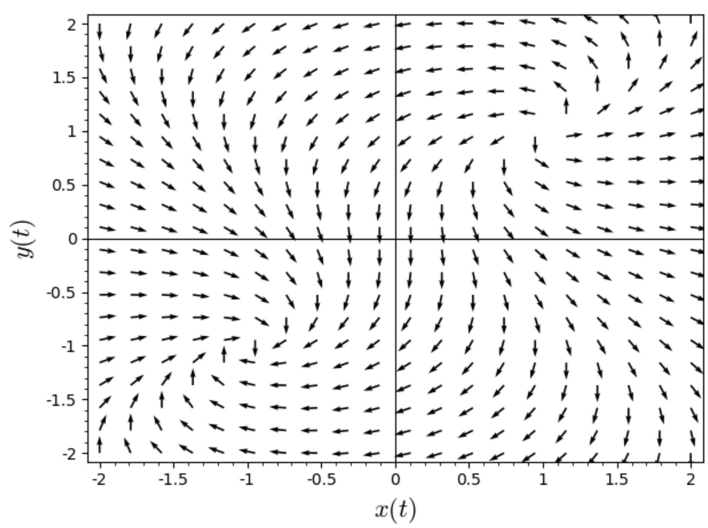
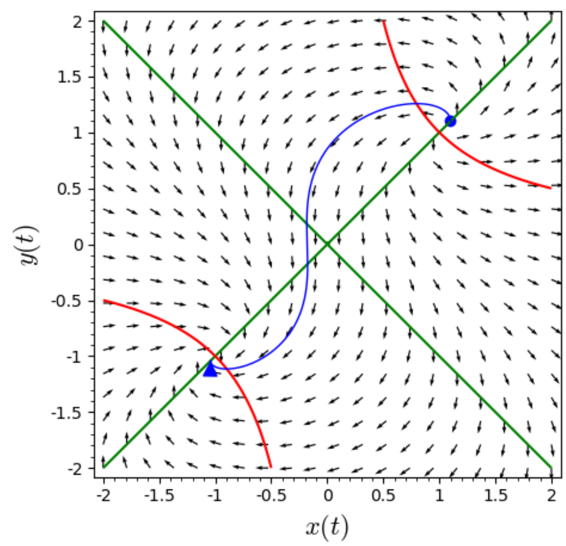


Figure 2: Phase Sketch

We can compare this to the vector field computed using Sage.



(a) Vector Field



(b) With solution

The green lines are the solutions to $x' = 0$ and the red curves are the solutions to $y' = 0$. So their intersections are the equilibrium points. The blue curve is a solution, the solution to an I.V.P. passing through the point $(1.1, 1.1)$.

Example 2

Analyze the system

$$\begin{aligned} x' &= x^2 - y^2 \\ y' &= y - 1 \end{aligned}$$

1. Find the **equilibrium points**:

$$x^2 = y^2 \quad \Rightarrow \quad x = \pm y$$

so the graph is the union of the lines $y = x$ and $y = -x$. Then the vertical line

$$y = 1$$

intersects these at equilibrium points

$$(x_1, y_1) = (1, 1) \quad \text{and} \quad (x_2, y_2) = (-1, 1).$$

2. Compute the **Jacobian matrix** of the system:

$$J(x, y) = \begin{bmatrix} 2x & -2y \\ 0 & 1 \end{bmatrix}$$

3. Compute the Jacobian at each equilibrium point:

$$J_1 = J(1, 1) = \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad J_2 = J(-1, 1) = \begin{bmatrix} -2 & -2 \\ 0 & 1 \end{bmatrix}$$

4. Analyze the phase plane at each equilibrium point:

(1) At $(1, 1)$,

$$J_1 \text{ has eigenvalues } \lambda = 2, 1$$

which is a **nodal source**. The eigenvectors are

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

which we will use in our sketch.

(2) At $(-1, 1)$,

$$J_2 \text{ has eigenvalues } \lambda = -2, 1$$

which is a **saddle**. The eigenvectors are

$$\vec{v}_{-2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_1 = \begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$$

which we will use in our sketch.

5. Sketch the vector field given by the system by sketching the phase portrait of each J_i at the respective equilibrium point.

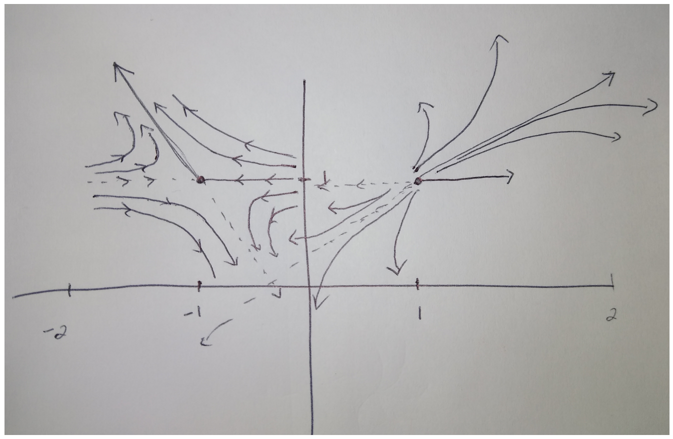
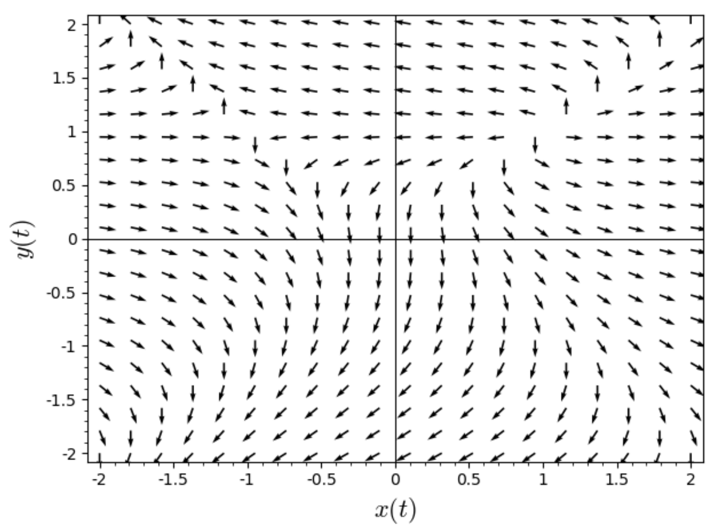
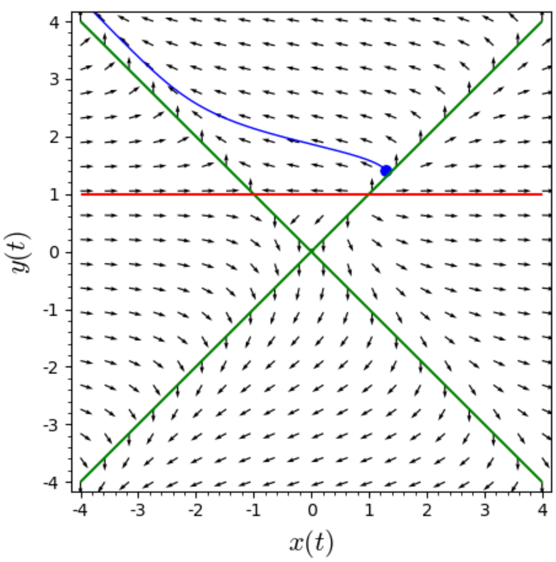


Figure 3: Phase Sketch

We can compare this to the vector field computed using Sage.



(a) Vector Field



(b) With solution

The green lines are the solutions to $x' = 0$ and the red line is the solution to $y' = 0$. So their intersections are the equilibrium points. The blue curve is a solution, the solution to an I.V.P. passing through the point $(1.3, 1.4)$.

Example 3

This is an example of the **Lotka-Volterra equations**, used to model predator-prey systems. Analyze the system

$$\begin{aligned} x' &= x - xy \\ y' &= xy - y \end{aligned}$$

1. Find the **equilibrium points**:

$$x - xy = x(1-y) = 0 \quad \Rightarrow \quad x = 0 \text{ or } y = 1$$

so the graph is the union of the lines $x = 0$ and $y = 1$. Then

$$xy - y = y(x-1) = 0 \quad \Rightarrow \quad y = 0 \text{ or } x = 1$$

so the graph is the union of the lines $y = 0$ and $x = 1$. These graphs intersect at equilibrium points

$$(x_0, y_0) = (0,0) \quad \text{and} \quad (x_1, y_1) = (1,1).$$

2. Compute the **Jacobian matrix** of the system:

$$J(x,y) = \begin{bmatrix} 1-y & -x \\ y & x-1 \end{bmatrix}$$

3. Compute the Jacobian at each equilibrium point:

$$J_0 = J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad J_1 = J(1,1) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

4. Analyze the phase plane at each equilibrium point:

(1) At $(0,0)$,

$$J_0 \text{ has eigenvalues } \lambda = 1, -1$$

which is a **saddle**. The eigenvectors are

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which we will use in our sketch.

(2) At $(1,1)$,

$$J_1 \text{ has eigenvalues } \lambda = \pm i$$

which is a **center**. Checking the path of a solution curve passing through $(1,0)$ has tangent vector

$$J_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we see the curve is moving to the left and upward as time passes. Thus the solutions are moving in closed orbits and counterclockwise.

5. Sketch the vector field given by the system by sketching the phase portrait of each J_i at the respective equilibrium point.

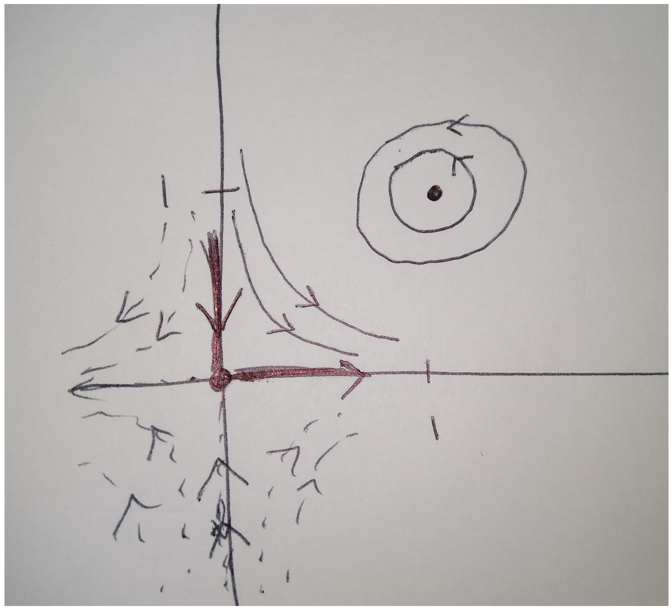
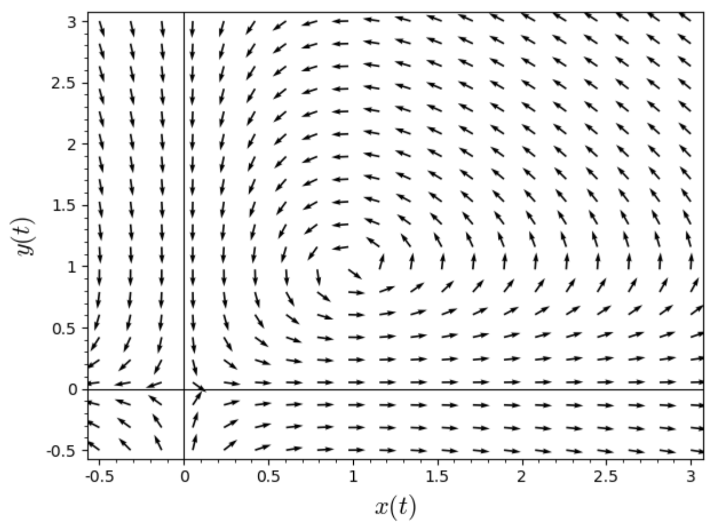
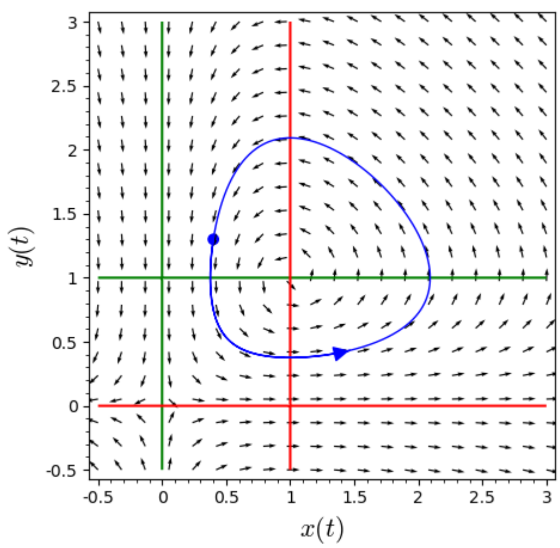


Figure 4: Phase Sketch

We can compare this to the vector field computed using Sage.



(a) Vector Field



(b) With solution

The green lines are the solutions to $x' = 0$ and the red lines are the solutions to $y' = 0$. So their intersections are the equilibrium points. The blue curve is a solution, the solution to an I.V.P. passing through the point $(.4, 1.3)$.

Example 4

This is an example of the predator-prey model where exponential growth of the prey ($x' = kx$) is replaced by logistic growth ($x' = kx(1 - x/N)$).

Analyze the system

$$\begin{aligned} x' &= x(1 - x/2) - xy \\ y' &= xy - y \end{aligned}$$

1. Find the **equilibrium points**:

$$xy - y = y(x - 1) = 0 \quad \Rightarrow \quad y = 0 \text{ or } x = 1, \text{ and}$$
$$x(1 - x/2) - xy = x(1 - x/2 - y) = 0.$$

   If $x = 1$ then

$$x(1 - x/2 - y) = 1(1 - 1/2 - y) = 0 \quad \Rightarrow \quad 1/2 - y = 0 \quad \Rightarrow \quad y = 1/2.$$

   So one fixed point is $(1, 1/2)$.

   If $y = 0$ then

$$x(1 - x/2 - y) = x(1 - x/2) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = 2.$$

   So two more fixed points are $(0, 0)$ and $(2, 0)$.

   Thus the equilibrium points are

$$(x_1, y_1) = (0, 0) \quad \text{and} \quad (x_2, y_2) = (1, 1/2) \quad \text{and} \quad (x_3, y_3) = (2, 0).$$

2. Compute the **Jacobian matrix** of the system:

$$J(x, y) = \begin{bmatrix} 1 - x - y & -x \\ y & x - 1 \end{bmatrix}$$

3. Compute the Jacobian at each equilibrium point:

$$J_1 = J(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; J_2 = J(1, 1/2) = \begin{bmatrix} -1/2 & -1 \\ 1/2 & 0 \end{bmatrix}, \text{ and } J_3 = J(2, 0) = \begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$$

4. Analyze the phase plane at each equilibrium point:

   (1) At $(0, 0)$,

$$J_1 \text{ has eigenvalues } \lambda = 1, -1$$

   which is a **saddle**. The eigenvectors are

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   (2) At $(1, 1/2)$,

$$J_1 \text{ has eigenvalues } \lambda = -\frac{1}{4} \pm i\frac{\sqrt{7}}{4}$$

   which is a **spiral sink**. Checking the path of a solution curve passing through $(1, 0)$ has tangent vector

$$J_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

   we see the curve is moving to the left and upward as time passes. Thus the solutions are moving in and counterclockwise.

(3) At $(2, 0)$,

$$J_3 \text{ has eigenvalues } \lambda = -1, 1$$

which is a **saddle**. The eigenvectors are

$$\vec{v}_{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

5. Sketch the vector field given by the system by sketching the phase portrait of each J_i at the respective equilibrium point.

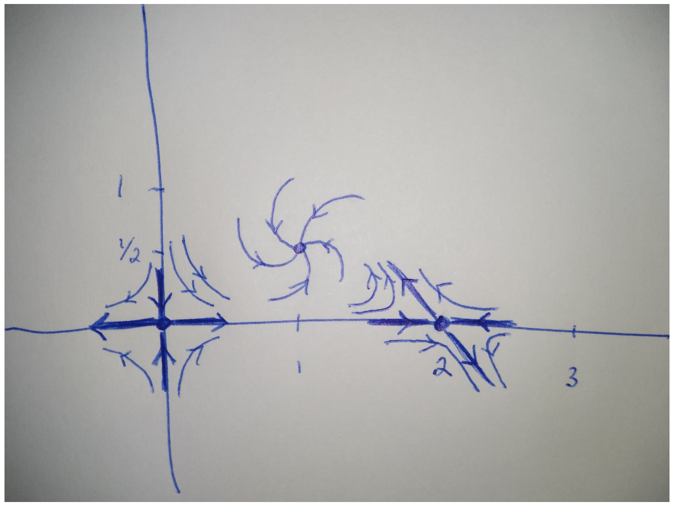
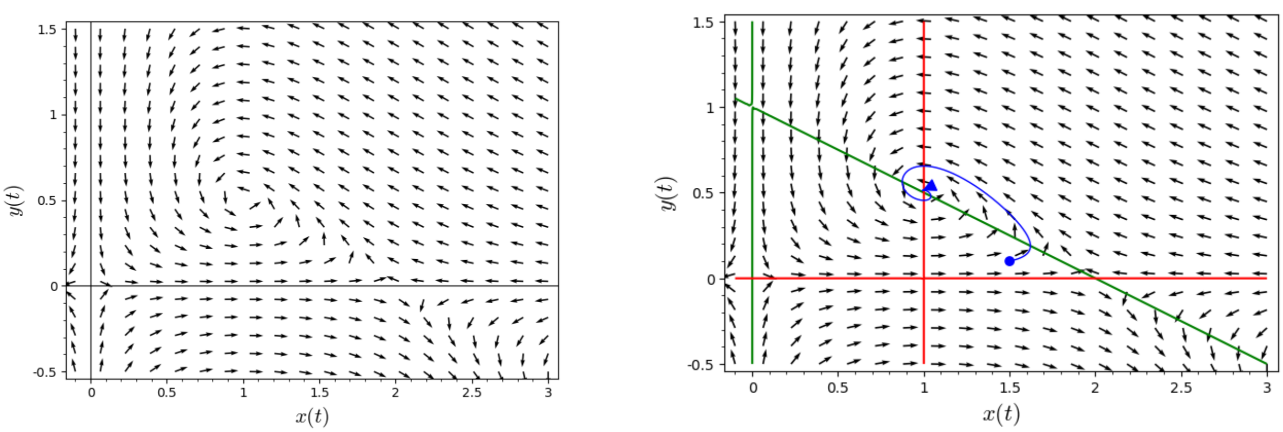


Figure 5: Phase Sketch

We can compare this to the vector field computed using Sage.



(a) Vector Field

(b) With solution

The green lines are the solutions to $x' = 0$ and the red lines are the solutions to $y' = 0$. So their intersections are the equilibrium points. The blue curve is a solution, the solution to an I.V.P. passing through the point $(1.5, .1)$.