Homework #2 Solutions

You will notice in the inductive proofs below that, in the inductive step, I have chosen to denote the particular integer for which the inductive hypothesis is assumed by k instead of n. The case can be made that using a different letter in the inductive step is more logically sound, and some mathematicians feel very strongly that one *must* use a different letter. You are free to use the same letter throughout (as our book seems to do), or to use some other letter in the inductive hypothesis. The crucial thing to keep in mind is that, in the inductive step, one assumes the statement in question is true for a particular integer, not for all integers.

1.3.2. Computing the sum $\sum_{j=1}^{n} 2j$ of the first n even integers for a few small values of n (and working under the reasonable assumption that an explicit formula for the sum will involve n), one is led to conjecture that $\sum_{j=1}^{n} 2j = n(n+1)$. We will prove this formula by induction on n. When $n = 1$, the lefthand side of the putative formula is 2 while the righthand side is $1 \cdot 2 = 2$, so the formula is valid in this case. Assume now that the formula holds for some positive integer $k \geq 1$. It follows that

$$\begin{aligned} \sum_{j=1}^{k+1} 2j &= \sum_{j=1}^{k} 2j + 2(k+1) \\ &= k(k+1) + 2(k+1) = (k+1)(k+2). \end{aligned}$$

Thus the formula holds for $k+1$, so, by induction, we may conclude that it holds for all positive integers n.

1.3.10. We argue by induction on n. When $n = 1$, we have

$$\sum_{j=1}^{n} (-1)^{j-1} j^2 = \sum_{j=1}^{1} (-1)^{j-1} j^2 = (-1)^{1-1}(1)^2 = 1,$$

while

$$(-1)^{n-1}\frac{n(n+1)}{2} = (-1)^{1-1}\frac{1 \cdot (1+1)}{2} = \frac{2}{2} = 1.$$

Thus the formula holds in this case. Assuming now that the formula holds for some positive integer $k \geq 1$, that is, that $\sum_{j=1}^{k} (-1)^{j-1} j^2 = (-1)^{k-1} k(k+1)/2$, we will prove that the

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formula holds for $k+1$. To this end, we compute

$$\begin{aligned}\sum_{j=1}^{k+1}(-1)^{j-1}j^2 &= \sum_{j=1}^{k}(-1)^{j-1}j^2 + (-1)^k(k+1)^2 \\ &= (-1)^{k-1}\frac{k(k+1)}{2} + (-1)^k(k+1)^2 \\ &= (-1)^{k-1}\frac{k(k+1)}{2} + (-1)^k\frac{2(k+1)^2}{2} \\ &= (-1)^{k-1}(k+1)\frac{k-2(k+1)}{2} \\ &= (-1)^{k-1}(k+1)\frac{-k-2}{2} = (-1)^{k-1}(k+1)\frac{-(k+2)}{2} = (-1)^k\frac{(k+1)(k+2)}{2}.\end{aligned}$$

Having proved that the formula is valid for $k+1$, we may conclude by induction that it is valid for all positive integers n.

1.3.22. When $n=1$, we have $(1+h)^n = 1+h = 1+1\cdot h$, so the inequality holds (it is in fact an equality in this case). Assume now that, for some positive integer $k \geq 1$, the inequality $(1+h)^k \geq 1+kh$ holds. It follows that

$$\begin{aligned}(1+h)^{k+1} &= (1+h)^k(1+h) \\ &\geq (1+kh)(1+h) \\ &= 1+kh+h+kh^2 \geq 1+kh+h = 1+(k+1)h,\end{aligned}$$

where for the final inequality we have used that $kh^2 \geq 0$ (this is true because k is a positive integer and the square of any real number is nonnegative). Note that we have also tacitly used the fact that multiplication by $1+h$ preserves weak inequalities, which holds due to the assumption that $h \geq -1$, i.e., $1+h \geq 0$. We have deduced the inequality for $k+1$ from the inequality for k, so, by induction, the inequality holds for all positive integers n.

The inequality in Exercise 1.3.22 is called *Bernoulli's inequality*, and is used frequently in mathematical analysis.

1.4.8. We will show by induction on n that $\sum_{j=1}^{n} f_{2j} = f_{2n+1} - 1$. When $n=1$, the lefthand side of the putative formula is $f_2 = 1$ while the righthand side is $f_3 - 1 = 2 - 1 = 1$. Thus the formula is valid in this case. We now assume that $\sum_{j=1}^{k} f_{2j} = f_{2k+1} - 1$ for some positive integer $k \geq 1$. It follows that

$$\begin{aligned}\sum_{j=1}^{k+1} f_{2j} &= \sum_{j=1}^{k} f_{2j} + f_{2(k+1)} \\ &= f_{2k+1} - 1 + f_{2(k+1)} \\ &= f_{2k+1} + f_{2k+2} - 1 = f_{2k+3} - 1 = f_{2(k+1)+1} - 1.\end{aligned}$$

Having proved that the validity of the formula for k implies the validity of the formula for $k+1$, we may conclude by induction that the formula is true for all positive integers n.

1.5.14. Since $a \mid b$, there is an integer j such that $b = aj$. Similarly, since $b \mid a$, there is an integer k such that $a = bk$. It follows that $a = bk = (aj)k = a(jk)$, and since $a \neq 0$, we may divide both sides of this equation by a to obtain $jk = 1$. Because j and k are integers which multiply to 1, we must have either $(j, k) = (1, 1)$ or $(j, k) = (-1, -1)$. In the first case, $a = b$, while in the second case, $a = -b$. Thus we conclude that $a = \pm b$.

1.5.16. Yes, there are integers a, b, and c such that $a \mid bc$ but $a \nmid b$ and $a \nmid c$. For example, we can take $a = 6$, $b = 2$, and $c = 3$.

1.5.23. Our assumption is that $a = bq + r$ with $q, r \in \mathbf{Z}$ and $0 \leq r < b$. Denote by q_1 and r_1 the quotient and remainder for division of $-a$ by b. We first consider the case in which $b \nmid a$. This is equivalent to the assumption that $r > 0$. We have

$$b(-(q + 1)) + (b - r) = -bq - b + b - r = -bq - r = -(bq + r) = -a.$$

Moreover, since $r < b$, $b - r > 0$, and since $r > 0$, $b - r < b$. We have shown that $-a = b(-(q + 1)) + (b - r)$ and that $0 < b - r < b$. By the uniqueness in the division algorithm, we may conclude that $q_1 = -(q + 1)$ and that $r_1 = b - r$, as desired.

We now consider the simpler case in which $b \mid a$, which is equivalent to $r = 0$. Thus $a = bq$, so we may multiply both sides by -1 to obtain $-a = -bq = b(-q) = b(-q) + 0$. Once again, the uniqueness in the division algorithm allows us to conclude that $q_1 = -q$ and $r_1 = 0$.

1.5.29. If $x < d$, then there are no positive integers divisible by d which are less than or equal to x. Moreover, since $x/d < 1$, $[x/d] = 0$. Thus the result holds in this case. (It is not strictly necessary to treat this case separately, but if one does not do so, I think the argument becomes somewhat awkward to write.) We will now assume that $x \geq d$. Let K be the largest positive integer such that $Kd \leq x$. Then the positive integers divisible by d which are less than or equal to x are the integers kd, $1 \leq k \leq K$. Thus there are exactly K such integers. To see that $K = [x/d]$, note that, by the maximality of K, we must have $x < (K+1)d$. We therefore have $Kd \leq x < (K+1)d$, and upon dividing throughout by d, we obtain $K \leq x/d < K + 1$. As $[x/d]$ is the unique integer satisfying $[x/d] \leq x/d < [x/d] + 1$, we may conclude that $K = [x/d]$.

1.5.30. For convenience, we will denote the number of positive integers divisible by a positive integer d which are less than or equal to 1000 by $M_{d,1000}$. Applying the result of Exercise 1.5.29, we have

$$\begin{aligned} M_{5,1000} &= [1000/5] = 200, \\ M_{25,1000} &= [1000/25] = 40, \\ M_{125,1000} &= [1000/125] = 8, \text{ and} \\ M_{625,1000} &= [1000/625] = 1. \end{aligned}$$