Solution to Braun, 3.9, Number 4
Solve the equation

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} \vec{x}$$

- The characteristic polynomial of A is

$$p(\lambda) = (1-\lambda)(\lambda^2+1)$$

- The eigenvalues are $\lambda = 1$ and $\lambda = \pm i$.
- For $\lambda = 1$, we find the eigenspace $E_\lambda = E_1 = \ker(A - I_3)$:

$$\text{RREF}(A - I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

gives us equations

$$\begin{array}{rcl} x & = & 0 \\ z & = & 0 \\ y \text{ is free} & & \end{array} \Rightarrow \begin{array}{rcl} x & = & 0 \\ y & = & y \\ z & = & 0 \end{array}$$

so the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so

$$\vec{v}_\lambda = \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } E_1 = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and

$$\vec{x}^{\,1}(t) = e^{\lambda t}\vec{v}_\lambda = e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- For $\lambda = i$, we find the eigenspace $E_\lambda = E_i = \ker(A - iI_3)$:

$$A - iI_3 = \begin{bmatrix} 1-i & 0 & 1 \\ 0 & 1-i & -1 \\ -2 & 0 & -1-i \end{bmatrix}$$

$$\text{RREF}(A - iI_3) = \begin{bmatrix} 1 & 0 & \frac{1}{2}(1+i) \\ 0 & 1 & -\frac{1}{2}(1+i) \\ 0 & 0 & 0 \end{bmatrix}$$

gives us equations

$$\begin{array}{rcl} x + \frac{1}{2}(1+i)z & = & 0 \\ y - \frac{1}{2}(1+i)z & = & 0 \\ z \text{ is free} & & \end{array} \qquad \begin{array}{rcl} x & = & -\frac{1}{2}(1+i)z \\ y & = & \frac{1}{2}(1+i)z \\ z & = & z \end{array}$$

so the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(1+i)z \\ \\ \frac{1}{2}(1+i)z \\ \\ z \end{bmatrix} = z \begin{bmatrix} -\frac{1}{2}(1+i) \\ \\ \frac{1}{2}(1+i) \\ \\ 1 \end{bmatrix}$$

and

$$\vec{v}_i = \begin{bmatrix} -\frac{1}{2}(1+i) \\ \\ \frac{1}{2}(1+i) \\ \\ 1 \end{bmatrix}$$

Now we can construct our complex solution

$$\vec{z}(t) = e^{it}\vec{v}_i = (\cos t + i \sin t) \left(\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \\ \cos t \end{bmatrix} + i \begin{bmatrix} -\frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \\ \sin t \end{bmatrix}$$

giving us the two real solutions

$$\vec{x}^2(t) = \begin{bmatrix} -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \\ \cos t \end{bmatrix} \quad \text{and} \quad \vec{x}^3(t) = \begin{bmatrix} -\frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \\ \sin t \end{bmatrix}$$

- The general solution is

$$\vec{x}(t) = c_1 e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \\ \frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \\ \cos t \end{bmatrix} + c_3 \begin{bmatrix} -\frac{1}{2}\cos t - \frac{1}{2}\sin t \\ \\ \frac{1}{2}\cos t + \frac{1}{2}\sin t \\ \\ \sin t \end{bmatrix}$$

Alternative Question
Solve the equation

$$\frac{d}{dt}\vec{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} \vec{x}$$

using the eigenvectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3-2i \\ 3+2i \\ 5-i \end{bmatrix}$$

- Instead of finding the eigenvalues, and *then* finding associated eigenvectors as bases for kernels, we already have the eigenvectors, so we just need to find the associated eigenvalue for each eigenvector. We do this by taking an eigenvector, $\vec{v}$, computing $A \cdot \vec{v}$, setting it equal to $\lambda\vec{v}$, and solving for λ.

- Using

  $$\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

  we get

  $$A\vec{v} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

  and we can see that the only solution is $\lambda = 1$, giving us our first solution

  $$\vec{x}^{\,1}(t) = e^{\lambda t}\vec{v}_\lambda = e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- Using

  $$\vec{v} = \begin{bmatrix} -3-2i \\ 3+2i \\ 5-i \end{bmatrix}$$

  we get

  $$A\vec{v} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} -3-2i \\ 3+2i \\ 5-i \end{bmatrix} = \begin{bmatrix} 2-3i \\ -2+3i \\ 1+5i \end{bmatrix} = \lambda \begin{bmatrix} -3-2i \\ 3+2i \\ 5-i \end{bmatrix}$$

  so we can find λ by solving any one of the following three equations:

  $$\begin{aligned} 2-3i &= \lambda\,(-3-2i) \\ -2+3i &= \lambda\,(3+2i) \\ 1+5i &= \lambda\,(5-i) \end{aligned}$$

  For example, using $2-3i = \lambda(-3-2i)$ we get

  $$\lambda = \frac{2-3i}{-3-2i} = \frac{(2-3i)(-3+2i)}{(-3-2i)(-3+2i)} = \frac{-6+4i+9i+6}{9+4} = \frac{13i}{13} = i$$

This gives us the eigenvalue i. Now we create our complex solution

$$\vec{z}(t) = e^{it} \begin{bmatrix} -3-2i \\ 3+2i \\ 5-i \end{bmatrix} = (\cos t + i \sin t) \left(\begin{bmatrix} -3 \\ 3 \\ 5 \end{bmatrix} + i \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -3\cos t + 2\sin t \\ 3\cos t - 2\sin t \\ 5\cos t + \sin t \end{bmatrix} + i \begin{bmatrix} -2\cos t - 3\sin t \\ 2\cos t + 3\sin t \\ -\cos t + 5\sin t \end{bmatrix}$$

giving us the two real solutions

$$\vec{x}^2(t) = \begin{bmatrix} -3\cos t + 2\sin t \\ 3\cos t - 2\sin t \\ 5\cos t + \sin t \end{bmatrix} \quad \text{and} \quad \vec{x}^3(t) = \begin{bmatrix} -2\cos t - 3\sin t \\ 2\cos t + 3\sin t \\ -\cos t + 5\sin t \end{bmatrix}$$

- The general solution is

$$\vec{x}(t) = c_1 e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3\cos t + 2\sin t \\ 3\cos t - 2\sin t \\ 5\cos t + \sin t \end{bmatrix} + c_3 \begin{bmatrix} -2\cos t - 3\sin t \\ 2\cos t + 3\sin t \\ -\cos t + 5\sin t \end{bmatrix}$$