Heat Equation and Sine Series

- There are three big equations in the world of second-order partial differential equations:

  1. The Heat Equation:
  $$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
  2. The Wave Equation:
  $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$
  3. Laplace's Equation (The Potential Equation):
  $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- We're going to focus on the heat equation, in particular, a boundary value problem involving the heat equation: Find $u(x,t)$ if
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < l$$
$$u(0,t) = u(l,t) = 0$$

- If $f(x)$ is any finite linear combination of functions of the form
$$f(x) = \sum_{n=1}^{N} b_n \sin\frac{n\pi x}{l}$$
then the function
$$u(x,t) = \sum_{n=1}^{N} \left(b_n \sin\frac{n\pi x}{l} \cdot e^{-\alpha^2 n^2 \pi^2 t/l^2} \right)$$
solves the heat equation.

- **IN FACT**, this is also true if $f(x)$ is an infinite series of such functions. If
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$
then the function
$$u(x,t) = \sum_{n=1}^{\infty} \left(b_n \sin\frac{n\pi x}{l} \cdot e^{-\alpha^2 n^2 \pi^2 t/l^2} \right)$$
solves the heat equation.

- Heat Equation:

  To solve the Heat Equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x),\ 0 < x < l$$

$$u(0,t) = u(l,t) = 0$$

  where f and f' are piecewise continuous on the interval $0 \leq x \leq l$, we compute the Sine Series for $f(x)$:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

  where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx, \qquad n = 1, 2, ...$$

  Then $u(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[b_n \cdot \sin \frac{n\pi x}{l} \cdot e^{-\left(\frac{\alpha n\pi}{l}\right)^2 t} \right]$$

- **NOTE:**

  Of the examples below, 2, 5, and 6 are the ones we will be working with. I left the others up... because the pictures are cool.

Links to converging Fourier Series:

1. `https://www.desmos.com/calculator/auavzwptdr`
   Braun, Section 5.4, Example 1
   Fourier Series
   $$f(x) = \begin{cases} 0 & -1 \leq x < 0 \\ 1 & 0 \leq x < 1 \end{cases}$$
2. `https://www.desmos.com/calculator/43jvcy9w6l`
   Braun, Section 5.5, Example 6
   Sine Series
   $$f(x) = \begin{cases} -e^{-x} & -1 < x < 0 \\ e^{x} & 0 < x < 1 \end{cases}$$
3. `https://www.desmos.com/calculator/wo0xwqagla`
   Braun, Section 5.4, Exercise 9
   Cosine Series
   $$f(x) = \begin{cases} e^{-x} & -1 < x < 0 \\ e^{x} & 0 < x < 1 \end{cases}$$
4. `https://www.desmos.com/calculator/mvwdtjfjyd`
   Braun, Section 5.4, Exercise 10
   Fourier Series
   $$f(x) = \left\{ \begin{array}{ll} e^{x} & -l < x < l \end{array} \right.$$
5. `https://www.desmos.com/calculator/rbjvydrnbg`
   Similar to
   Braun, Section 5.6, Example 1
   Braun, Section 5.6, Exercise 1 (a)
   Fourier Series AND Heat Distribution
   $$f(x) = \left\{ \begin{array}{ll} A & 0 < x < l \end{array} \right.$$
6. `https://www.desmos.com/calculator/epladkiwoe`
   Fourier Series AND Heat Distribution
   $$f(x) = \left\{ \begin{array}{ll} x & 0 \leq x < 1 \end{array} \right.$$

Homework:

1. Braun, Section 5.3, Numbers 1 and 2
   (See Braun, Section 5.3, Example 1)

2. Braun, Section 5.5, Numbers 7, 8, and 10.

   (a) For number 8, let $a = 1$.

   (b) For number 10, let $l = 1$.

   If you're bored, go back and do number 8 with $a = 2$ and number 10 with $l = 2$.
   Now you're twice as bored.

3. Braun, Section 5.6, Number 1 parts a, c, and d.

4. Now go back and use your Sine Series from the homework in Section 5.5 and do those like Heat Equation problems in 5.6.