Heat Equation and Fourier Series

- There are three big equations in the world of second-order partial differential equations:

  1. The Heat Equation:
  $$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
  2. The Wave Equation:
  $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$
  3. Laplace's Equation (The Potential Equation):
  $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

- We're going to focus on the heat equation, in particular, a boundary value problem involving the heat equation:
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = f(x),\ 0 < x < l$$
$$u(0,t) = u(l,t) = 0$$

- We'll start by solving the boundary value problem
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(0,t) = u(l,t) = 0$$

- To solve this equation we do something called *separation of variables*, which means we make a pretty big assumption:
$$u(x,t) = X(x)T(t)$$
In that case
$$\frac{\partial u}{\partial t} = X(x)T'(t), \text{ and } \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$
$$\Rightarrow \quad X(x)T'(t) = \alpha^2 X''(x)T(t) \quad \Rightarrow \quad \frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha^2 T(t)} = -\lambda$$
giving us two ordinary (not partial) differential equations:
$$X''(x) + \lambda X(x) = 0,\ X(0) = 0,\ X(l) = 0, \text{ and}$$
$$T'(t) + \alpha^2 \lambda T(t) = 0$$

- We saw in 5.1, example 1, that the solutions to the first equation

$$X''(x) + \lambda X(x) = 0,\ X(0) = 0,\ X(l) = 0, \text{ and}$$

come as pairs of eigenvalues and eigenfunctions:

For $n = 1, 2, 3, \ldots$ we have

$$\text{eigenvalue } \lambda_n = \frac{n^2\pi^2}{l^2} \text{ with solution eigenfunction } X_n(x) = \sin\left(\frac{n\pi}{l}x\right) = \sin\left(\sqrt{\lambda_n}\, x\right)$$

- For each of these eigenvalues λ_n, we get another ordinary differential equation

$$T'(t) + \alpha^2 \lambda_n T(t) = 0$$

which we solved in chapter 1:

$$\text{For } \lambda_n = \frac{n^2\pi^2}{l^2}, \text{ we get solution } T_n(t) = e^{-\frac{\alpha^2 n^2 \pi^2}{l^2}t} = e^{-\alpha^2 \lambda_n t}$$

- Thus, for each eigenvalue

$$\lambda_n = \frac{n^2\pi^2}{l^2},$$

we get solution

$$u_n(x,t) = X_n(x)T_n(t) = \sin\frac{n\pi x}{l} e^{-\alpha^2 n^2 \pi^2 t / l^2}$$

- For each of these solutions, we get

$$f(x) = u_n(x,0) = X_n(x)T_n(0) = \sin\frac{n\pi x}{l}$$

- Now, we're not limited to solving BVP's of the form

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = \sin\frac{n\pi x}{l},\ 0 < x < l$$

$$u(0,t) = u(l,t) = 0$$

(which, btw, would have solution $u_n(x,t)$ above).
Why?
LINEARITY.

- If $f(x)$ is any linear combination of functions of the form $X_n(x)$, i.e.

$$f(x) = \sum c_n \sin\frac{n\pi x}{l}$$

then the function

$$u(x,t) = \sum c_n \sin\frac{n\pi x}{l} e^{-\alpha^2 n^2 \pi^2 t / l^2}$$

solves the heat equation.

- This begs the question: which functions $f(x)$ can be written as a sum (or series!) of these funny sine functions?
  The answer: A LOT.
  How do we know which functions f can be written this way, and how do we find these coefficients c_n?
  Answer: Fourier Series, 5.4, and the c_n are called Fourier coefficients.

- Fourier Series: Let f and f' be piecewise continuous on the interval $-l \le x \le l$. Compute the numbers

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \qquad n = 0, 1, 2, ...$$

  and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx, \qquad n = 1, 2, ...$$

  then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right]$$

  and this is called the Fourier Series for f.

- Even and odd functions:
  - ⋆ A function $f(x)$ is *odd* if $f(-x) = -f(x)$ (like $\sin x$ and x^3).
  - ⋆ A function $f(x)$ is *even* if $f(-x) = f(x)$ (like $\cos x$ and x^4).
  - ⋆ If a function $f(x)$ is *odd*, its Fourier Series will consist of only sine functions.
  - ⋆ If a function $f(x)$ is *even*, its Fourier Series will consist of only cosine functions.
  - ⋆ If a function $f(x)$ is only defined on an interval $0 \le x \le l$, then it can be extended to the left (on $-l \le x \le 0$) so that it is *even* or *odd*. This gives the following expression of $f(x)$ on $0 \le x \le l$ as either a pure Sine Series or a pure Cosine Series.

- Fourier Series on a bar of length l: Let f and f' be piecewise continuous on the interval $0 \le x \le l$. Then, on this interval, $f(x)$ can be expanded in either a pure cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

  where

$$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \qquad n = 0, 1, 2, ...$$

  OR a pure sine series

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

  where

$$b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\frac{n\pi x}{l}\,dx, \qquad n = 1, 2, ...$$

- Heat Equation:

  To solve the Heat Equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

$$u(x,0) = f(x),\ 0 < x < l$$

$$u(0,t) = u(l,t) = 0$$

  where f and f' are piecewise continuous on the interval $0 \leq x \leq l$, we compute the Sine Series for $f(x)$:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

  where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx, \qquad n = 1, 2, ...$$

  Then $u(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[b_n \cdot \sin \frac{n\pi x}{l} \cdot e^{-\left(\frac{\alpha n \pi}{l}\right)^2 t} \right]$$

Links to converging Fourier Series:

1. https://www.desmos.com/calculator/auavzwptdr

   Braun, Section 5.4, Example 1

   Fourier Series

$$f(x) = \begin{cases} 0 & -1 \leq x < 0 \\ 1 & 0 \leq x < 1 \end{cases}$$

2. https://www.desmos.com/calculator/43jvcy9w6l

   Braun, Section 5.5, Example 6

   Sine Series

$$f(x) = \begin{cases} -e^{-x} & -1 < x < 0 \\ e^{x} & 0 < x < 1 \end{cases}$$

3. https://www.desmos.com/calculator/wo0xwqagla

   Braun, Section 5.4, Exercise 9

   Cosine Series

$$f(x) = \begin{cases} e^{-x} & -1 < x < 0 \\ e^{x} & 0 < x < 1 \end{cases}$$

4. https://www.desmos.com/calculator/mvwdtjfjyd

   Braun, Section 5.4, Exercise 10

   Fourier Series

$$f(x) = \left\{ \begin{array}{ll} e^{x} & -l < x < l \end{array} \right.$$

5. https://www.desmos.com/calculator/rbjvydrnbg

   Similar to

   Braun, Section 5.6, Example 1

   Braun, Section 5.6, Exercise 1 (a)

   Fourier Series AND Heat Distribution

$$f(x) = \left\{ \begin{array}{ll} A & 0 < x < l \end{array} \right.$$

6. https://www.desmos.com/calculator/epladkiwoe

   Fourier Series AND Heat Distribution

$$f(x) = \left\{ \begin{array}{ll} x & 0 \leq x < 1 \end{array} \right.$$