Arrow Diagrams and Generalized Eigen Vectors Part 2

Example 1

Consider the system $\frac{d}{dt}\vec{x} = A\vec{x}$ where A has exactly one eigenvalue, $\lambda = -1$,

$$A = \begin{bmatrix} -3 & 1 & -1 & 2 & 2 & 4 \\ -3 & 2 & -3 & 3 & 3 & 6 \\ -3 & 3 & -4 & 3 & 3 & 6 \\ 1 & 0 & 0 & -2 & -1 & -2 \\ -1 & -1 & 1 & 1 & 0 & 2 \\ -1 & 1 & -1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathrm{RREF}(A + I_6) = \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & -2 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the standard basis of E_{-1}?
(b) What is the homogeneous system obtained while answering part (c)?
(c) Give the solution(s) generated by the standard vector(s) $J \in \ker((A - \lambda I)^2) \setminus \ker(A - \lambda I)$.
(d) What is the arrow diagram for the system?

Solution:

(a)

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

For parts (b) and (c), we want to solve $(A + I_6)\vec{J} \in E_{-1}$:

$$\left[\begin{array}{cccccc|c} -2 & 1 & -1 & 2 & 2 & 4 & \beta+\gamma+2\delta \\ -3 & 3 & -3 & 3 & 3 & 6 & \alpha \\ -3 & 3 & -3 & 3 & 3 & 6 & \alpha \\ 1 & 0 & 0 & -1 & -1 & -2 & \beta \\ -1 & -1 & 1 & 1 & 1 & 2 & \gamma \\ -1 & 1 & -1 & 1 & 1 & 2 & \delta \end{array}\right] \quad \begin{cases} R_3 \to R_3 - R_2 \\ R_5 \to R_5 + R_4 \\ R_6 \to R_6 + R_4 \end{cases}$$

$$\left[\begin{array}{cccccc|c} -2 & 1 & -1 & 2 & 2 & 4 & \beta+\gamma+2\delta \\ -3 & 3 & -3 & 3 & 3 & 6 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & -2 & \beta \\ 0 & -1 & 1 & 0 & 0 & 0 & \beta+\gamma \\ 0 & 1 & -1 & 0 & 0 & 0 & \beta+\delta \end{array}\right] \quad \begin{cases} R_1 \to R_1 + R_5 \\ R_2 \to R_2 + 3R_5 \\ R_5 \to R_5 + R_6 \end{cases}$$

$$\left[\begin{array}{cccccc|c} -2 & 0 & 0 & 2 & 2 & 4 & 2\beta+2\gamma+2\delta \\ -3 & 0 & 0 & 3 & 3 & 6 & \alpha+3\beta+3\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & -2 & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\beta+\gamma+\delta \\ 0 & 1 & -1 & 0 & 0 & 0 & \beta+\delta \end{array}\right] \quad \begin{cases} R_1 \to R_1 + 2R_4 \\ R_2 \to R_2 + 3R_4 \end{cases}$$

$$\left[\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 4\beta+2\gamma+2\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha+6\beta+3\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & -1 & -2 & \beta \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\beta+\gamma+\delta \\ 0 & 1 & -1 & 0 & 0 & 0 & \beta+\delta \end{array}\right] \quad \{ \text{ swap rows}$$

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & -1 & -1 & -2 & \beta \\ 0 & 1 & -1 & 0 & 0 & 0 & \beta+\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & 4\beta+2\gamma+2\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha+6\beta+3\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\beta+\gamma+\delta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The solution to part (b) is

$$\begin{aligned} 4\beta+2\gamma+2\delta &= 0 \\ \alpha+6\beta+3\gamma &= 0 \\ 2\beta+\gamma+\delta &= 0 \end{aligned}$$

Now we find the solutions to this homogeneous system in parametric form:

$$\begin{bmatrix} 0 & 4 & 2 & 2 \\ 1 & 6 & 3 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{-1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ \frac{-1}{2} \\ 0 \\ 1 \end{bmatrix}$$

For $\alpha = 0, \beta = \frac{-1}{2}, \gamma = 1, \delta = 0$ we have

$$\vec{J}^1 = \begin{bmatrix} -1/2 \\ -1/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}^{\,1}_{-1} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ -1/2 \\ 1 \\ 0 \end{bmatrix}$$

For $\alpha = 3, \beta = \frac{-1}{2}, \gamma = 0, \delta = 1$ we have

$$\vec{J}^1 = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{v}^{\,1}_{-1} = \begin{bmatrix} 3/2 \\ 3 \\ 3 \\ -1/2 \\ 0 \\ 1 \end{bmatrix}$$

So the solution to part (c) is

$$\vec{x}^1(t) = e^{-t} \begin{bmatrix} \frac{-1}{2} + \frac{1}{2} \cdot t \\ \frac{-1}{2} \\ 0 \\ \frac{-1}{2}t \\ t \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{x}^2(t) = e^{-t} \begin{bmatrix} \frac{-1}{2} + \frac{3}{2} \cdot t \\ \frac{1}{2} + 3t \\ 3t \\ \frac{-1}{2}t \\ 0 \\ t \end{bmatrix}$$

The solution to part (d) is

$$-1 : \left\{ \begin{array}{cccc} & & \leftarrow & \leftarrow \\ \leftarrow & \leftarrow & \leftarrow & \leftarrow \end{array} \right.$$

Example 2

Consider the system $\frac{d}{dt}\vec{x} = A\vec{x}$ where A has exactly one eigenvalue, $\lambda = 2$,

$$A = \begin{bmatrix} -1 & 3 & 3 & -2 & -11 \\ -2 & 4 & 2 & 1 & -5 \\ -1 & 1 & 3 & -6 & -9 \\ 1 & -1 & -1 & 3 & 4 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad \text{RREF}(A - 2I_5) = \begin{bmatrix} 1 & -1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the standard basis of E_2?
(b) What is the homogeneous system obtained while answering part (c)?
(c) Give the solution(s) generated by the standard vector(s) $J \in \ker((A - \lambda I)^2) \setminus \ker(A - \lambda I)$.
(d) What is the arrow diagram for the system?

Solution:

(a)

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

For parts (b) and (c), we want to solve $(A - 2I_5)\vec{J} \in E_2$:

$$\left[\begin{array}{ccccc|c} -3 & 3 & 3 & -2 & -11 & \alpha + \beta - 3\gamma \\ -2 & 2 & 2 & 1 & -5 & \alpha \\ -1 & 1 & 1 & -6 & -9 & \beta \\ 1 & -1 & -1 & 1 & 4 & -\gamma \\ 0 & 0 & 0 & -1 & -1 & \gamma \end{array}\right] \qquad \left\{ \begin{array}{c} R_1 \to R_1 + 3R_4 \\ R_2 \to R_2 + 2R_4 \\ R_3 \to R_3 + R_4 \end{array} \right.$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 1 & 1 & \alpha + \beta - 6\gamma \\ 0 & 0 & 0 & 3 & 3 & \alpha - 2\gamma \\ 0 & 0 & 0 & -5 & -5 & \beta - \gamma \\ 1 & -1 & -1 & 1 & 4 & -\gamma \\ 0 & 0 & 0 & -1 & -1 & \gamma \end{array}\right] \qquad \left\{ \begin{array}{c} R_1 \to R_1 + R_5 \\ R_2 \to R_2 + 3R_5 \\ R_3 \to R_3 - 5R_5 \\ R_4 \to R_4 + R_5 \\ R_5 \to -R_5 \end{array} \right.$$

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 0 & \alpha + \beta - 5\gamma \\ 0 & 0 & 0 & 0 & 0 & \alpha + \gamma \\ 0 & 0 & 0 & 0 & 0 & \beta - 6\gamma \\ 1 & -1 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & -\gamma \end{array}\right] \qquad \{ \text{ swap rows}$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & -1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 & -\gamma \\ 0 & 0 & 0 & 0 & 0 & \alpha + \beta - 5\gamma \\ 0 & 0 & 0 & 0 & 0 & \alpha + \gamma \\ 0 & 0 & 0 & 0 & 0 & \beta - 6\gamma \end{array}\right]$$

The solution to part (b) is

$$\begin{aligned} \alpha + \beta - 5\gamma &= 0 \\ \alpha + \gamma &= 0 \\ \beta - 6\gamma &= 0 \end{aligned}$$

Now we find the solutions to this homogeneous system in parametric form:

$$\begin{bmatrix} 1 & 1 & -5 \\ 1 & 0 & 1 \\ 0 & 1 & -6 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \\ 1 \end{bmatrix}$$

For $\alpha = -1, \beta = 6, \gamma = 1$, we have

$$\vec{J} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 6 \\ -1 \\ 1 \end{bmatrix}$$

So the solution to part (c) is

$$\vec{x}(t) = e^{2t} \begin{bmatrix} 2t \\ -t \\ 6t \\ -1-t \\ t \end{bmatrix}$$

The solution to part (d) is

$$-1 : \begin{cases} & & \leftarrow \\ & & \leftarrow \\ \leftarrow & \leftarrow & \leftarrow \end{cases}$$