[Unofficial] UT Austin Algebraic Topology Past Prelim Exam Solutions

Compiled by Jemma Schroder

If you find any errors, please email me at jemma@utexas.edu so I may correct them. Claude and ChatGPT were used to help typeset handwritten solution and verify human generated solutions.

Contents

ALGEBRAIC TOPOLOGY PRELIM EXAM

Do all three problems on both sides of this paper. You have two hours. If you have any questions, please ask.

(1) Let X and Y be path-connected CW complexes with base points $*_x$ and $*_y$. Let $\tilde{Y}$ be a cover of Y with projection map $p : \tilde{Y} \to Y$ and base point $*_{\tilde{y}} \in p^{-1}(*_y)$. Let $H = p_*(\pi_1(\tilde{Y}, *_{\tilde{y}}))$ be the subgroup of $\pi_1(Y, *_y)$ associated with this cover. Prove that a continuous based map $f : (X, *_x) \to (Y, *_y)$ lifts to a continuous based map $\tilde{f} : (X, *_x) \to (\tilde{Y}, *_{\tilde{y}})$ if and only if $f_*(\pi_1(X, *_x)) \subset H$.

You may take the following two facts as given but should not assume any more advanced results:

- Every continuous path in Y starting at $*_y$ lifts to a unique continuous path in $\tilde{Y}$ starting at $*_{\tilde{y}}$.
- The (continuous) lift $\tilde{\gamma}$ of a (continuous) closed loop γ starting and ending at $*_y$ is a closed loop starting and ending at $*_{\tilde{y}}$ if and only if γ represents an element of H.

Note: The assumption that X and Y are CW complexes is overkill. All we really need is for X to be locally path-connected.

(2) As you surely know from reviewing recent prelims, the Hawaiian earring H is the union in $\mathbb{R}^2$ of the circles of radius $1/n$ centered at $(1/n, 0)$, where n ranges over the natural numbers $\mathbb{N}$. Let X be the wedge of countably many circles: $X = S^1 \times \mathbb{N}/\sim$, where a specific point on each circle (say, $\theta = 0$) is identified with the corresponding point on every other circle.

(a) Show that H does not have a universal cover.
(b) Construct a universal cover of X.

Taken together, these give yet another proof that H and X are not homotopy equivalent.

Date: January 2026.

(3) Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. Let $f : T^2 \to T^2$ be counterclockwise rotation by 90 degrees. Let $X = T^2 \times [0,1]/\sim$, where $(x,1) \sim (f(x),0)$. (This is often called a *mapping cylinder* or the *suspension* of the map f. Hatcher calls it a *mapping torus*.)

(a) Use Mayer-Vietoris to compute $H_1(X)$, $H_2(X)$ and $H_3(X)$. [Hint: It is convenient to take $U = T^2 \times (0.1, 0.9)$ and V to be the complement of $T^2 \times [0.2, 0.8]$. To get the right answer, you will need to track the actual maps in the MV sequence, not just the ranks of the groups.]

(b) Use van Kampen to obtain a presentation of $\pi_1(X)$. [Hint: Since $U \cap V$ is required to be connected, the sets you used for part (a) won't work. Instead, adjoin the circle $O \times [0,1]$ to both U and V, where O is the origin in $\mathbb{R}^2/\mathbb{Z}^2$.]

(c) If parts (a) and (b) are too difficult, you can get **half credit** by replacing T^2 with the circle $\mathbb{R}/\mathbb{Z}$ and f with the reflection $f(x) = -x$ and by computing H_1, H_2 and π_1 of the resulting space. To receive credit, you must obtain your answers with Mayer-Vietoris and van Kampen, not by recognizing the resulting space and using other methods. (You can do parts (a) and (b) **or** you can do part (c), but you should not submit work for both. If you want us to grade part (c), make this clear by crossing out any work on parts (a) and (b).)

January 2026

Solution 1.

$(\implies)$ The forwards direction is trivial: Assume f lifts to $\widetilde{f}$. Then $f = p \circ \widetilde{f} \implies f_* = p_* \circ \widetilde{f}_* \implies \text{im}(f_*) \subset \text{im}(p_*) = H$ $\square$

$(\impliedby)$ Assume $f_*(\pi_1(X, *_x)) \subset H$. Then every based loop in X must be sent to a based loop in Y which lifts to a loop in $\widetilde{Y}$. We use these lifts to construct a lift of f.

Take $x \in X$. Since X is path connected, $\exists \gamma : I \to X$ st $\gamma(0) = *_x$, $\gamma(1) = x$.

Then $f \circ \gamma$ is a path from $f(*_x) = *_y$ to $f(x)$ in Y, which lifts to a unique path $\widetilde{\gamma}$ by assumption 1.

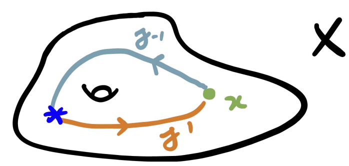
Define $\widetilde{f} : X \to Y$

$$x \mapsto \text{lift of } f(x) \text{ on } \widetilde{\gamma}$$

We show $\widetilde{f}$ is well-defined & continuous.

Well-defined: Let γ' be another path $\gamma' : I \to X$. We must show that $\widetilde{f}_\gamma = \widetilde{f}_{\gamma'}$.

Clearly $\gamma' \simeq \gamma'\gamma^{-1}\gamma$, and note that $\gamma'\gamma^{-1}$ forms a closed loop in X, thus $[\gamma'\gamma^{-1}] \in \pi_1(X, *_x)$. By assumption, $f_*([\gamma'\gamma^{-1}]) \in H \subset \pi_1(Y, *_y)$, so $f \circ \gamma'\gamma^{-1}$ lifts to a loop in $\widetilde{Y}$ by assumption 2. Thus the endpoint of the lift of $f \circ \gamma'\gamma^{-1}\gamma$ is the same as that for $f \circ \gamma$.

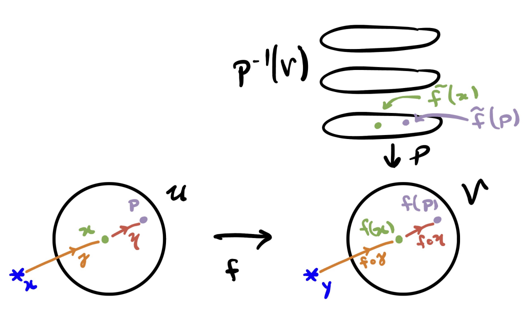


To show that $f \circ \gamma'$ has this same endpoint, note that a homotopy of paths in X gives a homotopy of lifted paths in $\widetilde{Y}$.

Since the lifts of $f(x)$ is a discrete set, the homotopy must preserve the endpoint.

Continuity: Take $U \subset X$ a nbhd of x.

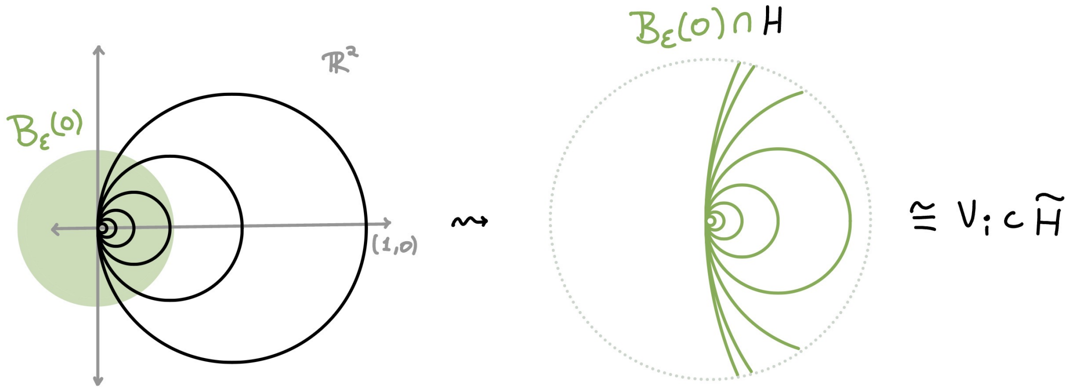
Since f and γ are continuous, $f \circ \gamma$ is too, so we must show that lifting $f \circ \gamma(x)$ is continuous, which can only fail if the lifts of $x, p \in U$ are contained in different components of $p^{-1}(f(U))$.



Since X is locally path connected, every point $p \in U$ has a path $\eta : I \to U$ st $\eta(0) = x, \eta(1) = p$. Thus the path γ for each $p \in U$ can be chosen to be the concatenation $\gamma\eta$, and by choosing U smaller if necessary, $V = f(U)$ is evenly covered by p, thus the lift of V is completely determined by the lift of x, so $\widetilde{f}(U)$ is contained in the same component of $p^{-1}(V)$ as $f(x)$. $\square$

Solution 2.

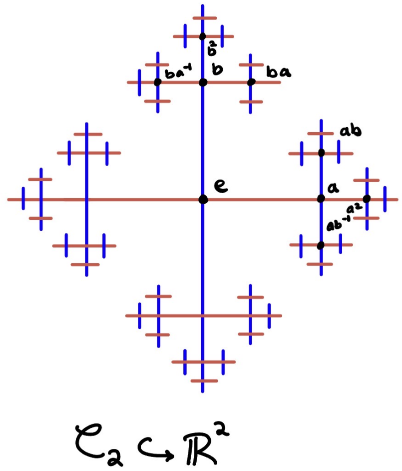
a) Recall that $H \subset \mathbb{R}^2$ inherits the subspace topology. Hence every open set in H is of the form $U \cap H$, for $U \subset \mathbb{R}^2$ an open set. Further note that for every $\varepsilon > 0$, the open epsilon ball about the origin $B_\varepsilon(0) \subset \mathbb{R}^2$ contains infinitely-many π_1 essential loops of H, one for each $n \in \mathbb{N}$ where $n > 1/\varepsilon$. Therefore if H had a universal cover $\widetilde{H}$, then there exists a neighborhood $U \in H$ of $(0,0)$ such that $p^{-1}(U) = \bigcup_{i \in I} V_i$ and V_i is homeomorphic to U $\forall i$. Since U must contain π_1-essential loops, each V_i must too. Since H contains no 2-cells killing those loops, $\widetilde{H}$ can't either, thus $\pi_1(\widetilde{H}) \neq 1$, so $\widetilde{H}$ cannot be a universal cover. $\square$

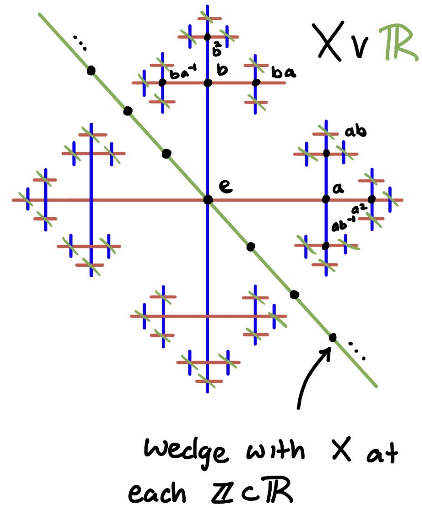


b) Recall that the universal cover of $S^1 \bigvee S^1$ is the infinite Cayley graph on 2 generators, which we denote by $\mathcal{C}_2$. Thus the universal cover of $\bigvee_{i=1}^{\infty} S^1$ is the ∞-Cayley graph on ∞-many generators; we call it $\mathcal{C}_\infty$. $\mathcal{C}_\infty$ is constructed inductively as follows:

- $\mathcal{C}_1$ = universal cover of $S^1 = \mathbb{R}$, with covering map $p(x) \mapsto x \bmod 1$
- $\mathcal{C}_2$: at every integer in $\mathcal{C}_1$, take the wedge with $\mathbb{R}$. Then, at every integer in each new $\mathbb{R}$, attach another copy of $\mathbb{R}$, and do this ∞-many times. The resulting space is $\mathcal{C}_2$, the universal cover of $\bigvee_{i=1}^{n} S^1$.
- $\mathcal{C}_{n+1}$: at each integer value of each $\mathbb{R}$ in $\mathcal{C}_n$, except for the point representing e, wedge with another copy of $\mathbb{R}$. Call this X. The identity point e can be thought of as the point $\underbrace{(0,0,\ldots,0)}_{n} \in \mathbb{R}^n$ via the n-dimensional analog of the embedding $\mathcal{C}_2 \hookrightarrow \mathbb{R}^2$ shown.

  Now, wedge with another $\mathbb{R}$ at e, and for all $p \in \mathbb{Z} \subset \mathbb{R}$, wedge with X.
- Do this process infinitely many times to get $\mathcal{C}\infty$. This infinite Cayley graph is clearly contractible, since we may contract each $\mathbb{R}$ to the point at which it was wedged on, and we may do these contractions at the same speed in $\mathcal{C}_\infty$ itself, before we embed in $\mathbb{R}^\infty$. Therefore $\pi_1(\mathcal{C}_\infty, 0) \cong 1$. The covering map $p_\infty : \mathcal{C}_\infty \to \bigvee_{i=1}^{\infty} S^1$ is defined inductively as. Send C_1 to the $S^1_1 \subset \bigvee_{i=1}^{\infty} S^1$ by the usual covering map $p_1 : \mathbb{R} \to S^1$. Therefore $\mathbb{Z} \subset \mathbb{R} \cong \mathcal{C}_1$ is sent to the wedge point p. Extend the map p_1 to a map $p_2 : C_2 \to \bigvee_{i=1}^{\infty} S^1$ by sending each new $\mathbb{R}$ factor to $S^1_2 \subset \bigvee_{i=1}^{\infty} S^1$ via

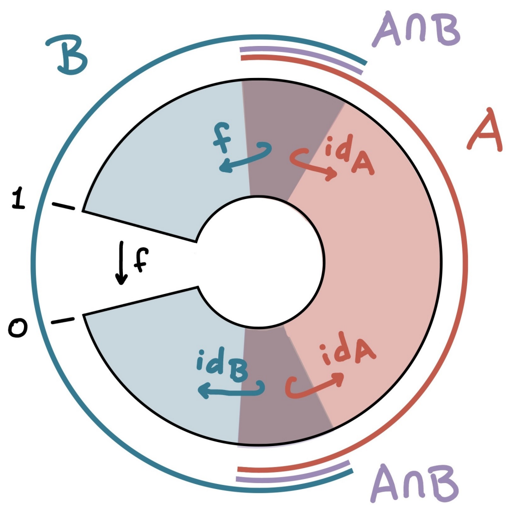




the usual covering map. Notice that every wedge point in each of the $\mathbb{R}$ factors is sent to p, and each wedge point corresponds to a word in ${}_2$. Doing this for all n, we get a map $p_\infty : \mathcal{C}_\infty \to \bigvee_{i=1}^{\infty} S^1$, and p has a unique lift to $\mathcal{C}_\infty$ for each word F_∞. $\square$

Solution 3.

a) Decompose $X = A \cup B$, where $A = T^2 \times (0.2, 0.8)$ and $B = X \setminus T^2 \times [0.3, 0.7]$



Note that $A \cap B = T^2 \times (0.2, 0.3) \cup T^2 \times (0.8, 0.9)$.
Therefore $B \simeq_{\text{h.e.}} A \simeq_{\text{h.e.}} T^2$ and $A \cap B \simeq_{\text{h.e.}} T^2 \sqcup T^2$, so we know that

$$H_n(A) \cong H_n(B) \cong \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

$$H_n(A \cap B) \cong H_n(T^2 \sqcup T^2) \cong \begin{cases} \mathbb{Z}^{\oplus 2} & n = 0, 2 \\ (\mathbb{Z} \oplus \mathbb{Z})^{\oplus 2}, & n = 1 \\ 0, & \text{else} \end{cases}$$

Since X is connected and no component has cells of dimension ≥ 3, we know that $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$ and $H_n(X; \mathbb{Z}) \cong 0$ for all $n \geq 4$.

We now apply Mayer-Vietoris:

$$H_3(A \cap B) \xrightarrow{\iota_3} H_3(A) \oplus H_3(B) \xrightarrow{j_3} H_3(X) \xrightarrow{\partial_3} H_2(A \cap B) \xrightarrow{\iota_2} H_2(A) \oplus H_2(B) \xrightarrow{j_2} H_2(X)$$

$$\xrightarrow{\partial_2} H_1(A \cap B) \xrightarrow{\iota_1} H_1(A) \oplus H_1(B) \xrightarrow{j_1} H_1(X) \xrightarrow{\partial_1} H_0(A \cap B) \xrightarrow{\iota_0} H_0(A) \oplus H_0(B) \xrightarrow{j_0} H_0(X) \to 0$$

Which we fill in as:

$$0 \oplus 0 \xrightarrow{\iota_3} 0 \oplus 0 \xrightarrow{j_3} H_3 \xrightarrow{\partial_3} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\iota_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_2} H_2$$

$$\xrightarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\iota_1} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_1} H_1 \xrightarrow{\partial_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\iota_0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_0} H_0 \to 0$$

We now compute ι_0 and ι_2. Since the inclusions $T^2 \sqcup T^2 \hookrightarrow A$, $T^2 \sqcup T^2 \hookrightarrow B$ send the 2-cells to the 2 cells at each end,

$$\iota_2 : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$
$$(1, 0) \longmapsto (1, 1)$$
$$(0, 1) \longmapsto (1, 1)$$

and the associated matrix is

$$\iota_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ which has}$$

$$\det = 0, \text{ so } \ker \iota_2 = \left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\rangle \cong \mathbb{Z},$$

$$\text{and } \operatorname{im}\iota_1 = \left\langle \begin{pmatrix}1\\1\end{pmatrix} \right\rangle \cong \mathbb{Z}\,.$$

Note this is the same matrix as the one for ι_0, so ι_0 has the same image and kernel as ι_2.

We can now compute H_3 by examining the subsequence

$$0 \oplus 0 \xrightarrow{j_3} H_3 \xrightarrow{\partial_3} \partial_3\, \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\iota_2} \mathbb{Z} \oplus \mathbb{Z}$$

Since $\mathbb{Z} = \ker \iota_2 = \operatorname{im} \partial_3 \implies \mathbb{Z} \subset H_3$ and
$0 = \operatorname{im} j_3 = \ker \partial_3$, $H_3 \cong \mathbb{Z}$.

To find H_2 and H_1, we need to compute ι_1.

The inclusions of the 1-cells of $T^2 \sqcup T^2$ are all the identity except the inclusion $T^2 \hookrightarrow B$, which is given by f: $f(a) = b$, $f(b) = -a$

Thus

$$\iota_1 : \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$1, 0,\ 0, 0 \mapsto 1, 0,\ 1, 0$$

$$0, 1,\ 0, 0 \mapsto 0, 1,\ 0, 1$$

$$0, 0,\ 1, 0 \mapsto 1, 0\ \ 0, 1$$

$$0, 0,\ 0, 1 \mapsto 0, 1\ \ -1, 0$$

$$\text{and the matrix for } \iota_1 \text{ is } \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix} = \left(\begin{array}{c|c} I_2 & I_2 \\ \hline I_2 & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{array} \right)$$

which has determinant 2.

Therefore $\ker \iota_1 = 0$, $\operatorname{im} \iota_1$ is an index 2 subgroup of $\mathbb{Z}^4$

We can now compute H_2:

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\iota_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_2} H_2 \xrightarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\iota_1} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

$$\operatorname{im} j_2 = \mathbb{Z} = \ker \partial_2 \implies \mathbb{Z} \subset H^2$$

$$\operatorname{im} \partial_2 = \ker \iota_1 = \{0\} \implies H^2 \cong \mathbb{Z}$$

and H_1:

$$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\iota_1} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_1} H_1 \xrightarrow{\partial_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\iota_0} \mathbb{Z} \oplus \mathbb{Z}$$

$\operatorname{im} \iota_1 = \text{index 2 subgp of } \mathbb{Z}^4 = \ker j_1 \implies \dfrac{\mathbb{Z}^4}{\text{subgp}} \cong \mathbb{Z}/2 \subset H_1$

$\operatorname{im} \partial_1 = \ker \iota_0 = \mathbb{Z} \implies \mathbb{Z} \subset H_1$

$\implies H_1 \cong \mathbb{Z} \oplus \mathbb{Z}/2$

Thus

$$H_n(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n = 0, 2, 3 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & n = 1 \\ 0, & \text{else.} \end{cases} \qquad \square$$

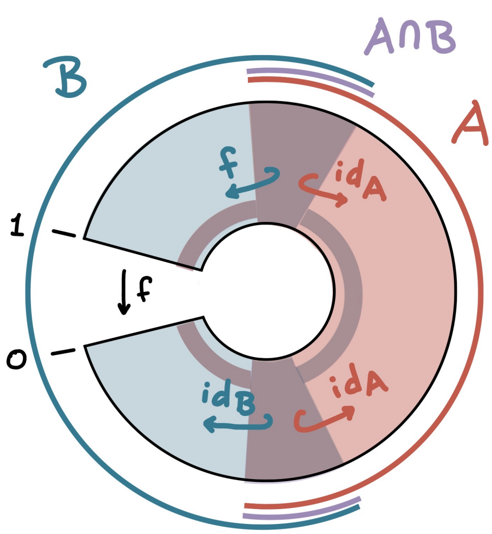
b) We decompose X as $A \cup B$, as before. However, note that $A \cap B$ is not path connected, so we cannot use Seifert-Van-Kampen with this decomposition. Instead, we must adjoin the circle $(0,0) \times S^1$ to A and B, which still gives $X = A \cup B$, and we instead have that

$$A = T^2 \times (0.2, 0.8) \cup (0,0) \times S^1 \simeq_{\text{h.e.}} T^2 \vee S^1$$
$$B = X \setminus T^2 \times [0.3, 0.7] \cup (0,0) \times S^1 \simeq_{\text{h.e.}} T^2 \vee S^1$$

$$A \cap B = T^2 \times (0.2, 0.3) \cup T^2 \times (0.8, 0.9) \cup (0,0) \times S^1 \simeq_{\text{h.e.}} T^2 \vee T^2 \vee S^1$$

Therefore

$$\pi_1(A) \cong \pi_1(T^2 \vee S^1) \cong \langle \alpha_1, \beta_1, \gamma_1 \mid [\alpha_1, \beta_1] \rangle$$
$$\pi_1(B) \cong \pi_1(T^2 \vee S^1) \cong \langle \alpha_2, \beta_2, \gamma_2 \mid [\alpha_2, \beta_2] \rangle$$
$$\pi_1(A \cap B) \cong \pi_1(T^2 \vee T^2 \vee S^1) \cong \langle a, b, c, d, e \mid [a, b], [c, d] \rangle$$



The generators of $A \cap B$ include into A and B as follows:

$$\alpha_2 \xleftarrow{\iota_B} a \xrightarrow{\iota_A} \alpha_1, \; := \alpha \text{ in } X$$
$$\beta_2 \longleftarrow b \longmapsto \beta_1, := \beta \text{ in } X$$
$$\gamma_2 \longleftarrow e \longmapsto \gamma_1, := \gamma \text{ in } X$$
$$\gamma_2 \beta_2 \gamma_2^{-1} \longleftarrow c \longmapsto \alpha_1$$
$$\gamma_2 \alpha_2^{-1} \gamma_2^{-1} \longleftarrow d \longmapsto \beta_1$$

By Seifert-Van-Kampen,

$$\pi_1(X) \cong \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B) \cong \left\langle \alpha, \beta, \gamma \;\middle|\; [\alpha, \beta], \quad \alpha = \gamma \beta \gamma^{-1}, \quad \beta = \gamma \alpha^{-1} \gamma^{-1} \right\rangle$$

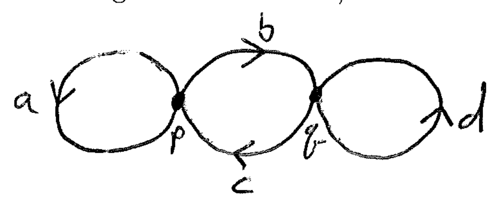
We may verify that this is correct by computing $\pi_1^{ab} \cong H_1$ and comparing with our result for H_1 from part a). Indeed,

$$\pi_1(X)^{ab} \cong \left\langle \alpha, \beta, \gamma \;\middle|\; [\alpha, \beta], [\alpha, \gamma], [\gamma, \beta] \quad \alpha = \gamma \beta \gamma^{-1}, \quad \beta = \gamma \alpha^{-1} \gamma^{-1} \right\rangle$$
$$\cong \left\langle \alpha, \beta, \gamma \;\middle|\; [\alpha, \beta], [\alpha, \gamma], [\gamma, \beta] \quad \alpha = \beta, \quad \beta = \alpha^{-1} \right\rangle$$
$$\cong \left\langle \beta, \gamma \;\middle|\; [\gamma, \beta], \quad \beta^2 \right\rangle \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \quad \square$$

ALGEBRAIC TOPOLOGY PRELIM EXAM

Do all three problems. You have three hours. If you have any questions, please ask.

(1) Let X be the figure shown below, with two vertices and four edges.



(a) Find explicit generators and relations for $\pi_1(X, p)$.
(b) Find two index-3 subgroups of $\pi_1(X, p)$, one normal and one not normal.
(c) Sketch the corresponding covering spaces of X (one regular and one not regular).

(2) Let X be a CW complex consisting of a vertex p, two edges a and b, and two 2-cells A and B, and with the following gluing maps:
- Both endpoints of a are identified with p, as are both endpoints of b.
- The boundary of A wraps around a, then b, then a. That is, $\partial A = aba$.
- $\partial B = ab^{-1}a$.

(a) Compute the fundamental group of X relative to the base point p.
(b) Compute the homology groups of X.
(c) Let Y be obtained from X by adding another 2-cell C with $\partial C = b$. Compute the fundamental group and homology groups of Y.

(3) Recall that a pair (X, A) has the *homotopy extension property* if and only if $\big(X \times \{0\}\big) \cup \big(A \times [0, 1]\big)$ is a retract of $X \times [0, 1]$. Also recall that the Hawaiian earring H is the union in $\mathbb{R}^2$ of the circles of radius $1/n$ centered at $(1/n, 0)$, where n ranges over the natural numbers $\mathbb{N}$. Let X be the wedge of countably many circles: $X = S^1 \times \mathbb{N}/\sim$, where a specific point on each circle (say, $\theta = 0$) is identified with the corresponding point on every other circle.

(a) Show that (X, p) has the homotopy extension property.
(b) Show that $(H, 0)$ does not have the homotopy extension property, where 0 is the origin in $\mathbb{R}^2$.

Date: August 2025.

August 2025

Solution 1.

a) First note that X deformation retracts onto the wedge of three S^1's, where the retraction sends the b edge to a point. Therefore $\pi_1(X,p) \cong \langle \alpha, \beta, \gamma \rangle$ the free product of three generators. To get explicit generators, we look at the pullback under the retraction, which gives

$$\pi_1(X,p) \cong F_3 \cong \langle [a] \rangle * \langle [bc] \rangle * \langle [bdb^{-1}] \rangle$$

$\square$

b) Recall that index 3 subgroups are in correspondence to degree 3 covers via the Galois correspondence, which sends

$$H \leftrightarrow p_*\big(\pi_1(\widetilde{X}_H, *)\big).$$

Further, recall that H is normal iff $\widetilde{X}_H$ is a normal cover.

Therefore to find two index three subgroups, we will find two degree three covers, with one normal, and then compute $p_*(\pi_1(\widetilde{X}_H, *))$ to give the subgroups

$$N := \Big\langle\, a^3,\quad abca^{-1},\quad abdb^{-1}a^{-1},\quad bc,\quad bdb^{-1},\quad a^{-1}bca,\quad a^{-1}bdb^{-1}a \,\Big\rangle < \pi_1(X, x_0)$$

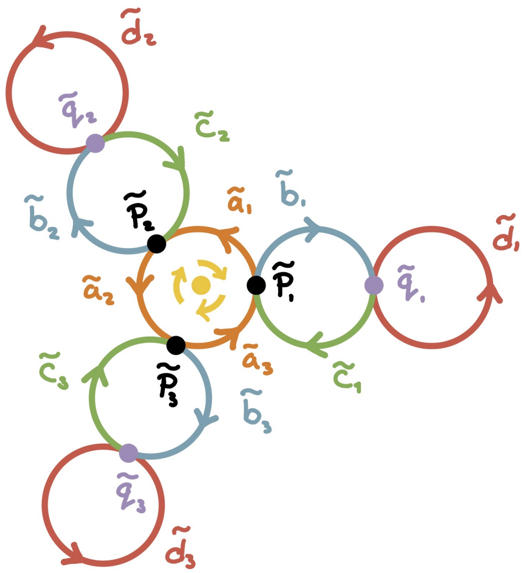
N is normal since $N \cong p_*(\pi_1(\widetilde{X}_N, *))$ which is a normal covering space, which we prove in c).

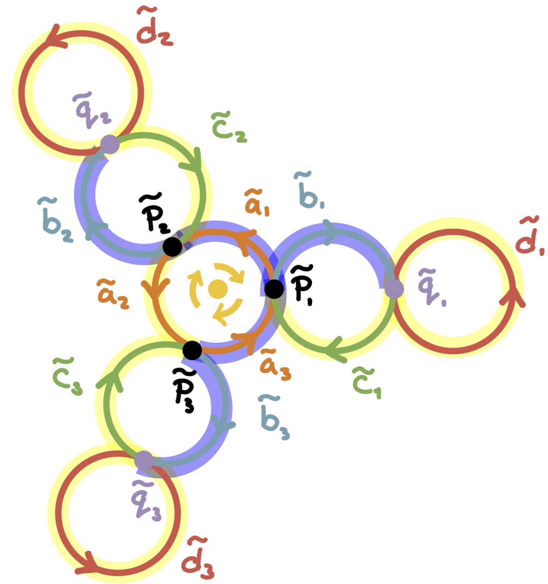
$$H := p_*\big(\pi_1(\widetilde{X}_H, \widetilde{p}_1)\big) \cong \Big\langle\, a,\quad bcbcbc,\quad bdb^{-1},\quad bcac^{-1}b^{-1}c^{-1}b^{-1},$$
$$bca^{-1}c^{-1}b^{-1}c^{-1}b^{-1},\quad bcbdb^{-1}c^{-1}b^{-1},\quad bcbcbdb^{-1}c^{-1}b^{-1}c^{-1}b^{-1} \,\Big\rangle < \pi_1(X, x_0)$$

H is not normal since $\widetilde{X}_H$ is not a normal covering space, which is proven in c).

c) (a) Normal:

Let $\widetilde{X}_N$ be as shown on the right below.





Note that there is a free action of $\mathbb{Z}/3$ on $\widetilde{X}_N$ given by rotation by 120 degrees around the center yellow point. Since the group is finite and the action is free, and the quotient gives a covering map $p : \widetilde{X}_N \to X \cong \widetilde{X}_N/(\mathbb{Z}/3)$. Thus $\widetilde{X}_N$ is normal.

We compute $N \subset \pi_1(X, q)$ associated to $\widetilde{X}_N$:

$$\pi_1(\widetilde{X}_N) \cong F_7 = \left\langle \widetilde{a}_1\widetilde{a}_2\widetilde{a}_3,\ \widetilde{a}_1\widetilde{b}_2\widetilde{c}_2\widetilde{a}_1^{-1},\ \widetilde{a}_1\widetilde{b}_2\widetilde{d}_2\widetilde{b}_2^{-1}\widetilde{a}_1^{-1},\ \widetilde{b}_1\widetilde{c}_1,\ \widetilde{b}_1\widetilde{d}_1\widetilde{b}_1^{-1},\ \widetilde{a}_3^{-1}\widetilde{b}_3\widetilde{c}_3\widetilde{a}_3,\ \widetilde{a}_3^{-1}\widetilde{b}_3\widetilde{d}_3\widetilde{b}_3^{-1}\widetilde{a}_3 \right\rangle$$

which we obtain by using the maximal spanning tree shown in blue in the right figure above.

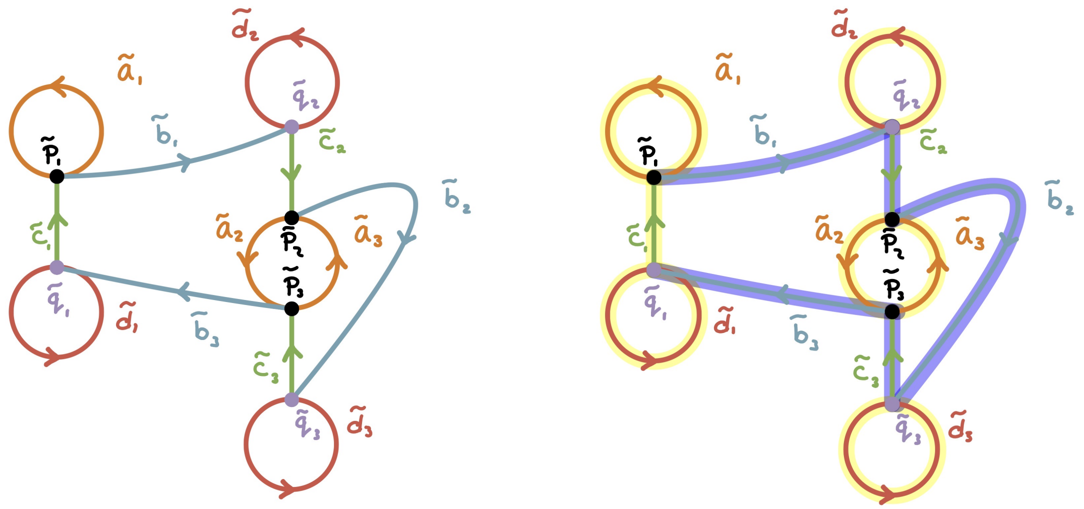
We now compute p_*:

$$p_*\Big(\left\langle \widetilde{a}_1\widetilde{a}_2\widetilde{a}_3,\ \widetilde{a}_1\widetilde{b}_2\widetilde{c}_2\widetilde{a}_1^{-1},\ \widetilde{a}_1\widetilde{b}_2\widetilde{d}_2\widetilde{b}_2^{-1}\widetilde{a}_1^{-1},\ \widetilde{b}_1\widetilde{c}_1,\ \widetilde{b}_1\widetilde{d}_1\widetilde{b}_1^{-1},\ \widetilde{a}_3^{-1}\widetilde{b}_3\widetilde{c}_3\widetilde{a}_3,\ \widetilde{a}_3^{-1}\widetilde{b}_3\widetilde{d}_3\widetilde{b}_3^{-1}\widetilde{a}_3 \right\rangle\Big)$$

$$\cong \left\langle a^3,\ \ abca^{-1},\ \ abdb^{-1}a^{-1},\ \ bc,\ \ bdb^{-1},\ \ a^{-1}bca,\ \ a^{-1}bdb^{-1}a \right\rangle =: N < \pi_1(X, x_0)$$

(b) Not normal:

Recall that for $\widetilde{X}$ normal, every pair of lifts must be related by a deck transform. We construct $\widetilde{X}_H$ such that there is no deck transform taking $\widetilde{p}_1$ to $\widetilde{p}_2$, thus $\widetilde{X}_H$ is not a normal covering space.
Let $\widetilde{X}_H$ be as shown below on the left.



If there was a deck transform $\varphi : \widetilde{X} \to \widetilde{X}$ sending $\widetilde{p}_1 \mapsto \widetilde{p}_2$, since φ is a self homeomorphism, $\widetilde{a}_1$ must be sent to a 1-cell with both endpoints at $\widetilde{p}_2$. However, every 1-cell attached to $\widetilde{p}_2$ has its second endpoint attached to a point which is not $\widetilde{p}_2$. Thus $\widetilde{X}_H$ is not regular, so $H = p_*\pi_1(\widetilde{X}_H, \widetilde{p}_1)$ is not a normal subgroup.
We now find H:

$$\pi_1(\widetilde{X}_H, \widetilde{p}_1) \cong F_7 \cong \big\langle\, \widetilde{a}_1,\ \ \widetilde{b}_1\widetilde{c}_2\widetilde{b}_2\widetilde{c}_3\widetilde{b}_3\widetilde{c}_1,\ \ \widetilde{b}_1\widetilde{d}_2\widetilde{b}_1^{-1},\ \ \widetilde{b}_1\widetilde{c}_2\widetilde{a}_2\widetilde{c}_3^{-1}\widetilde{b}_2^{-1}\widetilde{c}_2^{-1}\widetilde{b}_1^{-1},$$

$$\widetilde{b}_1\widetilde{c}_2\widetilde{a}_3^{-1}\widetilde{c}_3^{-1}\widetilde{b}_2^{-1}\widetilde{c}_2^{-1}\widetilde{b}_1^{-1},\ \ \widetilde{b}_1\widetilde{c}_2\widetilde{b}_2\widetilde{d}_3\widetilde{b}_2^{-1}\widetilde{c}_2^{-1}\widetilde{b}_1^{-1},\ \ \widetilde{b}_1\widetilde{c}_2\widetilde{b}_2\widetilde{c}_3\widetilde{b}_3\widetilde{d}_1\widetilde{b}_3^{-1}\widetilde{c}_3^{-1}\widetilde{b}_2^{-1}\widetilde{c}_2^{-1}\widetilde{b}_1^{-1}\,\big\rangle$$

Via the spanning tree: We now compute the image under p_*:

$$p_*\big(\pi_1(\widetilde{X}_H, \widetilde{p}_1)\big) \cong \big\langle\, a,\ \ bcbcbc,\ \ bdb^{-1},\ \ bcac^{-1}b^{-1}c^{-1}b^{-1},$$

$$bca^{-1}c^{-1}b^{-1}c^{-1}b^{-1},\ \ bcbdb^{-1}c^{-1}b^{-1},\ \ bcbcbdb^{-1}c^{-1}b^{-1}c^{-1}b^{-1}\,\big\rangle =: H$$

$H \subset \pi_1(X, q)$ is not normal since $\widetilde{X}_H$ is not. $\square$

Solution 2.

a) The fundamental group of a 2 dimensional cell complex where all 1-cells are attached to a single 0-cell has one generator for each 1-cell, and one relator for each 2-cell. Therefore $\pi_1(X) \cong \langle a, b \mid aba,\ ab^{-1}a \rangle$.

b) Since X has no cells of dim ≥ 3 and X is connected, $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0\ \ \forall n \geq 3$. To compute H_1 and H_2, we use cellular homology. The chain complex is:

$$0 \xrightarrow{\partial_3} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

With boundary maps given by the attaching maps of the cells:

$$(1,0) \xrightarrow{\partial_2} (2,1) = a + b + a$$

$$(0,1) \xrightarrow{\partial_2} (2,-1) = a - b + a$$

$$(1,0) \xrightarrow{\partial_1} 0 = v - v$$

$$(0,1) \xrightarrow{\partial_1} 0 = v - v$$

Since $\partial_2 : \mathbb{Z}^2 \mapsto \mathbb{Z}^2$, we may express it as a 2x2 matrix over $\mathbb{Z}$:

$$\implies \partial_2 = \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix}.$$

The determinant of this matrix is $-2 - 2 = -4 \neq 0$, therefore $\ker \partial_2 = 0$, so the map is injective, and $\operatorname{im} \partial_2$ is an index-4 subgroup of $\mathbb{Z} \oplus \mathbb{Z}$. Therefore

$$H_2(X) = \frac{\ker \partial_2}{\operatorname{im} \partial_3} = \frac{0}{0} = 0.$$

Since ∂_1 is the zero map, $\ker \partial_1 \cong \mathbb{Z} \oplus \mathbb{Z}, \quad \operatorname{im} \partial_1 = 0$. Therefore

$$H_1(X) = \frac{\mathbb{Z} \oplus \mathbb{Z}}{4\,\mathbb{Z} \oplus \mathbb{Z}} = \mathbb{Z}/4.$$

Explicitly, we can see this by doing the following change of basis. Let A and B denote the columns right to left of ∂_2, and a and b denote the rows top to bottom. The change of basis is:

$$\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \xrightarrow[B \mapsto B]{A \mapsto A+B} \begin{bmatrix} 4 & 2 \\ 0 & -1 \end{bmatrix} \xrightarrow[b \mapsto b]{a \mapsto a+2b} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot$$

And note that both change of basis matrices have determinant ± 1, so they are invertible over $\mathbb{Z}$.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & 2 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\implies H_1 \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\left(\begin{smallmatrix}4\\0\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}0\\-1\end{smallmatrix}\right)} \cong \mathbb{Z}/4$$

In full,

$$H_n(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}/4, & n=1 \\ 0, & \text{else} \end{cases}$$

c) The new fundamental group is obtained by the previous one by adding another relator corresponging to the new 2-cell.

$$\pi_1(Y) \cong \langle a, b \mid aba,\ ab^{-1}a,\ b\rangle \cong \langle a \mid a^2\rangle$$

Since $H_1(Y) \cong \mathrm{ab}(\pi_1(Y))$, we also know that $H_1(Y) \cong \mathbb{Z}/2$

Adding the 2-cell doesn't change connectedness or dimension, so $H_n(Y) = 0\ \forall n \geq 3$.

For H_2, we consider the new ∂_2 map:

$$0 \to \mathbb{Z}^3 \xrightarrow{\partial_2} \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

$$(1,0,0) \longmapsto (2,1)$$
$$(0,1,0) \longmapsto (2,-1)$$
$$(0,0,1) \longmapsto (0,1)$$

$$\implies \ker \partial_2 \cong \langle (1,-1,-2)\rangle \cong \mathbb{Z}$$

since $\partial_2(1,-1,-2) = (2-2+0,\ 1-(-1)-2) = (0,0)$

Verifying that this is indeed the entire kernel,

$$\begin{pmatrix} 2 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a+2b \\ a-b+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{if } a=-b \implies c=-2a$$

Thus

$$H_2(Y) \cong \frac{\mathbb{Z}}{0} \cong \mathbb{Z}$$

So

$$H_n(Y;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & n=0,2 \\ \mathbb{Z}/2, & n=1 \\ 0, & \text{else} \end{cases} \qquad \square$$

Solution 3. First, recall that a pair (X, A) is "good" if $A \subset X$ is closed and nonempty and admits a neighborhood $B \subset X$ such that B deformation retracts onto A. In other words, good pair are those in which, close enough to A, X has trivial topology rel A. Now, recall that (X, A) is a good pair if and only if $X \times I$ admits a retract onto $X \times \{0\} \cup A \times I$. Therefore it suffices to show that (X, p) is a good pair, but $(H, 0)$ is not.

a) For each $S^1_i \in \vee_{i=1}^{\infty} S^1$, take $\theta_i \in S^1_i \cong \mathbb{R} / \mathbb{Z}$. Then $(-\varepsilon, \varepsilon)_i \subset S^1_i$ is an open neighborhood of 0 in S^1_i. Let the wedge point p be the $\theta = 0$ point in each S^1 factor. Define the map $r_t : (-\varepsilon, \varepsilon) \times [0, 1] \to (-\varepsilon, \varepsilon)$ by the straight line homotopy $r_t(\theta) = (1 - t)\theta$. Clearly $r_0 = id$ and r_1 is the constant map sending every $\theta \in (-\varepsilon, \varepsilon)$ to 0. Now, define $B = \cup_{i=0}^{\infty}(-\varepsilon, \varepsilon)_i \subset \bigvee_{i=1}^{\infty} S^1$, and let $\vee_{i=1}^{\infty} r$ be the homotopy $\bigvee_{i=1}^{\infty} S^1 \times [0, 1] \to \bigvee_{i=1}^{\infty} S^1$ which does r_t on each wedge factor. This is a deformation retraction from B to A. Therefore $(\vee_{i=1}^{\infty} S^1, p)$ is a good pair. $\square$

b) We now show that $(H, 0)$ is not a good pair. Recall that $H \subset \mathbb{R}^2$ inherits the subspace topology. Hence every open set in H is of the form $U \cap H$, for $U \subset \mathbb{R}^2$ an open set. Further note that for every $\varepsilon > 0$, the open epsilon ball about the origin $B_\varepsilon(0) \subset \mathbb{R}^2$ contains infinitely-many π_1 essential loops of H, one for each $n \in \mathbb{N}$ where $n > 1/\varepsilon$. Therefore $\pi_1(U \cap H, 0) \ncong 1$ for every neighborhood U of 0, but $\pi_1(0, 0) \cong 1$. Therefore there is no neighborhood of the origin in H which deformation retracts to the origin, so $(H, 0)$ is not a good pair. $\square$

Preliminary Examination: Algebraic topology. August 20, 2024

Instructions: Answer all three questions. All questions (but not all subparts) carry equal weight

Time limit: 2 hours.

1. This problem is about constructing maps $S^n \to S^n$ of every possible degree.

(a) For each $k \in \mathbb{Z}$ construct a continuous map $f_k : S^1 \to S^1$ of degree k. Prove carefully that $\deg f_k = k$.

(b) Let $n \geq 1$, and let $f : S^n \to S^n$ be a continuous map. Explain how to use the suspension operation to construct a map $Sf : S^{n+1} \to S^{n+1}$ and prove that $\deg Sf = \deg f$. [*Hint*: It may helpful to consider the cone CS^n.]

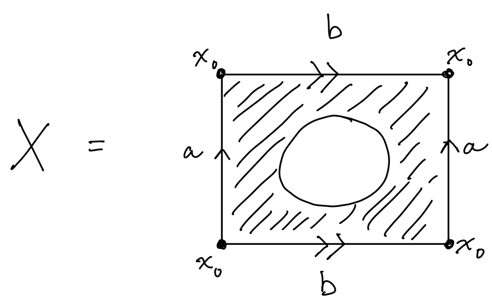
(c) Prove that for all $n \geq 1$ and $k \in \mathbb{Z}$ there exists a map $S^n \to S^n$ of degree k.

2. Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle. Fix a pair (p, q) of relatively prime integers. Consider the quotient space $X = S^1 \times S^1 \times [0,1]/\sim$, where the equivalence relation $\sim$ is given by $(x, y, 0) \sim (x', y', 0)$ whenever $x = x'$, and $(x, y, 1) \sim (x', y', 1)$ whenever $px + qy = px' + qy'$. For the following questions, your answer may depend on p, q.

(a) Compute $\pi_1(X)$.

(b) Compute all singular homology groups of X (with integer coefficients).

3. Let X be the orientable surface of genus one with one boundary component, i.e. X is a torus with a disk removed. The fundamental group of X is isomorphic to the free group of rank two, generated by elements a, b as indicated in the diagram: $\pi_1(X, x_0) = \langle a, b \rangle$.



Now consider the dihedral group of order eight, $D_8 = \langle x, y : x^2 = y^4 = 1, xyx^{-1} = y^{-1} \rangle$, and let $h : \pi_1(X, x_0) \to D_8$ be the homomorphism given by $h(a) = x,\ h(b) = y$. Let $H = \ker h < \pi_1(X, x_0)$. Let $p_H : \widetilde{X}_H \to X$ be the associated covering space.

(a) What is the degree of the cover p_H? What is the Euler characteristic of $\widetilde{X}_H$?

(b) The space $\widetilde{X}_H$ is an orientable surface of some genus g and some number of boundary components k. What are k and g?

January 2025

Solution 1.

a) We first identify $S^1 \cong \mathbb{R}/\mathbb{Z}$.

Define the map $F_k : \mathbb{R} \to \mathbb{R}$, $x \mapsto kx$. Since $F_k(\mathbb{Z}) = k\,\mathbb{Z} \subset \mathbb{Z}$, F_k descends to a map on the quotient:

$$p \circ F_k = f_k : S^1 \to S^1, \qquad x \mapsto kx \bmod 1$$

Now, let $\gamma : [0,1] \to S^1$ be a based (at 0) loop such that $[\gamma] = 1 \in \pi_1(S^1, 0) \cong \mathbb{Z}$.

Since $\pi_1([0,1], 0) \cong 1$, $\gamma_*(\pi_1([0,1], 0)) \subset 1 \subset \pi_1(S^1, 0)$, so γ lifts to a unique map $\widetilde{\gamma} : [0,1] \to \mathbb{R}$ st $\widetilde{\gamma}(0) = 0$.

Applying F_k to $\widetilde{\gamma}$, we get a map $F_k \circ \widetilde{\gamma} : [0,1] \to \mathbb{R}$ such that $F_k \circ \widetilde{\gamma}(0) = 0$, $F_k \circ \widetilde{\gamma}(1) = k$, and

$$F_k \circ \widetilde{\gamma}(t) = kt \quad \forall t \in [0,1].$$

Now, applying the covering projection $p : \mathbb{R} \to S^1$, we get a map

$$p \circ F_k \circ \widetilde{\gamma} : [0,1] \to S^1, \qquad t \mapsto kt \bmod 1$$

Each $x \in S^1$ clearly has k preimages under this composition, since

$$p \circ F_k \circ \widetilde{\gamma}(t) = kt \bmod 1 = kt + 1 \bmod 1 = k\Big(t + \tfrac{1}{k}\Big) \bmod 1 = p \circ F_k \circ \widetilde{\gamma}\Big(t + \tfrac{1}{k}\Big)$$

Thus $f_{k*} : \mathbb{Z} \to \mathbb{Z}$ sends $1 \mapsto k$. Thus $\deg(f_k) = k$.

b) Recall the long exact sequence in homology of the pair (CX, X):

$$\cdots \to H_{n+1}(X) \to H_{n+1}(CX) \to H_{n+1}(CX, X) \longrightarrow H_n(X) \to H_n(CX) \to \cdots$$

Since $X = S^n = 0\text{-cell} \cup n\text{-cell}$,

$$CX = S^n \times [0,1]/S^n \times \{1\} = 0\text{-cell} \cup 0\text{-cell} \cup 1\text{-cell} \cup n\text{-cell} \cup (n{+}1)\text{-cell},$$

so X is a subcomplex of CX so (CX, X) is a CW pair, thus is a good pair. Therefore $H_n(CX, X) \cong H_n(CX/X) \quad \forall n$.

Now, embed $X \hookrightarrow CX$ as $X \times \{0\} \subset CX$, so

$$CX/X \cong CX/X \times \{0\} \cong X \times [0,1] \Big/ \begin{matrix} X \times \{1\} \\ X \times \{0\} \end{matrix} \cong \Sigma X.$$

In our case, $\Sigma S^n \cong S^{n+1}$, $CS^n \cong D^n$.

Thus the LES becomes

$$\cdots \to H_{n+1}(S) \to H_{n+1}(D^n) \to H_{n+1}(S^{n+1}) \longrightarrow H_n(S^n) \to H_n(D^n) \to \cdots$$

$$\cdots \to 0 \to 0 \to \mathbb{Z} \longrightarrow \mathbb{Z} \to 0 \to \cdots$$

and by exactness, $H_{n+1}(S^{n+1}) \cong H_n(S^n)$.

Now, given $f_{n,k} : S^n \to S^n$ degree d, define $Sf_{n,k} := f_{n+1,k} : S^{n+1} \to S^{n+1}$ by:

$$(x,t) \longmapsto \begin{cases} (f_{n,k}(x), t), & t \neq 0,1 \\ x, t, & t = 0,1 \end{cases}$$

Since $Sf_{n,k}|_{S^n \times \{1/2\}} = f_{n,k}$, the following maps on homology commute:

$$\begin{array}{ccc} H_{n+1}(S^{n+1}) & \cong & H_n(S^n) \\ Sf_{n,k*} \downarrow & \partial & \downarrow f_{n,k*} \\ H_{n+1}(S^{n+1}) & \cong & H_n(S^n) \end{array}$$

Since $\deg f_{n,k} = k$, we must have that $\deg(Sf_{n,k}) = k$. $\square$

c) We proceed by induction on S^n:

- $n = 1$: by a), $\forall k \in \mathbb{Z}$, $\exists$ map $f_{1,k} : S^1 \to S^1$ st $\deg(f_{1,k}) = k$.
- Induction: assume we have a map $f_{n,k} : S^n \to S^n$ st $\deg(f_{n,k}) = k$.
- By b) we get a map $f_{n+1,k} : S^{n+1} \to S^{n+1}$ st $\deg(f_{n+1,k}) = k$. $\square$

Solution 2. Let

$$X = S^1 \times S^1 \times [0,1] \Big/ \begin{matrix} (x,y,0) \sim (x',y',0) & \text{if } x = x' \quad (*) \\ (x,y,1) \sim (x',y',1) & \text{if } px+qy = px'+qy' \quad (**) \end{matrix}$$

a) To apply Seifert-van-Kampen, we first decompose

$$S^1 \times S^1 \times [0,1] = \underbrace{S^1 \times S^1 \times [0,3/4)}_{A} \cup \underbrace{S^1 \times S^1 \times (1/4,1]}_{B}.$$

Then $X = S^1 \times S^1 \times [0,3/4]/(*) \cup S^1 \times S^1 \times [1/4,1]/(**) = A \cup B$, and the intersection $A \cap B$ is $S^1 \times S^1 \times (1/4, 3/4)$.

Let $* = (0,0,1/2) \in X$.

- We first find $\pi_1(A \cap B, *)$: Since $S^1 \times S^1 \times (1/4,3/4)$ deformation retracts onto $S^1 \times S^1 \times \{\ 1/2\}$, $\pi_1(A \cap B, *) \cong \pi_1(T^2, *) \cong \langle a, b | [a,b] \rangle$.
- We now find $\pi_1(A, *)$: First note that A deformation retracts onto $S^1 \times S^1 \times \{0\}/(*)$, thus $\pi_1(A,*) \cong \pi_1(S^1 \times S^1/(*), pt)$. Since $(x,y) \sim (x,y')$, the loop $\{x\} \times S^1$ in A is homotopic to a point. Thus $\pi_1(A,*) \cong \langle a \rangle$ where $a = [S^1 \times \{pt\}]$.
- We now find $\pi_1(B, *)$: Note that B deformation retracts onto $S^1 \times S^1 \times \{1\}/(**)$, so $\pi_1(B,*) \cong \pi_1(S^1 \times S^1/(**),\ pt)$. Identify $S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$, where $S^1 \times \{0\} \mapsto$ x-axis, $\{0\} \times S^1 \mapsto y$-axis.

  For $(x,y) \in \mathbb{R}^2$, $px + qy = n$ defines a line in $\mathbb{R}^2$ for each fixed n. Thus if $px + qy = px' + qy'$, (x,y) and (x',y') lie on the same line of slope $-p/q$ on $\mathbb{R}^2$. Since p and q are coprime integers, $px+qy = 0$ is solved by $(x,y) \in \mathbb{Z}^2$ st $x, y \neq 0$, thus the quotient under $\mathbb{Z}^2$ maps the line $px + qy = 0$ to a closed curve on T^2, along with each of its translates.

  We now quotient by the curves $px + qy = n\ \forall n \in \mathbb{Z}$. To see the effect on π_1, note that

  $$\big|\{px+qy=0\} \cap \{x\text{-axis}\}\big| = q$$
  $$\big|\{px+qy=0\} \cap \{y\text{-axis}\}\big| = p$$

  So quotienting sends the element of π_1 going q times around the x-axis generator & p times around the y-axis generator to 1.

  Thus $\pi_1(S^1 \times S^1/(**)) \cong \langle a, b \mid [a,b],\ a^q b^p \rangle$.

By van Kampen, the amaglamated product over $A \cap B$ identifies generators $a = a'$ and $b = b'$, and adds a commutator $[a,b]$ as a relation. Therefore $\pi_1(X,*) \cong \langle a, b \mid b \rangle * \langle a', b' \mid [a',b'],\ a'^q, b'^p \rangle \Big/_{\substack{a=a' \\ b=b'}} \cong \langle a \mid a^q \rangle \cong \mathbb{Z}/q\,\mathbb{Z}$ □

b) We compute homolgy. X is connected, $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$. For H_1, recall that $H_1(X;\mathbb{Z}) = \mathrm{ab}(\pi_1)$, therefore $H_1(X;\mathbb{Z}) \cong \mathbb{Z}/q\,\mathbb{Z}$.

To find H_n, we use Mayer–Vietoris on the same A, B as above. We know that both A and B are homotopy equivalent to S^1; for B, this S^1 is the line of slope $-\frac{1}{-p/q} = q/p$.

Further, $A \cap B$ deformation retracts to T^2. Since none of these objects have nonzero homology above n=2, we know that $H_n(X) = 0$ for all $n \geq 4$ The relevant part of the LES is:

$$H_3(A) \oplus H_3(B) \xrightarrow{j_3} H_3(X) \xrightarrow{\partial_3} H_2(A \cap B) \xrightarrow{\iota_2} H_2(A) \oplus H_2(B) \xrightarrow{j_2} H_2(X)$$

$$\xrightarrow{\partial_2} H_1(A \cap B) \xrightarrow{\iota_1} H_1(A) \oplus H_1(B) \xrightarrow{j_1} H_1(X) \xrightarrow{H}_0 (A \cap B)$$

Filling in the known homology of $A, B, A \cap B$, we get

$$0 \oplus 0 \xrightarrow{j_3} H_3(X) \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\iota_2} 0 \oplus 0 \xrightarrow{j_2} H_2(X) \xrightarrow{\partial_2} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\iota_1} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j_1} H_1(X) \xrightarrow{\mathbb{Z}}$$

Since $j_3$and ι_2 must be the zero map, $\operatorname{im}(j_3) \cong 0 \cong \ker(\partial_3)$ and $\operatorname{im}(\partial_3) \cong \mathbb{Z} \cong \ker(\iota_2)$. Therefore $H_3(X; \mathbb{Z}) \cong \mathbb{Z}$. Similarly, j_2 must be the zero map, so $\operatorname{im}(j_2) \cong 0 \cong \ker(\partial_2)$. It remains to examine the map ι_1.

Recall that the $\mathbb{Z} \oplus \mathbb{Z}$ in the domain of ι_1 represent the a and b curves on the middle torus, where $(1, 0) = [a]$ and $(0, 1) = [b]$. On the other hand, the $\mathbb{Z} \oplus \mathbb{Z}$ in the range represent the curves leftover after quotienting on both ends. Denote $(1, 0) = [S^1 \times \{pt\}] = [a]$ and $(0, 1) =$the line of slope q/p. Therefore $\iota_1((1, 0)) = (1, 0)$, and $\iota_1(p, q) = (0, 1)$, so $\iota_1 : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is given by the matrix $\begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}$. This matrix has determinant q, so it is injective unless q=0, with image an index q subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ and no kernel. Therefore $H_2(X) \cong 0$ when $q = 0$, and . Note that the image of ι_1 agrees with our previous $H_1(X)$ computation, since the index q subgroup must be the kernel of j_1 so $H_1(X) \cong \mathbb{Z} / p \mathbb{Z}$.

In full,

$$H_n(X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0, 3 \\ \mathbb{Z} / q \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \text{ and } q = 0 \\ 0 & n = 2 \text{ and } q \neq 0 \\ 0 & \text{else.} \end{cases}$$

Solution 3.

a) Recall that degree of a cover $p : \tilde{X} \to X$ is the index of the corresponding subgroup $p_*(\pi_1(\tilde{X}, \tilde{*})) \subset \pi_1(X, *))$ via the Galois correspondence. Further, if H is a normal subgroup, the index of X is the order G/H. Since H as given is the kernel of the a homomorphism $h : \pi_1(X, x_0) \to D_8$, H is a normal subgroup. Further, since $\pi_1(X, x_0) \cong \langle a, b \rangle$, $H = \langle a, b \mid a^2, b^4, aba^{-1} = b^{-1} \rangle \subset \langle a, b \rangle$, thus $H \cong D_8$, which we know has order 8. Thus degree $= 8$.
To find the Euler characteristic, recall that the Euler characteristic of a degree n cover is n times that of the base. Thus $\widetilde{\chi}_H = -8\chi(X)$. Since X deformation retracts to the wedge of two S^1's, $\chi(X) = 1 - 2 = -1$, so $\widetilde{\chi}_H = -8$. $\square$

b) Recall that the Euler characteristic of the genus g surface with k disks removed is $\chi(\Sigma_{g,k}) = 2 - 2g - k$. This is because we may choose a CW decomposition of $\mathbb{D}^2$ with k 2-cells, k 1-cells, and 1 0-cell, so removing k disks corresponds to not attaching those 2-cells. From a), we know that $\chi(\widetilde{X}) = -8$, thus $-8 = 2 - 2g - k$.

We first find the number of punctures, then use the above equation to find the genus. Recall that the cover must have punctures, since the base is a punctured torus, and the boundary curve must lift to a boundary curve in the cover. Thus we may find the number of punctures in the cover by finding the number of lifts of this boundary curve. The class $[\partial X] \in \pi_1(X, x_0)$ is $[aba^{-1}b^{-1}]$, and the number off lifts is the index of the subgroup $\langle aba^{-1}b^{-1} \rangle \subset D_8$.

Since $aba^{-1} = b^{-1} \in D_8$, $aba^{-1}b^{-1} = b^{-1} \cdot b^{-1} = b^{-2} \in D_8$.

Further, since $b^4 = 1 \in D_8$, b^{-2} has order 2.Thus the index is $8/2 = 4$, so $k = 4$. Solving for g,

$$\implies -8 = 2 - 2g - 4 \implies -4 = 2 - 2g \implies 2g = 6 \implies g = 3.$$

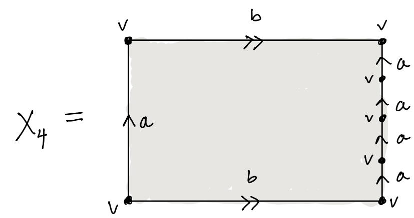
Therefore the corresponding covering space is the genus 3 surface with four disks removed. $\square$

Preliminary Examination: Algebraic topology. August 20, 2024

Instructions: Answer all three questions. Question 1 is worth 4 points. Questions 2 is worth 2 points. Question 3 is worth 2 points.

Time limit: 2 hours.

1. For each $d \in \mathbb{Z}$, consider the map $f_d : S^1 \to S^1$ that wraps the circle around itself d times: $f_d(e^{i\theta}) = e^{di\theta}$. Let $X_d = S^1 \times I / \sim$ be the mapping torus of f_d, defined by the relation $(x, 0) \sim (f_d(x), 1)$. The figure suggests a cell structure on X_d.

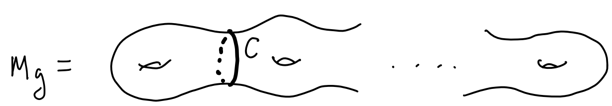


(a): For each $d \neq 1$, compute the homology groups $H_*(X_d)$.

(b): For which values of d does X_d retract onto the circle $S^1 \times \{1/2\}$?

(c): For which values of d is X_d homotopy equivalent to a surface?

(d): Now let the space $Y = Y_{d,g}$ be obtained from X_d and a closed orientable surface M_g of genus $g \geq 2$ by identifying $S^1 \times \{1/2\} \subset X_d$ with the curve $C \subset M_g$ shown in the figure. Use a Mayer-Vietoris sequence to compute the homology groups $H_*(Y)$.



2. Let $X = S^1 \vee S^1$ be the wedge of two circles, and let $x \in X$ be the point where the two circles meet (the "wedge" point). Then $\pi_1(X, x) \cong F_2 = \langle a, b \rangle$ is the free group on two generators.

(a): Describe all path-connected covering spaces of X of degree two. [*Note:* it is acceptable to describe a covering space by drawing a well-labeled picture.]

(b): Using your answer to (a), list all subgroups of index two in F_2 and give a geometric proof that every such subgroup is normal. [*Note:* It is a general fact that any index two subgroup in an arbitrary group G is normal, but do not use this general fact. The point is to prove this fact using covering space theory in the case $G = F_2$.]

3. Let G be a group.

(a): Briefly explain the general construction of a basepointed 2-dimensional cell complex (X_G, x_0), so that $\pi_1(X, x_0) = G$. Demonstrate the construction in the case $G = \langle a, b, c : a^2 b, c^3 \rangle$.

(b): Let (Y, y_0) be a path connected topological space with basepoint. Let $h : G \to \pi_1(Y, y_0)$ be a group homomorphism. Briefly describe a general construction of a continuous map $f : (X_G, x_0) \to (Y, y_0)$ such that $f_* = h$, where here f_* denotes the induced map on fundamental group.

August 2024

Solution 1.

a) We use cellular homology:

$$X_d = 0\text{-cell } \{v\} \cup 1\text{-cells } \{a, b\} \cup 2\text{-cell } \{F\}$$

with attaching maps

$$\partial a = \partial b = v - v = 0$$

$$\partial F = a + b - da - b = (1-d)a.$$

Therefore $H_n(X_d; \mathbb{Z}) \cong 0 \quad \forall n \geq 3$, and we get the following chain complex

$$0 \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

with boundary maps given by $\operatorname{im}\partial_2 = \langle -(d-1)a \rangle$, $\ker \partial_2 = 0$

$\operatorname{im}\partial_1 = 0$, $\ker \partial_1 = \langle a, b \rangle$ Computing the homology of this chain complex, we get the following:

$$H_2(X_d; \mathbb{Z}) \cong \frac{\ker \partial_2}{\operatorname{im}\partial_3} \cong 0$$

$$H_1(X_d; \mathbb{Z}) \cong \frac{\ker \partial_1}{\operatorname{im}\partial_2} \cong \frac{\mathbb{Z}\langle a \rangle \oplus \mathbb{Z}\langle b \rangle}{\mathbb{Z}\langle (1-d)a \rangle} \cong \mathbb{Z}/(1-d)\,\mathbb{Z} \oplus \mathbb{Z}$$

$$H_0(X_d; \mathbb{Z}) = \frac{\ker \partial_0}{\operatorname{im}\partial_1} \cong \frac{\mathbb{Z}}{0} \cong \mathbb{Z}$$

Therefore for $d \neq 1$,

$$H_n(X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/(1-d)\,\mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n \geq 2 \end{cases}$$

$\square$

b) If there exists a retraction onto $S^1 \times \{1/2\}$, then the inclusion map $\iota : S^1 \times \{1/2\} \hookrightarrow X_d$, must induce an injection on homology: $\iota_* : H_1(S^1; \mathbb{Z}) \cong \mathbb{Z} \hookrightarrow H_1(X_d; \mathbb{Z})$ must be injective.

The circle $S^1 \times \{1/2\}$ is homotopy equivalent to $S^1 \times \{0\}$, and $[S^1 \times \{0\}] = a \in H_1(X_d; \mathbb{Z})$.

Thus $\iota_*(1) = (1, 0) \in \mathbb{Z}/(1-d) \oplus \mathbb{Z}$

Since $\mathbb{Z}/(1-d)\,\mathbb{Z}$ is torsion $\forall d \neq 1$, any map $\mathbb{Z} \to \mathbb{Z}/(1-d)\,\mathbb{Z}$ has nonzero kernel for $d \neq 1$. Thus $d = 1$ is the only option for which there can be a retraction. When $d = 1$, $X_d \cong T^2$.

Define

$$f : T^2 \to T^2$$

$$(x, y) \to (x, 1/2)$$

This map clearly sends T^2 to $S^1 \times \{1/2\}$, and fixes $S^1 \times \{1/2\}$. Further, since $T^2 \cong S^1 \times S^1$ with the product topology,

$$f^{-1}(V) = f^{-1}(V \cap S^1 \times 1/2) = V \cap S^1 \times 1/2 \times S^1 \quad \text{which}$$

is open in T^2. $\square$

c) By the classification of surfaces, since

$$\chi(X_d) = 1 - 2 + 1 = 0,$$

if X_d is a surface, it must be T^2 or K.

$$H_1(T^2; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z} \implies d = 1$$

If $X_d \cong K$, since

$$H_2(K; \mathbb{Z}) \cong 0, \qquad H_1(K; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/2, \qquad H_0(K; \mathbb{Z}) \cong \mathbb{Z},$$

$$|1 - d| = 2 \implies d = -1 \text{ or } 3.$$

If $d = 3$, $\pi_1(X_d) \cong \langle a, b \mid ab\bar{a}^3 b^{-1} = 1 \rangle$

which is not isomorphic to $\pi_1(K) \cong \pi_1(X_{-1}) \cong \langle a, b \mid abab^{-1} = 1 \rangle$

Thus X_d is homotopy equivalent to a surface only when $d = \pm 1$. $\square$

d) Recall

$$H_k(M_g; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 2, 0 \\ (\mathbb{Z} \oplus \mathbb{Z})^g & k = 1 \\ \mathbb{Z}, & k = \text{else} \end{cases}$$

Since $Y_{d,g}$ only needs cells of dim ≤ 2, $H_n(Y_{d,g}; \mathbb{Z}) = 0 \ \forall n \geq 3$. Since it is connected, $H_0(Y_{d,g}; \mathbb{Z}) \cong \mathbb{Z}$

Assume $d \neq 1$. By Mayer Vietoris,

$$\begin{array}{ccccccccc} 0 \to & H_2(S^1) & \to & H_2(M_g) \oplus H_2(X_d) & \to & H_2(Y_{d,g}) & \to & H_1(S^1) \\ & 0 & & \mathbb{Z} & & 0 & & \mathbb{Z} \end{array}$$

$$\begin{array}{ccccc} \to H_1(M_g) \oplus & H_1(X_d) & \to H_1(Y_{d,g}) & \to H_0(S^1) \\ \mathbb{Z}^{2g} & \mathbb{Z}/(1-d) \oplus \mathbb{Z} & & \end{array}$$

So we get the LES

$$0 \xrightarrow{\iota_2} \mathbb{Z} \oplus 0 \xrightarrow{j_2} H_2 \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\iota_1} \mathbb{Z}^{2g} \oplus \mathbb{Z}/(1-d) \oplus \mathbb{Z} \xrightarrow{j_1} H_1 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\iota_0} \mathbb{Z} \oplus \mathbb{Z}$$

We now compute the image and kernel of the map $\iota_1 : \mathbb{Z} \mapsto \mathbb{Z}^{2g} \oplus \mathbb{Z}/(1-d) \oplus \mathbb{Z}$. Since the splitting curve C is a commutator in $\pi_1(M_g, *)$, $[C] = 0 \in H_1(M_g)$. Therefore iota must send 1 to 0 times the generator of the $\mathbb{Z}^{2g}$ summand. Since C is identified with the $S^1 \times \{1/2\}$ curve in X_d, which is homotopy equivalent to the a curve, which gives the torsion summand in $H_1(X_d)$, the generator of $H_1(S^1 \times \{1/2\})$ is sent only to the generator of the torsion summand. Thus

$$\iota_1(1) = (0, 1, 0)$$

which, by exactness, gives that

$$\operatorname{im} \iota_1 = \mathbb{Z}/(1-d) = \ker j_1$$

$$\ker \iota_1 = (1-d)\,\mathbb{Z} \cong \mathbb{Z} = \operatorname{im} \partial_2$$

Therefore $H_1 \cong \mathbb{Z}^{2g} \oplus \mathbb{Z} \cong \mathbb{Z}^{2g+1}$.
For H_2 note that ι_2 must be the zero map, thus

$$\operatorname{im} \iota_2 = 0 = \ker j_2 \implies H_2 \cong \mathbb{Z} \oplus \mathbb{Z}\,.$$

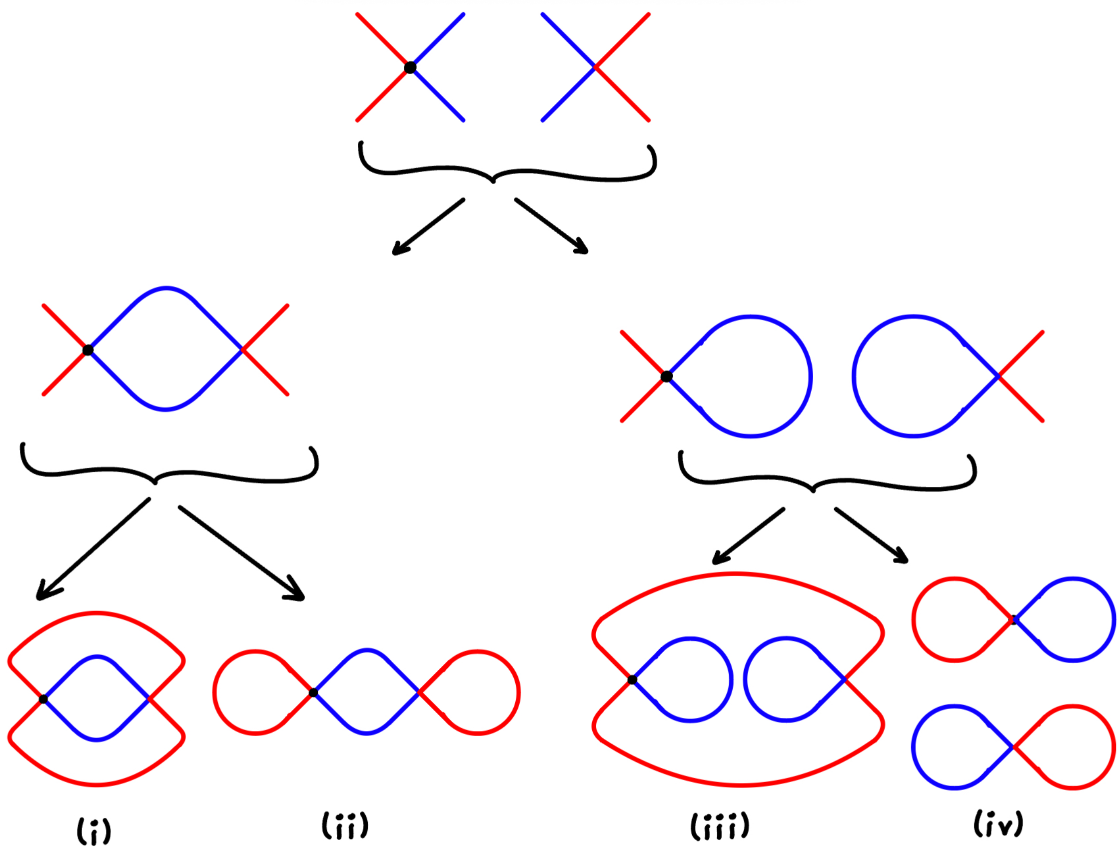
In full, the homology of $Y_{d,g}$ is

$$\implies H_n(Y_{d,g}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^{2g+1} & n = 1 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 2 \\ 0 & n \geq 3 \end{cases}$$

Solution 2. Let $X = S^1 \vee S^1$ be the wedge of two circles, and let $x \in X$ be the point where the two circles meet (the "wedge" point). Then $\pi_1(X, x) \cong F_2 = \langle a, b \rangle$ is the free group on two generators.

a) Describe all path-connected covering spaces of X of degree two. [Note: it is acceptable to describe a covering space by drawing a well-labeled picture.]

Since a degree n cover has n lifts of the base point, we can start by drawing the two lifts of the basepoint and their corresponding neighborhoods, which consist of the beginning and ends of the a and b 1-cells. We now enumerate all the possible ways to lift the entire 1-cells. Starting with a, there are two options: the cell starting at each basepoint ends at the same basepoint (right), or it connects to the other one (left). For b, we have the same two options, resulting in the following four degree two covers. However, note that the cover (iv) resulting from connecting both the a and b paths to



their original starting point results in a disconnected covering space. Thus there are only three degree 2 path connected covering spaces of X.

b) Using your answer to (a), list all subgroups of index two in F_2 and give a geometric proof that every such subgroup is normal. [Note: It is a general fact that any index two subgroup in an arbitrary group G is normal, but do not use this general fact. The point is to prove this fact using covering space theory in the case $G = F_2$.]

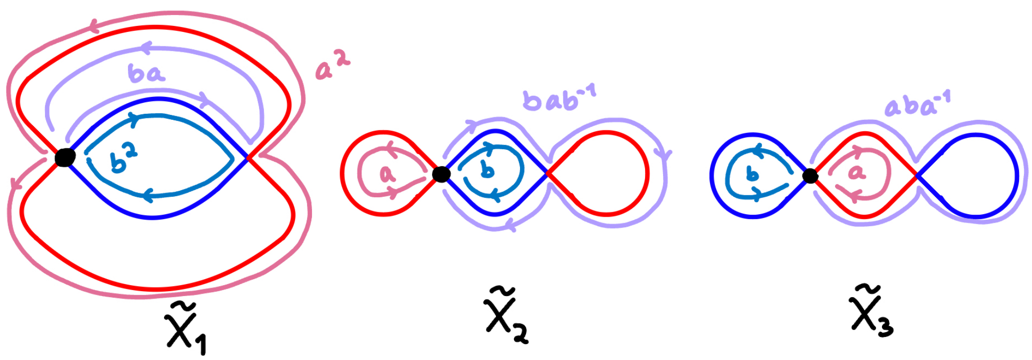
Recall that the Galois correspondence gives an bijection between the conjugacy classes of index n subgroups of $\pi_1(X, x_0)$ and degree-n covers of X. Given a cover $\tilde{X}$, we can recover the corresponding subgroup by looking at the image of the induced map p_* of the projection $p : \tilde{X} \to X$ on π_1.

For each of the three covers, the image of the induced map is

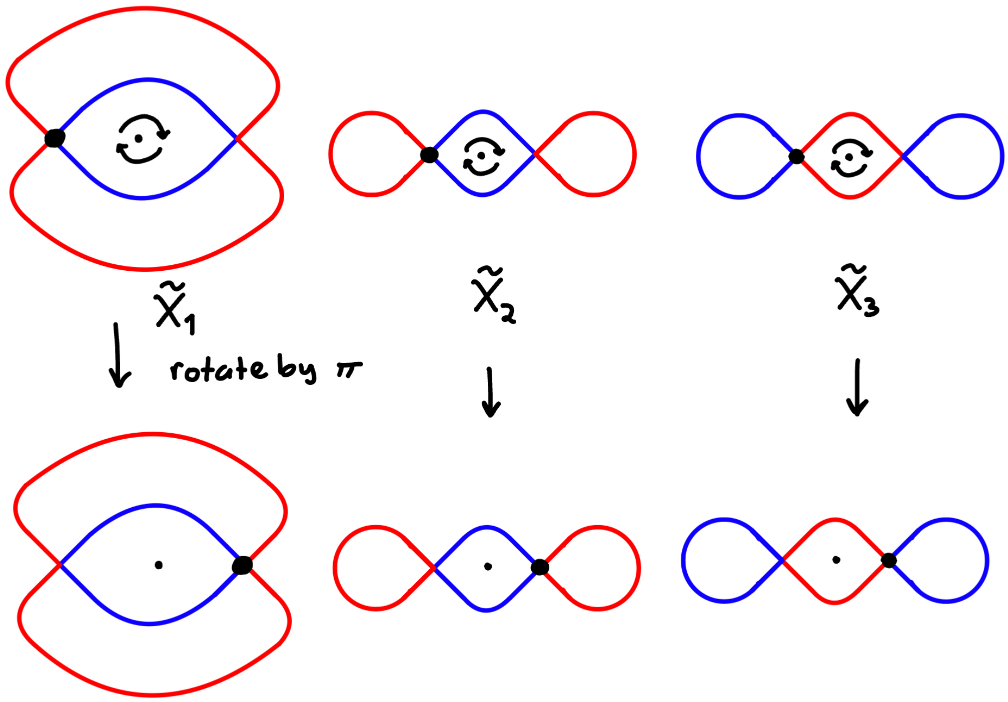
$$p_*(\pi_1(\tilde{X}_1, \tilde{x}_0) = \langle a^2, b^2, ba \rangle \subset \pi_1(X, x_0)$$

$$p_*(\pi_1(\tilde{X}_2, \tilde{x}_0) = \langle a, b, bab^{-1} \rangle \subset \pi_1(X, x_0)$$

$$p_*(\pi_1(\tilde{X}_3, \tilde{x}_0) = \langle a, b, aba^{-1} \rangle \subset \pi_1(X, x_0)$$



Thus the three index two subgroups of $F_2 = \langle a, b \rangle$ are $\langle a^2, b^2, ba \rangle$, $\langle a, b, bab^{-1} \rangle$, and $\langle a, b, aba^{-1} \rangle$ To show that these subgroups are normal, recall that normal subgroups correspond to normal covering spaces. Thus it suffices to demonstrate a deck transformation of each covering space that sends the the lifts of the basepoint of X to each other. After choosing a point in the center of the center-most loop, we can rotate about that point by π. This rotation defines a deck transformation of each cover, which sends the two lifts of the basepoint of X to each other as desired. Thus all three subgroups are normal.



Solution 3.

a) We build X_G as follows: Let $\langle g_1, \ldots, g_n \mid r_1, \ldots, r_k \rangle$ be a presentation for G.

$$X_G^0 = 1 \text{ 0-cell, denoted } r$$

$$X_G^1 = X_G^0 \bigcup_{i=1}^{n} e_i^1 = X_G^0 \cup n \text{ 1-cells along maps}$$

$$\varphi_i^1 : \partial e_i^1 \to X_G^0$$

$$0 \longmapsto r$$

$$1 \longmapsto r$$

Thus $X_G^1 \cong \bigvee_{i=1}^{n} S^1$ the wedge of n circles, where $n = \#$ generators.
and $\pi_1(X_G^1, r) = F_n \cong \langle g_1, \ldots, g_n \rangle$ the free group on n letters.

$$X_G^2 = X_G^1 \bigcup_{i=1}^{k} e_i^2 = X_G^1 \cup k \text{ 2-cells along maps}$$

$$\varphi_i^2 : \partial e_i^2 \to X_G^1$$

$$S^1 \longmapsto \gamma_i \text{ where } [\gamma_i] = r_i \text{ in } \pi_1(X_G^1, r)$$

On π_1, attaching these 2-cells corresponds to quotienting by the elements of $\pi_1(X^1_G, r)$ given by the relations of G.

Thus $\pi_1(X^2_G, r) \cong \langle g_1, \ldots, g_n \mid r_1, \ldots, r_k \rangle$ as desired.

For $G = \langle a, b, c \mid a^2b,\ c^3 \rangle$, $X^1_G = S^1 \vee S^1 \vee S^1$.
In coordinates, let θ, γ, and $\phi \in [0, 1)$ be angles on each $S^1 \cong \mathbb{R} \bmod 1$ where $\theta = 0, \gamma = 0, \phi = 0$ are identified as the wedge point.

Then $X^2_G = X^1_G \cup_{\varphi^2_1} e^2_1 \cup_{\varphi^2_2} e^2_2$ where $\varphi^2_1 : S^1 \to X^1_G$ is given by

$$\eta \mapsto \begin{cases} \theta = 3\eta \bmod 1, & 0 \leq \eta < 2/3 \\ \gamma = 3\eta \bmod 1, & 2/3 \leq \eta < 1 \end{cases}$$

and $\varphi^2_2 : S^1 \to X^1_G$ is given by

$$\eta \mapsto \gamma = 3\eta \bmod 1$$

Since every curve γ where $[\gamma] = a^2b$ or c^3 in π_1 is nullhomotopic,

$$\pi_1(X^2_G, r) \cong \langle a, b, c \mid a^2b, c^3 \rangle \text{ as desired.} \quad \square$$

b) Let $\langle g_1, \ldots, g_n \mid r_1, \ldots, r_k \rangle$ be a presentation for G.

Construct f by first defining f_0 a map from X^0_G, then defining f_1 from X^1_G, then defining $f = f_2$ from $X_G = X^2_G$. Since f is a based map, there is only one option for f_0: $f_0(x_0) = y_0$. Clearly f_0 is continuous.

For each generator $g_i \in \pi_1(X_G, x_0)$, pick a based continuous loop $\gamma_i : [0, 1] \to Y$ such that $[\gamma_i] = h(g_i) \in \pi_1(Y, y_0)$. Since the g_i correspond to 1-cells e^1_i in X^1_G, define $f_1 : X^1_G \to Y$ by: $x \mapsto \gamma_i(x)$, $x \in e^1_i$

Note that $f_1(x_0) = f_0(x_0)$ since $\gamma_i(x_0) = y_0\ \forall i$.

Since each γ_i is continuous, $f_1^{-1}(\nu(y)) = \gamma_i^{-1}(\nu(y)) = (a, b) \in e^1_i$ for $y \in \gamma_i \setminus y_0$

$$f_1^{-1}(\nu(y_0)) = \bigcup_{i=1}^{n} \gamma_i^{-1}\big(\nu(y) \cap \operatorname{im}(\gamma_i)\big) = \bigcup (a_i, b_i) \in X^1_G$$

thus f_1 is continuous.

Now, for each relation $r_i \in \pi_1(X_G, x_0)$, since h is a group homomorphism, $h(r_i) = 1 \in \pi_1(Y, y_0)$. Thus, every representative $\eta_i : I \to Y$ for r_i is homotopic in Y to a point; thus $\operatorname{im}(\eta_i) = \partial(D_i)$ a disk tracing the nullhomotopy in Y.

Define $f : X_G \to Y$ by extending over each nullhomotopy, i.e.,

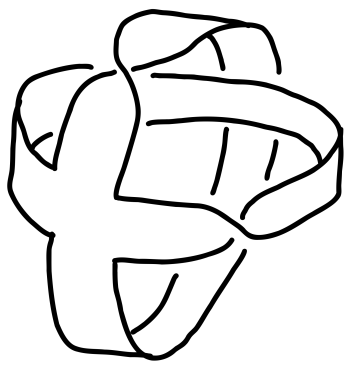
$$f(e^2_i) = D_i$$

By construction, $f_*([g_i]) = [\gamma_i] = h(g_i)$, and $f_*(r_i) = [\eta_i] = 1 = h(r_i) = h(1)$. $\quad \square$

Algebraic Topology Prelim Exam
Jan 11, 2024, 9:00–11:00

Solve all three problems.

Problem 1. Let X be the space pictured; it consists of two Möbius bands identified "across each other".



(a) What surface is obtained by capping off its boundary components with disks?
(b) Draw the orientation cover, indicating the action of the deck group.

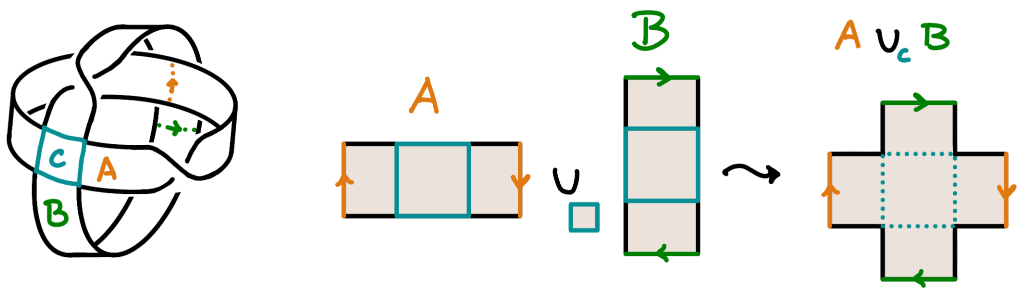
Problem 2. Let X be the 2-torus T with a 3-cell attached. Compute $H_*(X)$. (In particular, the answer is independent of the attaching map.)

Problem 3. Let $X = S^2$, and let Y be the wedge sum of X with another 2-sphere. Show that Y retracts to X, but does not deformation-retract to it.

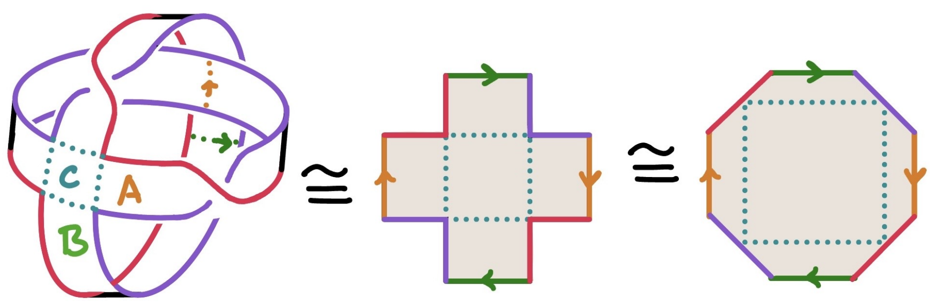
January 2024

Solution 1.

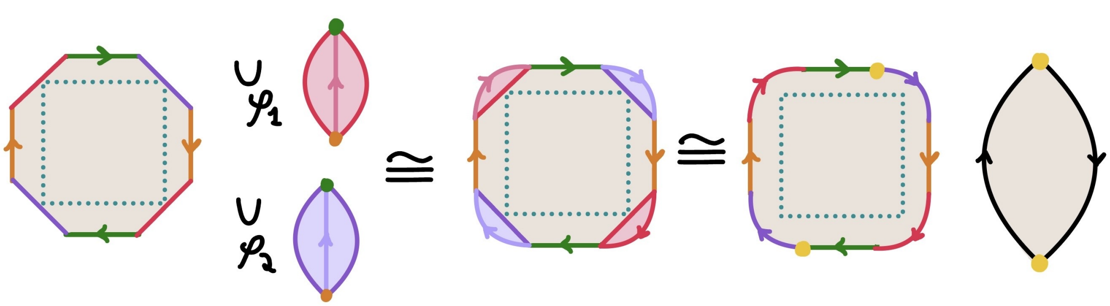
a) Let A be the "horizontal" Mobius band and B be the "vertical" Mobius band. Then $X = A \cup_C B$, where C is the center square, as shown below. We can now more clearly



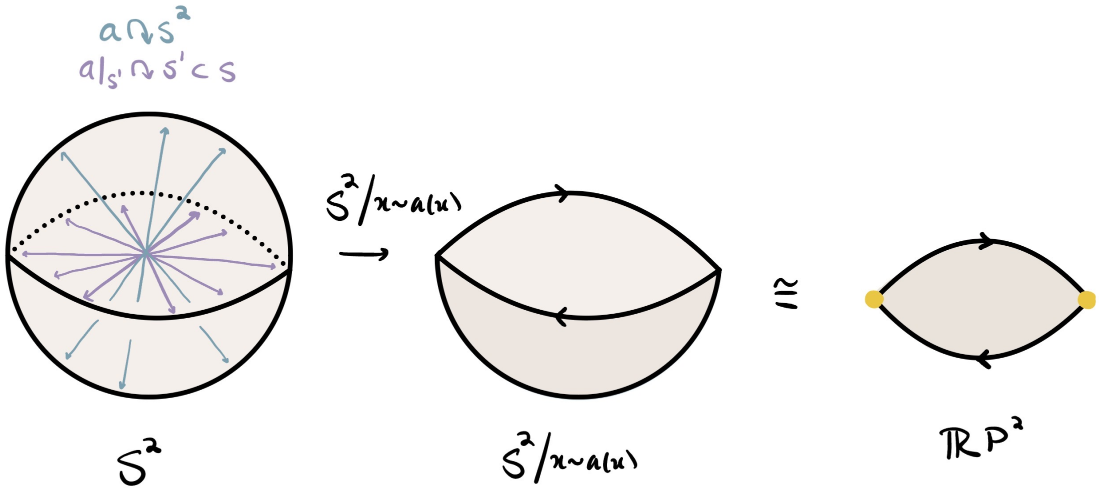
identify the two boundary components as $\partial X = Y_1 \sqcup Y_2$, shown in red and purple below, and isotope the surface as shown.



Attaching two disks D_1, D_2 to X via the maps $\varphi_1 : \partial D_1 \to Y_1$, $\varphi_2 : \partial D_2 \to Y_2$ can be equivalently viewed as identifying the red sides and the blue sides in a way that agrees with the identification of the green and orange vertices. By splitting each disk in half, we can glue the boundaries of the disks to the boundaries of X, remembering the identifications of the disks. Since the orientation and positions of the four sets of edges we must identify agree across opposite sides, we can instead think of the identification as identifying opposite sides of X, resulting space we know as $\mathbb{RP}^2$.



b) Let $S^2 \subset \mathbb{R}^3$ be the unit circle, and let $a : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be the antipodal map. Since the antipodal map preserves norm, it induces a map $a : S^2 \to S^2$. Further, since the only fixed point of the antipodal map on $\mathbb{R}^3$ is the origin, which is not in the unit circle, and since the $a^2 = id$, a gives a fixed-point free $\mathbb{Z}/2$ action on S^2. Therefore the quotient $S^2/x \sim a(x)$ is a smooth surface which is double covered by S^2, and $\pi_1(S^2/x \sim a(x), *) \cong \mathbb{Z}/2\,\mathbb{Z}$. Finally, since the antipodal map is orientation reversing on $\mathbb{R}^3$, the quotient is nonorientable. By the classification of surfaces, we conclude that $S^2/x \sim a(x) \cong \mathbb{R}\mathrm{P}^2$; therefore S^2 is the orientation double cover of $\mathbb{R}\mathrm{P}^2$, and the action of the deck group is the antipodal map.



Solution 2. Let $f : \partial D^3 \to T$ be the attaching map of the three cell, and denote the three cell in $T \cup_f D^3$ by K. Recall that T admits a cellular decomposition into one 0-cell v, two 1-cells a and b attached by $\partial_1(a) = \partial_1(b) = v - v = 0$, and one 2-cell F attached by $\partial_2(F) = a + b - a - b = 0$.
We now calculate cellular homology:

$$0 \xrightarrow{\partial_4} \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^2 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0$$

Since $\partial_4, \partial_2, \partial_1$, and ∂_0 are the zero map,

$$H_0 = \frac{\ker \partial_0}{\operatorname{im} \partial_1} \cong \frac{\langle v \rangle}{\langle v - v \rangle} \cong \mathbb{Z}$$

$$H_1 = \frac{\ker \partial_1}{\operatorname{im} \partial_2} \cong \frac{\langle a, b \rangle}{\langle a + b - a - b \rangle} \cong \mathbb{Z}^2$$

$$H_2 = \frac{\ker \partial_2}{\operatorname{im} \partial_3} \cong \frac{\langle F \rangle}{\operatorname{im} \partial_3}$$

$$H_3 = \frac{\ker \partial_3}{\operatorname{im} \partial_4} \cong \frac{\ker \partial_3}{0}$$

It now remains to compute ∂_3, which sends $\partial D^3 \cong S^2$ to the two-cell complex T. We claim that this map is null homologous. To see this, we use the lifting criterion. Recall that $\tilde{T} \cong \mathbb{R}^2$ and T is path-connected and locally path-connected. Since $\pi_1(S^2, x) = 1$, $f_*(\pi_1(S^2, x)) \subset p_*(\pi_1(\tilde{T}, \tilde{y}))$, thus the attaching map $f : S^2 \to T$ admits a lift $\tilde{f} : S^2 \to \mathbb{R}^2$ and $f = p \circ \tilde{f}$. However, since $H_*(\mathbb{R}^2) = 0$, the induced map on homology factors through zero, f_* must be zero as well. Thus $\ker \partial_3 = K$ and ,$\operatorname{im} \partial_3 = 0$ so

$$H_2 = \frac{\ker \partial_2}{\operatorname{im} \partial_3} \cong \frac{\langle F \rangle}{0} = \mathbb{Z}$$

$$H_3 = \frac{\ker \partial_3}{\operatorname{im} \partial_4} \cong \frac{\langle K \rangle}{0} = \mathbb{Z}$$

Therefore the homology of X is:

$$H_n(X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n = 0, 2, 3 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & n \geq 4 \end{cases}$$

$\square$

Solution 3. Let $\iota : X \to Y$ be the inclusion map, and identify X with $\iota(X)$. Let $p \in X \subset Y$ be the wedge point. Define a map $f : Y \to X$ by

$$f(y) = \begin{cases} \mathrm{id}, & x \in X \\ p, & x \notin X \end{cases}$$

Clearly f is continuous: take $U \subseteq X$ any open set. If $p \notin l]U$, then $f^{-1}(U) = U$ which is open. If $p \in U$, then $f^{-1}(U) = U \cup Y \backslash X$, which is open is the wedge topology. Thus f is a retraction.
On the other hand, if there was a deformation retraction $g : Y \times I \to X$, it would induce an isomorphism on homology $g_* : H_*(Y) \to H_*(X)$. However recall that since $Y = X \vee S^2$, $H_2(Y) = H_2(X) \oplus H_2(S^2) = \mathbb{Z} \oplus \mathbb{Z} \ncong \mathbb{Z} = H_2(X)$. Since no isomorphism exists, a deformation retraction g cannot exist either.

Algebraic Topology Prelim Exam

August 15, 2023, 12:30–14:30

Solve all three problems.

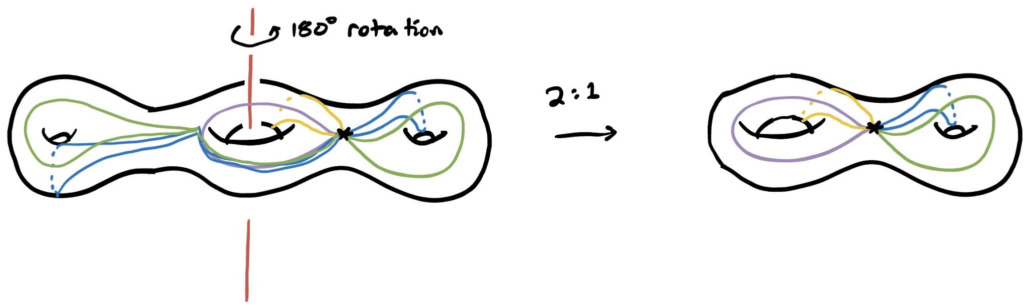
Problem 1. Let X be an orientable surface of genus 2, and Y be the connected sum of four copies of $\mathbb{R}P^2$. Find a common covering space, together with explicit covering maps from it to X and Y.

Problem 2. $\mathbb{Z}/2$ is the only nontrivial group that can act freely on S^n, for n even.

Problem 3. $S^a \times \mathbb{R}P^b$ and $S^b \times \mathbb{R}P^a$ are homeomorphic if and only if $a = b$.

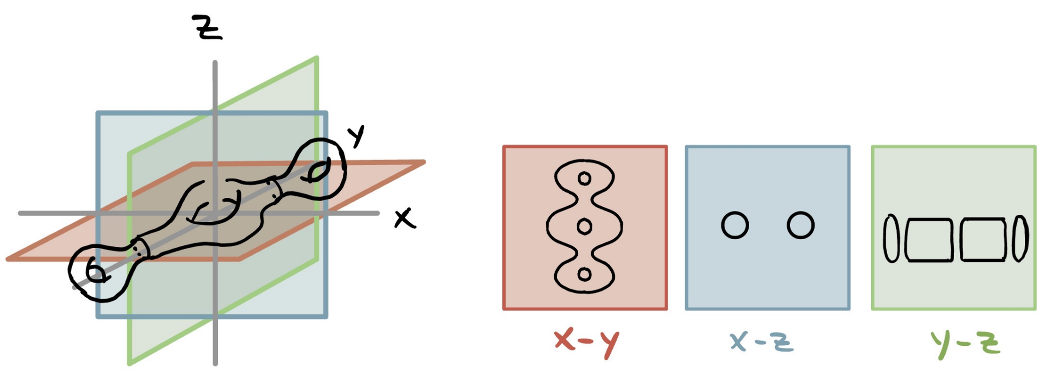
August 2023

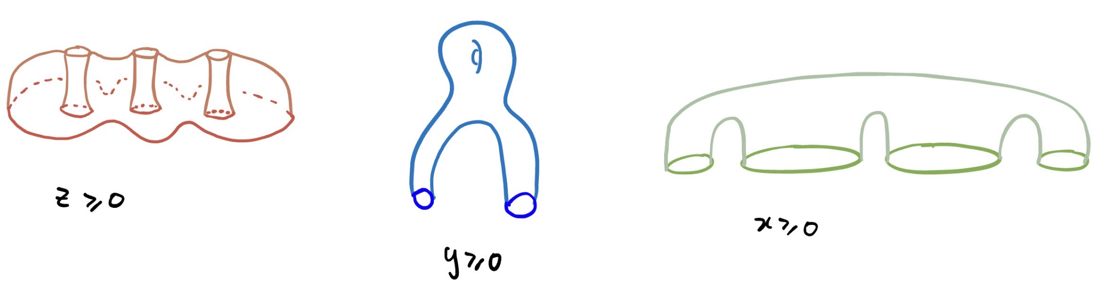
Solution 1. Recall that $\chi(\Sigma_2) = 2 - 2 * 2 = -2$ and $\chi(RP^2) = 1 \implies \chi(\#_4 RP^2) = 4 * 1 + 3 * 2 = -2$. Thus a common cover will have the same number of sheets for each surface. Let $\widetilde{X} = \Sigma_3$, and recall that the standard cover $p_0 : \widetilde{X} \to \Sigma_2$ is given by the quotient map of the properly discontinuous free $\mathbb{Z}/2$ action on Σ_3 shown below: This action



is fixed point free, as the fixed set of rotation about an axis in R^3 is the axis itself, and the axis does not intersect Σ_3, Further, $\mathbb{Z}/2$ is finite. Thus the projection to the quotient $p_0 : \tilde{X} \to \tilde{X}/r(x) \sim x$ is a covering map, and since this action is orientation-preserving, by the classification of surfaces, $\tilde{X}/r(x) \sim x$ is homeomorphic to the oriented surface of genus $(2 - 2 * 3)/2$, which is exactly Σ_2.

We now show that Σ_3 also covers $\#_4 RP^2$. Consider the following embedding of $\Sigma_3 \hookrightarrow \mathbb{R}^3$ which is symmetric about all three coordinate planes, as shown below. Since the image





of the embedding is symmetric about all three coordinate planes, the maps $r_i : \mathbb{R}^3 \to \mathbb{R}\,3$ which switches the sign of the ith coordinate, for $i = 1, 2, 3$, send Σ_3 to itself. Since the antipodal map on $\mathbb{R}^3$ is the composition of those three reflections, it descends to an involution $a : \Sigma_3 \to \Sigma_3$. Note that the only fixed point in $\mathbb{R}^3$ of the antipodal map is the origin, which is not in the image of the embedding of Σ_3, so the antipodal map on Σ_3 is fixed point free. Since $a^2 = \mathrm{id}$, the antipodal map gives a free $\mathbb{Z}/2$ action on Σ_3 whose quotient $\Sigma_3/x \cong a(x)$ is a manifold, thus we have a covering map $p : \Sigma_3 \to \Sigma_3/x \cong a(x)$. Since the antipodal map is orientation reversing, the quotient must be orientable. Since this action also has order 2, the Euler characteristic of the quotient must be -2. By the classification of surfaces, we conclude that $\Sigma_3/x \cong a(x) \cong \#_4 RP^2$ as desired. $\square$

Solution 2. Recall that a group action $G \circlearrowright S^n$ defines a homeomorphism $\varphi_g : S^n \to S^n$ for each $g \in G$. Since homeomorphisms have inverses and since the degree of a map is multiplicative under composition,

$$1 = \deg(\mathrm{id}_{S^n}) = \deg(\varphi_g^{-1} \circ \varphi_g) = \deg(\varphi_g^{-1})\deg(\circ\varphi_g).$$

Thus $\deg(\varphi_g) = \pm 1$, so a group action $G \circlearrowright S^n$ defines a homomorphism $f : G \to \mathbb{Z}/2 \cong \{\pm 1\}$. Now, recall that every fixed-point free map from the sphere to itself is homotopic to the antipodal map. The antipodal map has degree $\deg(a) = (-1)^{n+1}$ since it is the product of $n+1$ reflections of $S^n \hookrightarrow \mathbb{R}^{n+1}$. If n is even, $\deg(a) = -1$, thus $\deg(\varphi_g) = -1$ for all nontrivial $g \in G$. Thus G is a nontrivial subgroup of $\mathbb{Z}/2$, so $G \cong \mathbb{Z}/2$ $\square$

Solution 3.

One direction is obvious. If $a = b$, then $S^a \times RP^b = S^a \times RP^a$ and $S^b \times RP^a = S^a \times RP^a$ which are clearly homeomorphic.

For the other direction, we now show that $H_*(S^a \times RP^b) \cong H_*(S^b \times RP^a)$ only when $a = b$. Since the induced map of a homeomorphism on homology is an isomorphism, the conclusion follows.

Recall that $H_n(S^a) \cong \begin{cases} \mathbb{Z}, & n = 0, a \\ 0, & \text{else} \end{cases}$, and the homology of $\mathbb{R}\mathrm{P}^a$ also depends on whether a is even or odd:

a odd:

$$H_n(\mathbb{R}\mathrm{P}^a) \cong \begin{cases} \mathbb{Z}, & n = a, 0 \\ \mathbb{Z}/2, & 0 < n < a,\ n \text{ odd} \\ 0, & \text{else} \end{cases}$$

a even:

$$H_n(\mathbb{R}\mathrm{P}^a) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2, & 0 < n < a,\ n \text{ odd} \\ 0, & \text{else} \end{cases}$$

and the Künneth theorem: $H_n(X \times Y) = \bigoplus_{i+j=n} H_i(X) \otimes_{\mathbb{Z}} H_j(Y)$.

First we consider when $a = 1$:

$$\pi_1(S^1 \times \mathbb{R}\mathrm{P}^b) \cong \pi_1(S^1) \times \pi_1(\mathbb{R}\mathrm{P}^b) \cong \begin{cases} \mathbb{Z} \times \mathbb{Z}, & b = 1 \\ \mathbb{Z} \times \mathbb{Z}/2, & b \geq 2 \end{cases}$$

$$\pi_1(S^b \times \mathbb{R}\mathrm{P}^1) \cong \pi_1(S^b) \times \pi_1(\mathbb{R}\mathrm{P}^1) \cong \begin{cases} \mathbb{Z} \times \mathbb{Z}, & b = 1 \\ \mathbb{Z}, & b \geq 2 \end{cases}$$

$$\implies S^1 \times \mathbb{R}\mathrm{P}^b \cong S^b \times \mathbb{R}\mathrm{P}^1 \text{ only when } a = b = 1.$$

Now, assume $a, b \geq 2$. By the Künneth theorem, $H_a(S^a \times \mathbb{R}\mathrm{P}^b)$ and $H_b(S^b \times \mathbb{R}\mathrm{P}^a)$ are as follows:

If a is odd:

$$H_a(S^a \times \mathbb{R}\mathrm{P}^b) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2, & a < b \\ \mathbb{Z} \oplus \mathbb{Z}, & a = b \\ \mathbb{Z}, & a > b \end{cases} \qquad H_a(S^b \times \mathbb{R}\mathrm{P}^a) = \begin{cases} \mathbb{Z}, & a < b \\ \mathbb{Z} \oplus \mathbb{Z}, & a = b \\ \mathbb{Z}, & a > b \end{cases}$$

If a is even:

$$H_a(S^a \times \mathbb{R}\mathrm{P}^b) = \begin{cases} \mathbb{Z}, & a < b \\ \mathbb{Z}, & a = b \\ \mathbb{Z}, & a > b \end{cases} \qquad H_a(S^b \times \mathbb{R}\mathrm{P}^a) = \begin{cases} 0, & a < b \\ \mathbb{Z}, & a = b \\ 0, & a > b \end{cases}$$

Thus if a is odd, $H_a(S^a \times \mathbb{R}\mathrm{P}^b) \cong H_a(S^b \times \mathbb{R}\mathrm{P}^a)$ when $a = b$ or $a > b$, and if a is even, $H_a(S^a \times \mathbb{R}\mathrm{P}^b) \cong H_a(S^b \times \mathbb{R}\mathrm{P}^a)$ only when $a = b$.

However, when a is odd and $a > b$,

$$H_b(S^a \times \mathbb{R}\mathrm{P}^b) \cong \mathbb{Z}, \quad \text{but} \quad H_b(S^b \times \mathbb{R}\mathrm{P}^a) \cong \mathbb{Z} \oplus \mathbb{Z}/2$$

Thus $H_n(S^a \times \mathbb{R}\mathrm{P}^b) \cong H_n(S^b \times \mathbb{R}\mathrm{P}^a)$ for all n only when $a = b$. Thus$S^a \times \mathbb{R}\mathrm{P}^b \cong S^b \times \mathbb{R}\mathrm{P}^a \iff a = b$. $\square$

Algebraic Topology Prelim, January 2023

Do all three questions. The questions are weighted equally.

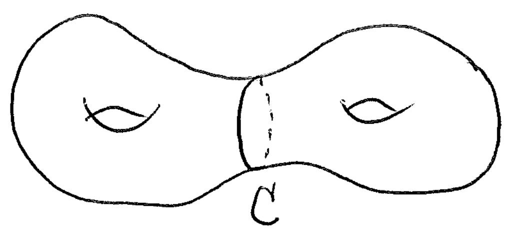
1. (a) Show that every non-surjective map from S^n to S^n has a fixed point.

(b) Show that (a) is false if S^n is replaced by T^n, $n \geq 2$.

2. Let S be a surface of genus 2, and let $C \subset S$ be the circle shown in the figure. Let X be the space obtained by gluing a Möbius band B to S by identifying the boundary of B with C by some homeomorphism.

(a) Compute the singular homology of X.

(b) Show that there is no retraction from X to the core circle of B.



3. Let X be the connected sum of three copies of $\mathbb{R}P^2$. Explicitly describe two connected 2-fold covering spaces $X_1 \to X$ and $X_2 \to X$, where X_1 is orientable and X_2 is non-orientable.

January 2023

Solution 1.

a) Let $f : S^n \to S^n$ be non-surjective. Then $\mathrm{im}(f)^n \backslash \{p\}$, so f factors as

$$S^n \to S^n \backslash \{p\} \cong^n \to S^n.$$

Thus the induced map on H_n factors as

$$H_n(S^n) \cong \mathbb{Z} \to H_n(^n) \cong 1 \to H_n(S^n) \cong \mathbb{Z},$$

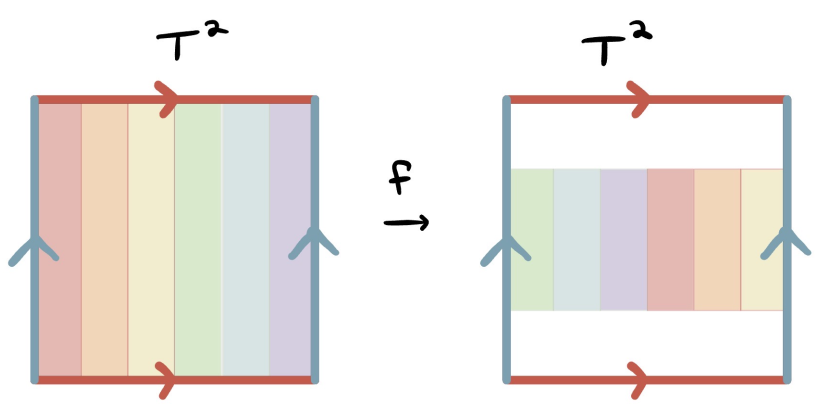
thus f_* must be trivial and so $\deg(f) = 0$.
Now, assume torwards contradiction that f is fixed point free. If we consider $S^n \subset \mathbb{R}^{n+1}$ as the unit circle, for every x^n, we can define $\gamma_x : [0,1] \to^{n+1}$ which sends $t \mapsto (1-t)f(x)+t(a(x))$, where $a(x) = -x$ is the antipodal map. Notice that the image of γ_x does not pass through zero. If it did, $(1-t)f(x) = tx$ for some t, so $f(x) = \frac{t}{1-t}x$, and since x and $f(x)$ are on the unit circle, this would imply that $f(x) = \pm x$. If $f(x) = x)$, we contradict our hypothesis, and if $f(x) = -x = a(x)$, the image of γ_x is $-x \neq 0$. Since every point on the line segment for each x has nonzero norm, we can divide by the norm at each point, which gives a homotopy $F_t(x) : S^n \times I \to S^n$, $F_t(x) = \frac{\gamma_x(t)}{||\gamma_x(t)||}$ between f and a. Therefore $\deg(f) = \deg(a) = (-1)^{n+1} \neq 0$, which is a contradiction. Thus f must have a fixed point.

b) Identify $T^n \cong \mathbb{R}^n \,/\, \mathbb{Z}^n$, and define $f : T^n \to T^n$ by

$$(x_1, \ldots, x_n) \mapsto (x_1 + \frac{1}{2} \pmod 1, \frac{1}{2}x_i + \frac{1}{4}) \text{ for } i = 2, \ldots, n$$

. First note that the shift map $x_1 \mapsto x_1 + \frac{1}{2} \pmod 1$ is fixed point free, so f cannot have a fixed point. Note further that the point $(0, \ldots, 0) \notin \mathrm{im}(f) \subset T^n$, since $0 \neq \frac{1}{2}x + \frac{1}{4}$ when $x \in [0,1]$. Since both the shift and compression operations are continuous on each factor, we get a continuous, nonsurjective, fixed point free map from the torus to itself.

Ex: n=2:



Solution 2.

a) Let $\varphi\colon \partial B \to C$ be the homeomorphism which attaches the boundary to of the Möbius band to the curve $C \subset S$. Since $\partial B \cong S^1 \cong C$, φ is determined by an integer k, which is the degree. However, since φ is a homeomorphism, $k = \pm 1$.

Since S has no cells of dimension ≥ 3, $H_n(S) = 0 \quad \forall n \geq 4$. Further, since S is connected, $H_0(S) \cong \mathbb{Z}$.

We use now apply Mayer-Vietoris, using the obvious decomposition $X = S \cup_C B$. Recall that

$$H_n(\Sigma_2;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 2,0 \\ \mathbb{Z}^4, & n = 1 \\ 0, & \text{else} \end{cases} \quad \text{and} \quad H_n(M;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 1,0 \\ 0, & \text{else} \end{cases}$$

Since S and B are glued along an S^1,

$$H_n(\Sigma_2 \cap M) \cong H_n(S^1) = \begin{cases} \mathbb{Z}, & n = 0,1 \\ 0, & \text{else.} \end{cases}$$

Mayer–Vietoris gives:

$$\begin{aligned} H_3(S^1) \xrightarrow{i_3} H_3(M) \oplus H_3(\Sigma_2) \xrightarrow{\partial_3} H_3(X) \longrightarrow H_2(S^1) \xrightarrow{i_2} H_2(M) \oplus H_2(\Sigma_2) \xrightarrow{\partial_2} H_2(X) \\ \longrightarrow H_1(S^1) \xrightarrow{i_1} H_1(M) \oplus H_1(\Sigma_2) \xrightarrow{\partial_1} H_1(X) \longrightarrow H_0(S^1) \xrightarrow{i_0} H_0(M) \oplus H_0(\Sigma_2) \end{aligned}$$

which becomes

$$\begin{aligned} 0 \xrightarrow{i_3} 0 \oplus 0 \xrightarrow{j_3} H_3(X) \xrightarrow{\partial_3} 0 \xrightarrow{i_2} 0 \oplus \mathbb{Z} \xrightarrow{j_2} H_2(X) \xrightarrow{\partial_2} \mathbb{Z} \\ \xrightarrow{i_1} \mathbb{Z} \oplus \mathbb{Z}^4 \xrightarrow{j_1} H_1(X) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{i_0} \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

Therefore $H_3(X) \cong 0$.

We now find H_2.
Since $\operatorname{im} i_2 \cong 0 \cong \ker j_2$, we know that $\mathbb{Z} \subset H_2$ and thus $\operatorname{im} j_2 \cong \mathbb{Z} \cong \ker \partial_2$.

Now we must find i_1. Recall that the inclusion of $\partial M \hookrightarrow M$ induces the map $1 \mapsto 2$ on homology, since the boundary of the Möbius band double covers its core circle, which generates the homology. On the other side, since C is the $aba^{-1}b^{-1}$ curve on Σ_2,

$$[C] = 0 \in H_1(\Sigma_2).$$

Thus

$$\begin{gathered} i_1\colon \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}^4, \qquad 1 \longmapsto (2,0). \\ \Rightarrow \quad \ker i_1 = 0, \qquad \operatorname{im} i_1 = 2\mathbb{Z} \subset \mathbb{Z}. \\ \text{Thus } \operatorname{im} \partial_2 = 0, \text{ so } H_2(X) \cong \mathbb{Z}. \end{gathered}$$

Finally, for H_1, since $\operatorname{im} i_1 = \ker j_1 = 2\mathbb{Z}$, we know that $\mathbb{Z}/2 \oplus \mathbb{Z}^4 \subset H_1$, so $\mathbb{Z}/2 \oplus \mathbb{Z}^4 = \operatorname{im} j_1 = \ker \partial_1$. Therefore $\operatorname{im} \partial_1 = \ker i_0 = 0$, so $H_1(X) \cong \mathbb{Z}^4 \oplus \mathbb{Z}/2$.

In full,

$$\Rightarrow \quad H_n(X) = \begin{cases} \mathbb{Z}, & n = 0, 2 \\ \mathbb{Z}^4 \oplus \mathbb{Z}/2, & n = 1 \\ 0, & \text{else} \end{cases}$$

$\square$

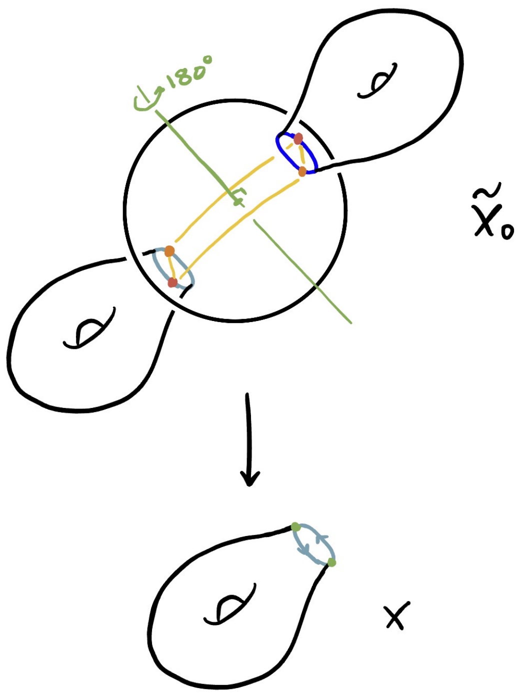
b) Assume there was a retraction of X to the core circle of B. Denote this circle γ. The inclusion $\gamma \hookrightarrow X$ comes from the inclusion $\gamma \hookrightarrow B \hookrightarrow X$, and must be injective on homology. As we just computed, $H_1(\gamma) \cong \mathbb{Z}$, and the inclusion of the core circle induces the map $\mathbb{Z} \cong H_1(\gamma) \to H_1(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^4$. Since $2[\gamma] \cong [\partial B]$, which is nullhomologous in X, i_γ must send $2 \mapsto (0, 0)$. Therefore the map induced by the inclusion is

$$i_\gamma : \mathbb{Z} \to \mathbb{Z}/2 \oplus \mathbb{Z}^4, \qquad 1 \mapsto (1, 0)$$

which has kernel $2\mathbb{Z} \subset \mathbb{Z}$, thus is not injective.

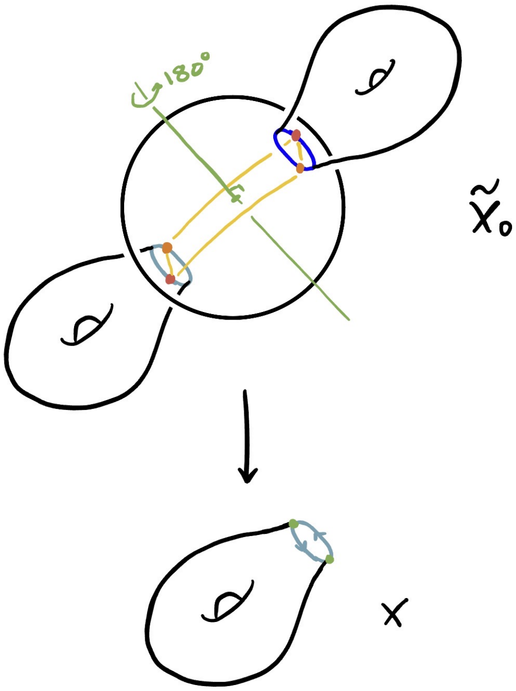
Thus no retraction can exist. $\square$

Solution 3. Recall that $\chi(RP^2) = 1 - 1 + 1 = 1$, and $\chi(X\$Y) = \chi(x) + \chi(Y) - 2$ for surfaces. Thus $\chi(\#_3\,\mathbb{R}\mathrm{P}^2) = (1 + 1 - 2) + (1) - 2 = -1$. Further, recall that if $\tilde{(}X)$ is a double cover of X, $\chi(\tilde{X} = 2\chi(X)$. Thus for $\tilde{X}$ a double cover of $\#_3\,\mathbb{R}\mathrm{P}^2$, $\chi\tilde{X} = -2$.
Let $\tilde{X}_o$ be the orientable 2:1 cover. By the classification of orientable surfaces, $\chi(\tilde{X}_o = 2 - 2g$, so $g = 2$ and $\tilde{X}_o \cong \Sigma_2$. To see the covering map $p_o : \tilde{X}_o \to X$, first recall that $X \cong T^2\#RP^2$, and view Σ_2 as $S^2\#T^2\#T^2$, where the disks D_1, D_2 removed from S^2 in the connect sum operation are such that $a(D_1) = D_2$ where a is the antipodal map on S^2. Further, viewing ∂D^1 as an oriented embedding of S^1 into S^2, note that the image of ∂D_1 under a is the same as the image under 180 degree rotation around the axis perpendicular to the plane through two opposite points on ∂D_1 and their antipodes on D_2; see the figure.



Extending this map to $\mathbb{R}^3$ and viewing S^2 as the unit circle, $S^2\#T^2\#T^2$ can be embedded such that the rotation on R^3 interchanges the two T^2 connect summands, and agrees with a on the connect sum region. This allows us to define a map f from $S^2\#T^2\#T^2$ to itself, which is the antipodal map on $S^2\backslash D_1 \sqcup D_2$ and rotation on the rest. Note that $f^2 = \mathrm{id}$ and f is fixed point free, so we get a free $\mathbb{Z}/2$ action on $\tilde{X}_o$. Quotienting by this involution collapses the two $T^2\backslash D_i$'s into one, and sends $S^2\backslash D_1 \sqcup D^2 \cong S^1 \times I$ to $S^1\tilde{\times} I \cong M \cong RP^2\backslash^2$. Since $\tilde{X}_o/x \sim f(x)$ is non-orientable and is double covered by Σ_2, by the classification of surfaces, $\tilde{X}_o/x \sim f(x) \cong X$, so we have constructed an orientable double covering space of X.

For the nonorientable cover $\tilde{X}_n$, we adjust our previous idea by using that $\#_3RP^2 \cong K\#RP^2$, and take $\tilde{X}_n = K\#K$. Indeed, $\chi(K) = 1-2+1 = 0$, so $\chi(K\#K) = 0+0-2$ as desired. Since T^2 and K can both be constructed from $S^1 \times I$ by using either the orientation preserving or reversing map of the boundaries, we can produce a new covering map $p : \tilde{X}_n \to X$ from the previous one by removing an $S^1 \times I$ from each T^2 summand and choosing the orientation-reversing boundary identification. Since the quotient map given above restrict to a quotient map on $\tilde{X}_o\backslash S^1 \times I \sqcup S^1 \times I$, and we may choose twisted boundary identifications on each $S^1 \times I$ factor which are equivariant with respect to the rotation map, we get a well defined quotient $\tilde{X}_n/f \cong X$.



It remains to show that $K\#K$ is nonorientable. Since it has the same Euler characteristic as Σ_2, by the classification of surfaces, it is enough to show that they are not homeomorphic. Indeed, we compare the abelianizations of their π_1's:

$$\pi_1(K\#K,*) \cong \langle a,b,c,d | abab^{-1} = cdcd^{-1}\rangle$$

$$\implies \pi_1(K\#K,*)^{ab} \cong \langle a,b,c,d | a^2 = c^2, \text{all commutators}\rangle$$

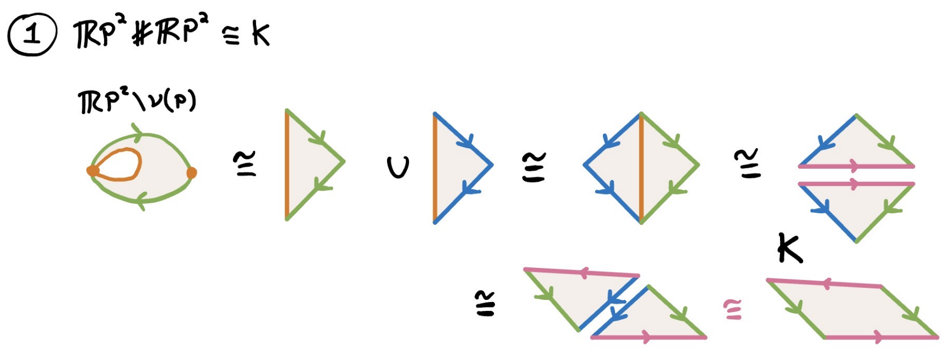
$$\cong \mathbb{Z}/2 \oplus \mathbb{Z}^3$$

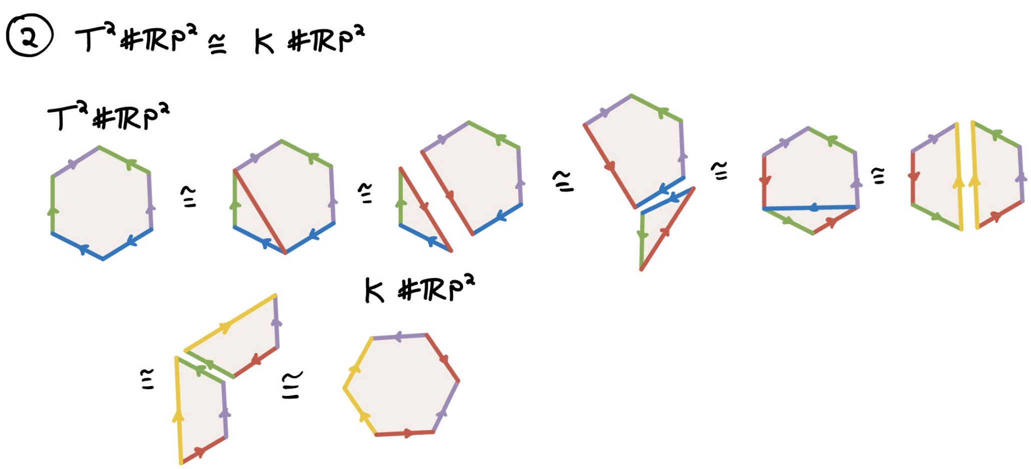
$$\pi_1(\Sigma_2,*) \cong \langle a,b,c,d | aba^{-1}b^{-1} = cdc^{-1}d^{-1}\rangle$$

$$\implies \pi_1(\Sigma_2,*)^{ab} \cong \langle a,b,c,d | \text{all commutators}\rangle \cong \mathbb{Z}^4$$

Since their fundamental groups have different abelianizations, KK is not homeomorphic to Σ_2. Therefore $K\#K$ is a nonorientable double cover of $\#_3^2$.

Finally, we prove the claim that $\#_3RP^2 \cong T^2\#RP^2 \cong K\#RP^2$.





Algebraic Topology Prelim, August 2022

Do all three questions. The questions are weighted equally.

1. Let M_1 and M_2 be copies of the Möbius band, and let n be a positive integer. Parametrizing ∂M_1 and ∂M_2 as S^1, the unit circle in the complex plane, let $f : \partial M_1 \to \partial M_2$ be the map $f(z) = z^n$. Let X be the space obtained from the disjoint union of M_1 and M_2 by gluing ∂M_1 to ∂M_2 by the map f. Compute the homology of X.

2. (a) Let X be the wedge of two circles. Describe two connected 2-fold covering spaces of X that are not homeomorphic.

(b) Let S be a closed orientable surface of positive genus. Show that any two connected 2-fold covering spaces of S are homeomorphic.

3. Let X and Y be topological spaces, and let $x_0 \in X$, $y_0 \in Y$.
Show that $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

August 2022

Solution 1.
We use Mayer-Vietoris: Decompose $X = M_1 \cup_{S^1} M_2$, so $M_1 \cap M_2 \cong S^1$.

Since X is connected, $H_0(X) \cong \mathbb{Z}$ and since $\dim(X) = 2$, $H_n(X) = 0\ \forall n \geq 3$. Further, recall that the Möbius band $M := [0,1] \times [0,1]/(1,t) \sim (0, 1-t)$ deformation retracts onto its core circle $S^1 \times \{1/2\}$, so $H_n(M;\mathbb{Z}) \cong H_n(S^1;\mathbb{Z})$. Therefore the relevant part of the long exact sequence is:

$$0 \to H_2(S^1) \xrightarrow{\iota_2} H_2(M_1) \oplus H_2(M_2) \xrightarrow{j_2} H_2(X) \xrightarrow{\partial_1} H_1(S^1) \to 0$$

$$0 \quad \longrightarrow \quad 0 \oplus 0 \quad \longrightarrow \quad H_2 \quad \longrightarrow \quad \mathbb{Z}$$

$$\xrightarrow{\iota_1} H_1(M_1) \oplus H_1(M_2) \xrightarrow{j_1} H_1(X) \xrightarrow{\partial_1} H_0(S^1)$$

$$\mathbb{Z} \oplus \mathbb{Z} \quad \longrightarrow \quad H_1 \quad \longrightarrow \quad \mathbb{Z} \quad \longrightarrow 0$$

We now must find the image and kernel of the map ι_1.

$$\iota_1 : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$

$$1 \longmapsto (2, 2n)$$

Since the boundary ∂M_i double covers the core circle which generates H_1, and since we are identifying n-times the boundary of M_2 with M_1. This tells us that

$$\operatorname{im} \iota_1 = 2\,\mathbb{Z} \oplus 2n\,\mathbb{Z} = \ker j_1 \text{ and } \ker \iota_1 = 0 = \operatorname{im} \partial_2$$

Since j_2 is the zero map, $H_2 \cong 0$. Finally, since ι_0 is the zero map, $\ker \iota_0 = \mathbb{Z} = \operatorname{im} \partial_1$, so $H_1 \cong \dfrac{\mathbb{Z} \oplus \mathbb{Z}}{2\,\mathbb{Z} \oplus -2n\,\mathbb{Z}} \oplus \mathbb{Z}$

iTo better identify $H_1(X;\mathbb{Z})$, first note that the kernel of $\mathbb{Z} \oplus \mathbb{Z}$ consists of vectors in the span of $(2, -2n)$

Now, apply the change of basis $a \mapsto a$, $b \mapsto b + na$, which in matrix form is left multiplication by $\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$. This matrix is invertible over $\mathbb{Z}$ since $\det = 1 \cdot 1 - 0 \cdot n = 1$.

This change of basis applied to the vector $(2, -2n)$ gives

$$\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2n \end{bmatrix} = \begin{bmatrix} 2 \\ 2n - 2n \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

thus the quotient becomes

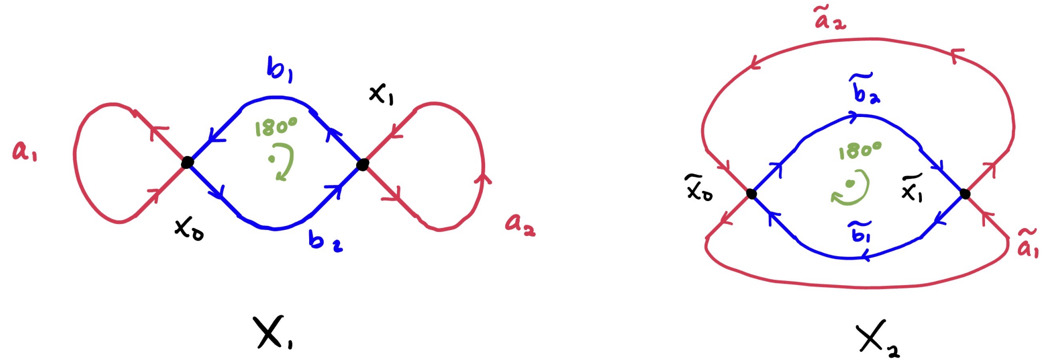
$$\frac{\mathbb{Z} \oplus \mathbb{Z}}{2\,\mathbb{Z} \oplus -2n\,\mathbb{Z}} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{2\,\mathbb{Z} \oplus 0} \cong \mathbb{Z}/2 \oplus \mathbb{Z}.$$

$$\implies H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2$$

Therefore $H_n(X) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2, & n = 1 \\ 0, & n \geq 2 \end{cases}$ $\square$

Solution 2.

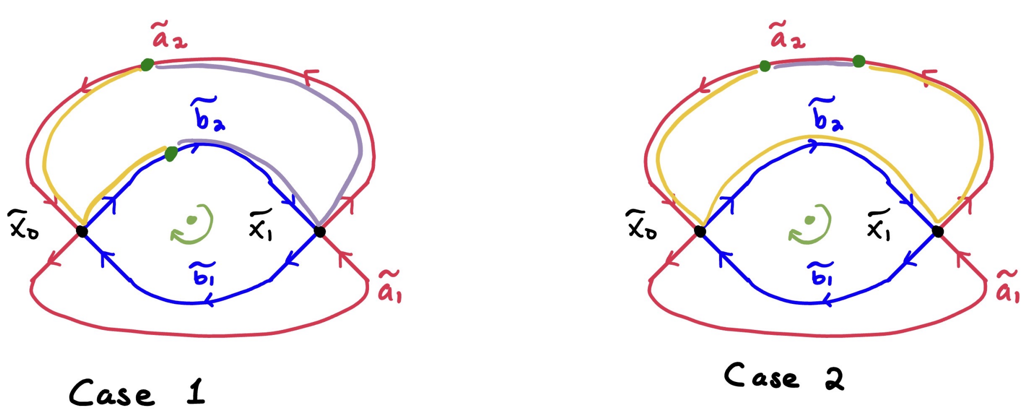
a) Consider the covering spaces of $S^1 \vee S^1$ shown below. Note that the involution on



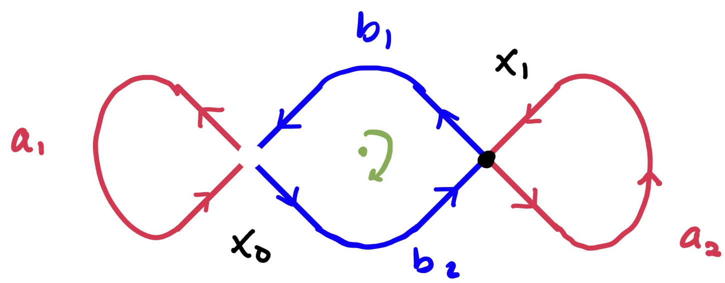
each $\widetilde{X}$ given by 180 degree rotation about the green point is fixed point free, giving a free $\mathbb{Z}/2\mathbb{Z}$ action which we may quotient by to get $S^1 \vee S^1$. Thus X_1 & X_2 are degree-2 covering spaces of $S^1 \vee S^1$. Note that both X_1, X_2 are homotopy equivalent to $S^1 \vee S^1 \vee S^1$, so they are homotopy equivalent. We now show that X_1 and X_2 are not homeomorphic. Note that we cannot use the fundamental group to obstruct a homeomorphism since they are homotopy equivalent. We begin by first assuming there exists a homeomorphism $f : X_1 \to X_2$.

Note that $X_1 \backslash x_0$ is disconnected, and f restricts to a homeomorphism $f|_{X_1 \backslash x_0} : X_1 \backslash x_0 \to X_2 \setminus f(x_0)$, so $X_2 \setminus f(x_0)$ must be disconnected. We show there is no $p \in X_2$ st $X_2 \setminus p$ is disconnected. Since every 1-cell is attached along both x_0 and x_1, for any two points q_1, q_2 we have two cases:

- if q_1 and q_2 are on different 1-cells, there is a path between them through x_0, and a path between them through x_1, which are disjoint away from the ends.
- if q_1 and q_2 are on the same 1-cell, there is a path between them through both x_0 and x_1, and a path through neither, which are disjoint away from the ends.



Thus for any point $p = f(\widetilde{x}_0)$, $X_2 \backslash p$ is still path connected, hence connected. However, $X_1 \setminus x_0$ is clearly disconnected, since every path between $q_1 \in a_1$ and $q_2 \in X_1 \setminus a_1$ must pass through x_0, as a_1 is attached by sending both ends to x_0. Thus there can be no homeomorphism $f : X_1 \to X_2$. $\square$



b) Recall that if $\widetilde{X} \xrightarrow{2:1} \Sigma_g$, $\chi(\widetilde{X})$, and that $\chi(\widetilde{X}) = 2 - 2g$. Therefore

$$\begin{aligned} \chi(\widetilde{X}) &= 2(2-2g) \\ &= 4 - 4g \\ &= 2 - 2(2g-1) \end{aligned}$$

Since $\widetilde{X}$ is connected & orientable, and since every genus g surface is homeomorphic to the standard one, $\widetilde{X} \cong \Sigma_{2g-1}$ for every 2-fold cover of Σ_g. Thus they are all homeomorphic. $\square$

Solution 3. Define $\varphi : \pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$ by

$$[f] \longmapsto (p_{x*}([f]),\ p_{y*}([f]))$$

We must show φ is surjective & injective.

Surjective: Take $(\alpha, \beta) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$, and choose based loops $a : I \to X$ and $b : I \to Y$ such that $\alpha = [a]$, $\beta = [b]$.

Define $c : I \to X \times Y$ by $c(t) = (a(t), b(t))$.

Clearly $c(0) = (a(0), b(0)) = (x_0, y_0) = c(1) = (a(1), b(1))$, so c is a based loop in $X \times Y \implies [c] = \gamma \in \pi_1(X \times Y, (x_0, y_0))$, and φ is surjective.

Injective: Let $c, \widetilde{c} : I \to X \times Y$ be based loops such that $\varphi([c]) = \varphi([\widetilde{c}])$. Then $\big(p_{x*}([c]),\ p_{y*}([c])\big) = \big(p_{x*}([\widetilde{c}]),\ p_{y*}([\widetilde{c}])\big)$, so

$$p_{x*}([c]) = p_{x*}([\widetilde{c}]) \text{ and } p_{y*}([c]) = p_{y*}([\widetilde{c}]),$$

$$\text{therefore } [p_x(c)] = [p_x(\widetilde{c})] \text{ and } [p_y(c)] = [p_y(\widetilde{c})].$$

$\implies p_x(c)$ is homotopic to $p_x(\widetilde{c})$, and same for p_y.

Let H_x and H_y be those homotopies. Then $H_x \times H_y$ is a based homotopy between c and $\widetilde{c}$, so $[c] = [\widetilde{c}]$, and thus φ is injective. $\square$

Algebraic Topology Prelim Exam
January 12, 2022

Problem 1. Suppose $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ is not surjective. Then

(a) the induced map $f_* : \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2)$ is trivial (Hint: what is $\mathbb{R}P^2 - \{\text{point}\}$?);

(b) f is homotopic to a constant map. (You may use the conclusion of (a), whether you prove it or not.)

Problem 2. In this problem, all surfaces are closed, connected and orientable.

(a) Suppose S is a surface of genus 2. Show that for every $g > 1$, the surface of genus g occurs as a covering space of S.

(b) Now suppose S has genus 3. For which values of g does the surface of genus g occur as a covering space of S?

Problem 3. Compute $\pi_1(X)$, where X is obtained from two tori $S^1 \times S^1$, by identifying $\{\text{point}\} \times S^1$ in the first torus with the diagonal of the second torus.

January 2022

Solution 1.

a) Recall $\mathbb{RP}^2 = M \cup D^2$. Thus $\mathbb{RP}^2 \setminus \{pt\} \cong M$.
If $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ is not surjective, then it must factor as

$$\mathbb{RP}^2 \longrightarrow \mathbb{RP}^2 \setminus \{pt\} \cong M \hookrightarrow \mathbb{RP}^2$$

Since the Möbius band $M := [0,1] \times [0,1]/(1,t) \sim (0, 1-t)$ deformation retracts onto its core circle $S^1 \times \{1/2\}$,

$$\pi_1(M, *) \cong \pi_1(S^1, *) \cong \mathbb{Z}\,.$$

Therefore $f_* : \pi_1(\mathbb{RP}^2) \to \pi_1(\mathbb{RP}^2)$ factors as

$$\pi_1(\mathbb{RP}^2) \to \mathbb{Z} \to \pi_1(\mathbb{RP}^2).$$

Since $\mathbb{RP}$ is the quotient of S^2 under the antipodal map, which is a free $\mathbb{Z}/2\mathbb{Z}$ action, $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$, so f_* is an automorphism of $\mathbb{Z}/2\mathbb{Z}$ which factors through $\mathbb{Z}$, and hence must be the trivial map. Therefore every nonsurjective f must be nullhomotopic.

$\square$

b) Since $\pi_1(S^2, *) \cong 1$ As shown above, S^2 covers $\mathbb{RP}^2$. Since $\pi_1(S^2, *) \cong 1$ and f_* is trivial, $f_*(\pi_1(\mathbb{RP}^2)) = 1 \subset p_*\pi_1(S^2) \subset \mathbb{RP}^2$. Hence f satisfies the lifting criterion, so f lifts to a map $\widetilde{f} : \mathbb{RP}^2 \to S^2$, and $f = p \circ \widetilde{f}$.

$$\begin{tikzcd} & S^2 \arrow[d, "p"] \\ \mathbb{R}P^2 \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & \mathbb{R}P^2 \end{tikzcd}$$

Since S^2 is simply connected, and $\mathbb{RP}^2$ is of dimension 2, any map $\widetilde{f} : \mathbb{RP}^2 \to S^2$ is homotopy equivalent to a map sending the 1-cell of $\mathbb{RP}^2$ to the 0-cell of S^2, which, by extending the nullhomotopy over the 2-cell of $\mathbb{RP}^2$, is homotopy equivalent to the constant map. Thus $\widetilde{f} \simeq c$, and since $p(c) = c$,

$$f = p \circ \widetilde{f} \simeq p \circ c \simeq c \qquad \square$$

Solution 2.

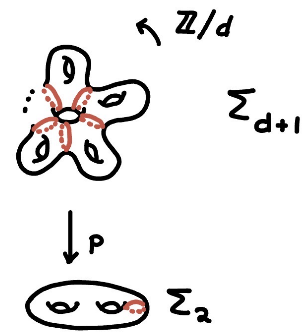
Recall that for $X =$ CW complex with $n_i = \#$ i-cells, a degree d cover of X must have $d \cdot n_i$ i-cells. Further recall that Σ_g admits a CW-decomposition with 1 0-cell, $2g$ 1-cells, and 1 2-cell. Thus the Euler characteristic of a degree d cover of Σ_g is $\chi(\widetilde{X}_d) = d\chi(\Sigma_g)$. But this cover must itself be a closed surface of some genus h. Therefore $\chi(\widetilde{X}_d) = 2 - 2h$. Combining these equalities and solving for h, we get

$$\begin{aligned} 2 - 2h &= d(2 - 2g) \\ h &= 2 - d(2 - 2g) \\ h &= 1 - d(1 - g) \end{aligned}$$

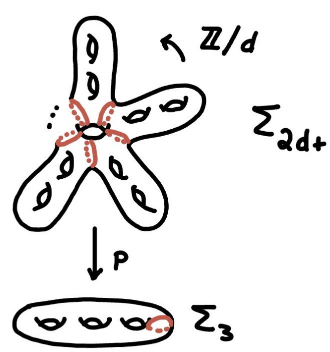
Thus the genus of a degree-d cover of Σ_g is $1 - d(1 - g)$.

When $g = 2$, a degree-d cover is the closed and oriented surface of genus $1 - d(1 - 2) = 1 + d$. Note that Σ_2 is a cover of itself by the identity map, so and no smaller genus surfaces can cover it by our Euler characteristic calculation, so to determine the admissible genera, we need only to find the $h \in \mathbb{N}_{\geq 2}$ which can be written as $h = d + 1$ for $d \in \mathbb{N}_{\geq 1}$, which is clearly all of $\mathbb{N}_{\geq 2}$.

To construct the covering space, start with $T^2 \in \mathbb{R}^3$ embedded symmetrically across all coordinate planes, and let z be the axis through the hole. Then, for every $d \geq 1$, the degree-d cover is $T^2 \#_d T^2$, where the $d + 1$ tori are attached symmetrically such that the rotation on R^3 given by rotating by $2\pi/d$ radians about the z axis permutes the connect summands. Clearly this rotation gives a free $\mathbb{Z}/d$ action on $T^2 \#_d T^2$, which we can quotient by to recover the surface of genus 2.



b) When $g = 3$, a degree-d cover is the closed and oriented surface of genus $1 - d(1 - 3) = 1 + 2d$. Since the smallest genus is 3, with Σ_3 covering itself by the identity map, the values of g which Σ_g covers Σ_3 are the odd numbers larger than or equal to 3. Similar to the case above, construct Σ_{2d+1} as $T^2 \#_{2d}$, where the $2d$ tori are arranged in pairs radiating outwards, and are exchanged by the same $\mathbb{Z}/d$ action as above. $\square$



Solution 3. Let T_1, T_2 be two copies of $S^1 \times S^1$.

Choose the following presentations: $\pi_1(T_1, pt) \cong \langle a, b \mid [a,b] \rangle$, $\quad \pi_1(T_2, pt) \cong \langle c, d \mid [c,d] \rangle$ where

$$a = [S^1 \times pt] \text{ in } T_1, \qquad b = [pt \times S^1] \text{ in } T_1,$$
$$c = [S^1 \times pt] \text{ in } T_2, \qquad d = [pt \times S^1] \text{ in } T_2.$$

Note that the identification of $\{pt\} \times S^1 \subset T_1$ and $\Delta \subset T_2$ induces the relation $\langle b = cd \rangle$. Thus by Seifert-van-Kampen,

$$\pi_1(X, pt) \cong \langle a, b \mid [a,b] \rangle * \langle c, d \mid [c,d] \rangle \big/ \langle b = cd \rangle$$
$$\cong \langle a, b, c \mid [a,b], [c,b] \rangle \qquad \square$$