DERIVATIVE ESTIMATES FOR VISCOUS SCALAR CONSERVATION LAWS

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In this short note, we will give a proof that solutions to the PDE:

$$\begin{cases} u_t + [f(u)]_x = u_{xx} & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1}$$

are smooth for any $t > 0$, assuming that $u_0 \in L^\infty(\mathbb{R})$. The theory developed will generalize to higher spacial dimensions for domains that are products of copies of $\mathbb{R}$ or $\mathbb{T}$. One might expect this because solutions to the heat equation

$$\begin{cases} u_t = u_{xx} & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases}$$

are smooth for any $t > 0$. Indeed, this is easily seen from dispersion estimates for the heat kernel. Denoting $e^{\Delta t} u := H_t * u$, where H_t is the heat kernel

$$H_t(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{\frac{-|x|^2}{4t}},$$

one has sharp dispersion estimates

$$\begin{aligned} ||e^{\Delta t} u||_p &\leq c_{p,q} t^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} ||u_0||_q, \quad p \geq q, \\ ||\nabla_x e^{\Delta t} u||_p &\leq c'_{p,q} t^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{1}{2}} ||u_0||_q, \quad p \geq q, \end{aligned} \tag{2}$$

where $c_{p,p} = 1$.

The standard existence theory for (1) is as follows. One views the nonlinearity as a perturbation of the heat semigroup, and represents the solution using Duhamel's formula

$$u(t) = e^{\Delta t} u_0 - \int_0^t e_x^{\Delta(t-s)} f(u(s)) \, \mathrm{d}s. \tag{3}$$

Constructing a solution to (1) amounts to finding a fixed point for the map

$$L(u) = e^{\Delta t} u_0 - \int_0^t e_x^{\Delta(t-s)} f(u(s)) \, \mathrm{d}s.$$

It is easy to show that the map L is a contraction on a closed ball in $L^\infty(\mathbb{R} \times [0,T])$ assuming that $T = T(||u_0||_\infty)$ is sufficiently small. Then, one desires to iterate the local existence starting with initial data $u(T,\cdot)$ to obtain a solution for all time. However, one needs to show that the time T of existence does not degenerate, which amounts to showing an estimate $||u(T,\cdot)||_\infty \leq ||u_0||_\infty$. This is the well-known maximum principle, which will hold for u as long as u is smooth (or at least, say, C^2). So we need to show that our local solution, defined on the interval $[0,T]$, is smooth. This is what we aim to address in this note.

There is hope that the solution may be smooth due to the smoothness of solutions to the heat equation, which follow directly from the representation of the solution as a convolution with a smooth function. Further, the dispersion estimates give us even more information. Indeed, with $p = q = \infty$, we have an estimate

$$||(\partial_x)^n e^{\Delta t} u||_\infty \lesssim t^{-\frac{n}{2}} ||u_0||_\infty, \tag{4}$$

which gives decay rates for derivative of the solution for long times. We will be able to show an analogous estimate to (4) for the solution to (1), but only for short times. Indeed, one should not expect that the derivatives go to 0 in the asymptotic limit. For example, if $f'' > 0$, (1) supports traveling wave solutions in which the derivatives of the solution are nonzero and the L^∞ norm is constant in time. In these cases, the flux exhibits a nonlinear compressive effect that is perfectly balanced with the diffusion to prevent decay.

Anyways, let's begin with the derivative estimate. The first naive strategy, which is to use dispersion estimates (2) directly on the Duhamel formula (3), fails. Indeed, if one attempts to differentiate the Duhamel formula, there is nowhere good to put the derivative in the integral term. The heat kernel cannot support two derivatives as then the integral would not converge, and the flux term cannot support a derivative as we have no information on the derivatives of u_0, so any Gronwall-type argument is not available.

The idea is to push the estimate through the Picard iteration that produced the local solution u. Recall that the solution u may be derived as the L^∞-limit of the following sequence

$$\begin{aligned} v_0 &= e^{\Delta t} u_0, \\ v_{n+1} &= L(v_n). \end{aligned} \tag{5}$$

The first estimate we will need is a uniform L^∞ estimate on the sequence $\{v_n\}$. Without loss of generality, we will henceforth assume that $f(0) = 0$.

Lemma 0.1. *There exists $T_1 = T_1(||u_0||_\infty) > 0$ such that for all n*

$$||v_n(t)||_\infty \leq 2||u_0||_\infty, \quad \textit{for } 0 \leq t \leq T$$

Proof. Firstly, for any $t > 0$, we have the following.

$$||v_0(t)||_\infty \leq ||u_0||_\infty.$$

We also have

$$v_1(t) = e^{\Delta t} u_0 - \int_0^t e_x^{\Delta(t-s)} f(v_0(s))\, \mathrm{d}s.$$

Let $C_1 := \max(||f'||_{L^\infty([-4||u_0||_\infty, 4||u_0||_\infty])}, ||f''||_{L^\infty([-4||u_0||_\infty, 4||u_0||_\infty])})$. Then, we have an estimate

$$||v_1||_\infty(t) \leq ||u_0||_\infty + C_1 ||u_0||_\infty \int_0^t \frac{1}{\sqrt{t-s}}\, \mathrm{d}s.$$

Choosing $T_1 = \frac{1}{4C_1^2}$, for $t \leq T_1$, we obtain

$$||v_1||_\infty(t) \leq ||u_0||_\infty + \frac{1}{2}||u_0||_\infty.$$

Iterating, we obtain an estimate

$$||v_n||_\infty(t) \leq ||u_0||_\infty \sum_{k=0}^{n} 2^{-k} \leq 2||u_0||_\infty.$$

□

Now, we can show the first derivative estimate.

Lemma 0.2. *There exists* $T_2 = T_2(||u_0||_\infty) > 0$ *and* $C_2 > 0$ *such that for all* n

$$||v_n'(t)||_\infty \leq \frac{C_2||u_0||_\infty}{\sqrt{t}} \qquad \text{for } 0 < t \leq T_2$$

Proof. The proof is again by iteration. Firstly, we have an estimate

$$||v_0'||_\infty(t) \leq \frac{c'_{\infty,\infty}}{\sqrt{t}}||u_0||_\infty.$$

Further, differentiating (5), we obtain a representation for v_1. Taking the L^∞ norm

$$||v_1'||_\infty(t) \leq \frac{c'_{\infty,\infty}||u_0||_\infty}{\sqrt{t}} + \frac{c'_{\infty,\infty}||u_0||_\infty}{\sqrt{t}}\left(C_1 c'_{\infty,\infty}\pi\sqrt{t}\right),$$

as long as $t \leq T_1$. Iterating the estimate gives

$$||v_n'||_\infty(t) \leq \frac{c'_{\infty,\infty}||u_0||_\infty}{\sqrt{t}} \sum_{k=0}^{n} \left(C_1 c'_{\infty,\infty}\pi\sqrt{t}\right)^k.$$

So, picking $T_2 = \min(T_1, \frac{1}{(2C_1 c'_{\infty,\infty}\pi)^2})$, we obtain

$$||v_n'||_\infty(t) \leq \frac{2c'_{\infty,\infty}||u_0||_\infty}{\sqrt{t}},$$

for any $0 < t \leq T_2$. □

In the limit, this implies for the solution

$$||u'||_\infty(t) \leq \frac{2c'_{\infty,\infty}||u_0||_\infty}{\sqrt{t}},$$

for any $0 < t \leq T_2$.

Now, we move on to the higher derivative estimates. These are a little more delicate as we will run into issues with too many derivatives in the integral term if we attempt to iterate in the same way as in the first derivative estimate. So, as an intermediate step, we need to first iterate to get an estimates starting from a time $t_0 > 0$.

Lemma 0.3. *There exists* $T_3 = T_3(||u_0||_\infty)$ *and* $C_3 > 0$ *such that for any* $0 < t_0 \leq T_2 - T_3$*,*

$$||v_n''||_\infty(t) \leq \frac{C_3}{\sqrt{t-t_0}\sqrt{t_0}} \qquad \text{for any } t_0 \leq t \leq t_0 + T_3.$$

Proof. Fix $0 < t_0 \leq T_2$. Then, by Lemma 0.2, we have

$$||u'||_\infty(t_0) \leq \frac{C_2||u_0||_\infty}{\sqrt{t_0}}.$$

Further, using Duhamel again, we can represent the solution u to (1) as the L^∞-limit of the following sequence $\{w_n\}$ for $t \geq t_0$:

$$w_0 = e^{\Delta(t-t_0)}u(t_0), \tag{6}$$
$$w_{n+1} = e^{\Delta(t-t_0)}u(t_0) - \int_{t_0}^{t} e_x^{\Delta(t-s)} f(w_n(s))\,\mathrm{d}s.$$

Re-doing the proofs of Lemmas 0.1 and 0.2, we obtain the existence of $T_4 = T_4(||u_0||_\infty)$ such that for any $t_0 \leq t \leq t_0 + T_4$:

$$||w_n||_\infty(t) \leq 4||u_0||_\infty,$$
$$||w_n'||_\infty(t) \leq \frac{2C_2||u_0||_\infty}{\sqrt{t_0}}$$

Now, we start iterating. First, the second derivative estimate on w_0''

$$||w_0''||_\infty(t) \leq \frac{c'_{\infty,\infty}C_2||u_0||_\infty}{\sqrt{t-t_0}\sqrt{t_0}}.$$

Then, differentiating (6), we obtain for w_1''

$$w_1''(t) = e_x^{\Delta(t-t_0)}u'(t_0) - \int_{t_0}^{t} e_x^{\Delta(t-s)} \left(f''(w_0(s))w_0'(s)^2 + f'(w_0(s))w_0''(s)\right)\mathrm{d}s.$$

Taking the L^∞ norm,

$$||w_1''||_\infty(t) \leq \frac{c'_{\infty,\infty}C_2||u_0||_\infty}{\sqrt{t-t_0}\sqrt{t_0}} + C_1\int_{t_0}^{t} \frac{c'_{\infty,\infty}}{\sqrt{t-s}}\left(\frac{(C_2||u_0||_\infty)^2}{\sqrt{s}\sqrt{t_0}} + \frac{c'_{\infty,\infty}C_2||u_0||_\infty}{\sqrt{s-t_0}\sqrt{t_0}}\right)\mathrm{d}s.$$

Iterating, we again find the existence of T_3 so that the geometric series will converge for $t_0 \leq t \leq t_0 + T_3$, and we obtain an estimate in the limit

$$||u''||_\infty(t) \leq \frac{C_3||u_0||_\infty}{\sqrt{t-t_0}\sqrt{t_0}},$$

for $t_0 \leq t \leq t_0 + T_3$. □

One may put this in a more palatable form as follows:

Lemma 0.4. *We have an estimate*

$$||u''||_\infty(t) \leq \frac{2C_3||u_0||_\infty}{t}$$

for any time $0 < t \leq T_3$.

Proof. Fix $t \in (0, T_3]$. Then, let $t_0 = \frac{t}{2}$. Then, Lemma 0.3 gives

$$||u''||_\infty(t) \leq \frac{C_3||u_0||_\infty}{\sqrt{\frac{t}{2}}\sqrt{\frac{t}{2}}} = \frac{2C_3||u_0||_\infty}{t}.$$

□

This is a "decay estimate" exactly the same as for solutions to the heat equation, at least for short times (of course, one should not expect it to hold for long times in this setting, cf. the above discussion of traveling waves). Higher derivative estimates can be shown following the same strategy. This grants the smoothness of the local solution u for short times, which allows for the use of the maximum principle to iterate and obtain a solution for all time. To show smoothness at later times, one can either repeat the strategy above, or close Gronwall-type arguments as the "initial data" at time $t_0 > 0$ is actually smooth now rather than just L^∞.

NB: Thanks to Paul Blochas who outlined to me the strategy above. This note contains nothing new; it is well-known that solutions to (1) are smooth for positive times. In fact, it is so well-known that I cannot find a detailed proof in the literature anywhere. This note arose from reading the notes of Denis Serre L^1-stability of nonlinear waves in scalar conservation laws.