Kenneth DeMason, June 20, 2023

Contents

Measure Theory.

Problem 1 (Spring 2019). Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz transformation. Show that if A is a set of Lebesgue measure zero, then $T(A)$ also has Lebesgue measure zero.

Solution: Since A is measurable with $|A| = 0$, for any $\epsilon > 0$ there exists a family of intervals $\{I_k\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} |I_k| < |A| + \epsilon = \epsilon.$$

Let $I_k = [a_k, b_k]$ for some $a_k, b_k \in \mathbb{R}$. By definition, since T is Lipschitz there exists a constant $\mathrm{Lip}(T) < \infty$ such that

$$|T(x) - T(y)| \leq \mathrm{Lip}(T)|x - y|$$

for any $x, y \in \mathbb{R}$. It follows that

$$|T(I_k)| = |T(b_k) - T(a_k)| \leq \mathrm{Lip}(T)|b_k - a_k| = |I_k|$$

so

$$\sum_{k=1}^{\infty} |T(I_k)| \leq \mathrm{Lip}(T) \sum_{k=1}^{\infty} |I_k| < \epsilon.$$

Now, if $y \in T(A)$ then there exists an $x \in A$ such that $T(x) = y$. Because $\{I_k\}_{k=1}^{\infty}$ covers A, we know that $x \in I_k$ for some k. Hence, $y \in T(I_k)$ for some k and $\{T(I_k)\}_{k=1}^{\infty}$ cover $T(A)$. But by monotonicity,

$$|T(A)| \leq \sum_{k=1}^{\infty} |T(I_k)| < \epsilon.$$

This holds for any $\epsilon > 0$, and thus $|T(A)| = 0$.

NB: In $\mathbb{R}^n$, you have to be a little careful adapting the above idea (a constant depending only on n enters into play, if I remember correctly). If you define Lebesgue measure with balls, the same idea generalizes to $\mathbb{R}^n$ without edits.

Problem 2 (Spring 2016). For any $r \geq 0$ and any $x \in \mathbb{R}^2$, define the closed unit ball $B_r(x) := \{y \in \mathbb{R}^2 \mid |y - x| \leq r\}$. Let $0 < c < 1$. Let E be a measurable subset of the unit square $Q = [0,1]^2 \subset \mathbb{R}^2$ with the property that for every $x \in Q$ and every $r > 0$ there exists a $y \in B_r(x)$ such that $B_{c|x-y|}(y) \subset E$. Prove that $Q \setminus E$ has Lebesgue measure zero.

Solution (courtesy of Joe Miller): Let $\epsilon > 0$. Then, choose an open set O, with $Q \setminus E \subset O$, such that $\lambda(O \setminus [Q \setminus E]) < \epsilon \implies \lambda(O) < \lambda(Q \setminus E) + \epsilon$. Then, let $\mathcal{B}$ denote that set of balls $B_{|x-y|}(y)$ for $x \in Q \setminus E$ and y chosen such that $B_{c|x-y|}(y) \subset E$, where $y \in B_r(x)$ for some $r > 0$ sufficiently small such that $B_{|x-y|(y)} \subset O$. This is easily seen to be a Vitali cover of $Q \setminus E$. So, by the Vitali Covering Theorem (NOT the covering lemma), there exists a countable subcollection $\{B_k\}_{k=1}^{\infty}$ of pairwise disjoint balls in $\mathcal{B}$ such that:

$$\lambda(Q \setminus E \setminus \cup_{k=1}^{\infty} B_k) = 0$$

This implies:

$$0 = \lambda(Q \setminus E \setminus \cup_{k=1}^{\infty} B_k) < \lambda(O \setminus \cup B_K) = \lambda(O) - \lambda(\cup B_k) < \lambda(Q \setminus E) + \epsilon - \lambda(\cup B_k) \implies \lambda(\cup B_k) < \lambda(Q \setminus E) + \epsilon$$

as the B_k are pairwise disjoint. Note that as $\cup_{k=1}^{\infty} cB_k$ is contained in E, we have that $A \subset \cup_{k=1}^{\infty}(B_k \setminus cB_k)$. Furthermore, the collection $\{B_k \setminus cB_k\}$ is still disjoint. So:

$$\lambda(Q \setminus E) = \lambda(Q \setminus E \setminus \cup_{k=1}^{\infty} B_k) \leq \lambda(\cup_{k=1}^{\infty}(B_k \setminus cB_k)) \leq$$

$$\sum_{k=1}^{\infty} \lambda(B_k \setminus cB_k) = (1 - c^2) \sum_{k=1}^{\infty} \lambda(B_k) \leq (1 - c^2)\lambda(O) < (1 - c^2)[\lambda(Q \setminus E) + \epsilon]$$

Taking $\epsilon \to 0$, we see that $\lambda(Q \setminus E) = (1 - c^2)\lambda(Q \setminus E)$, a contradiction unless $\lambda(Q \setminus E) = 0$.

Problem 3 (Spring 2016). Let (X, d) be a compact metric space. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of positive Borel measures on X that converge in the weak$*$ topology to a finite positive Borel measure μ. Show that for every compact $K \subset X$,

$$\mu(K) \geq \limsup_{n\to\infty} \mu_n(K).$$

Solution: Let $K \subseteq X$ be compact. In particular, K is closed. Now, let $K_j := \{x \in X : d(K, x) < \frac{1}{j}\}$. By Urysohn's lemma, there exists a continuous function ϕ_j such that ϕ_j is 1 on K, 0 on K_j^c, and $0 \leq 1$ in between. So we have:

$$\mu_n(K) \leq \int \phi_j \mu_n \to \phi_j \mu \leq \mu(K_j)$$

So this implies:

$$\limsup_{n\to\infty} \mu_n(K) \leq \mu(K_j)$$

Taking $j \to \infty$, we obtain the result.

Problem 4 (Spring 2015, Spring 2012, Spring 2022). Let Z be a subset of $\mathbb{R}$ with measure zero. Show that the set $A = \{x^2 \mid x \in Z\}$ also has measure zero.

Solution: A quick way to prove this is to note that $f(x) = x^2$ is locally Lipschitz, and thus if A is bounded we have $|A| = 0$ implies $|f(A)| = 0$. But, $f(A) = Z$. If A is not bounded we can define $A_n = A \cap [-n, n]$ and note that A_n is bounded, so $|f(A_n)| = 0$. Consequently,

$$|Z| = \left| Z \cap \bigcup_{n=1}^{\infty} [0, n^2] \right| = \left| \bigcup_{n=1}^{\infty} Z \cap [0, n^2] \right| = \left| \bigcup_{n=1}^{\infty} A_n \right| \leq \sum_{n=1}^{\infty} |A_n| = 0.$$

Let's prove this from first principles instead. We can still use the same localization procedure – namely if $\{E_n\}_{n=1}^{\infty}$ is a sequence of measurable sets such that $|E_n| < \infty$ for all n and $\bigcup_{n=1}^{\infty} E_n = \mathbb{R}$ (remark: any measure space satisfying this is called σ-finite) then we just need to show $|Z \cap E_n| = 0$ for all n. Then,

$$|Z| = \left| Z \cap \bigcup_{n=1}^{\infty} E_n \right| = \left| \bigcup_{n=1}^{\infty} Z \cap E_n \right| \leq \sum_{n=1}^{\infty} |Z \cap E_n| = 0.$$

Choose $E_n = [-n^2, n^2]$. It suffices then to show that if $A \subset [0, n]$ then $|Z| = 0$ (you really need to show that it holds if $A \subset [-n, n]$, but you need to do $[-n, 0]$ and $[0, n]$ separately. The proofs are the same). Since A is measurable for any $\epsilon > 0$ there exist closed intervals $\{I_k\}_{k=1}^{\infty}$ covering A such that

$$\sum_{k=1}^{\infty} |I_k| \leq |A| + \epsilon = \epsilon.$$

Without loss of generality we may assume $[a_k, b_k] = I_k \subset [0, n]$. When we square this, the length is

$$|b_k^2 - a_k^2| = |b_k - a_k|(b_k + a_k) \leq 2n|b_k - a_k| = 2n|I_k|$$

since we assumed $0 \leq a_k, b_k \leq n$. Denote this squared interval by $\overline{I}_k$. Then,

$$|Z| \leq \left| \bigcup_{k=1}^{\infty} \overline{I}_k \right| \leq \sum_{k=1}^{\infty} |\overline{I}_k| \leq 2n \sum_{k=1}^{\infty} |I_k| = 2n\epsilon.$$

since the $\overline{I}_k$ cover Z.

Problem 5 (Spring 2015, Spring 2012). Let $E \subset \mathbb{R}$ be a measurable set such that $0 < |E| < \infty$. Prove that for every $\alpha \in (0,1)$ there is an open interval I such that

$$|E \cap I| \geq \alpha |I|.$$

Solution: We prove the contrapositive. Suppose there exists an $\alpha \in (0,1)$ such that every open interval I satisfies $|E \cap I| < \alpha |I|$. Since $E \subset \mathbb{R}$ is Lebesgue measurable for every $\epsilon > 0$ there exists a covering $\{I_k\}_{k=1}^{\infty}$ of E by open intervals such that

$$\sum_{k=1}^{\infty} |I_k| \leq |E| + \epsilon.$$

Since $E \subset \bigcup_{k=1}^{\infty} I_k$, applying the above bound we have

$$|E| = \left| E \cap \left(\bigcup_{k=1}^{\infty} I_k \right) \right| = \left| \bigcup_{k=1}^{\infty} (E \cap I_k) \right| \leq \sum_{k=1}^{\infty} |E \cap I_k| < \alpha \sum_{k=1}^{\infty} |I_k| \leq \alpha(|E| + \epsilon).$$

Thus, $|E| < \alpha(|E| + \epsilon)$, and taking $\epsilon \to 0$ we get $|E| \leq \alpha |E|$. If $|E| \neq \infty$, it follows that $|E| = 0$. Hence either $|E| = 0$ or $|E| = \infty$.

Problem 6 (Fall 2013). Assume that μ is a finite Borel measure on $\mathbb{R}^n$, and that there exists a constant $0 < R < \infty$ such that the k-th moments of μ satisfy the bound

$$\int |x|^k \, d\mu < R^{k^r} \qquad \forall k \in \mathbb{N},$$

for some $0 < r \leq 1$. Prove that μ has bounded support contained in $\{x \in \mathbb{R}^n \mid |x| \leq R\}$ if $r = 1$ and in $\{x \in \mathbb{R}^n \mid |x| \leq 1\}$ if $0 < r < 1$.

Solution: First suppose $r = 1$. Then the k-th moments satisfy the bound

$$\int |x|^k \, d\mu < R^k \qquad \forall k \in \mathbb{N}$$

for some $0 < R < \infty$. To show that $\operatorname{spt}(\mu) \subset B_R(0)$ we can show that

$$B_R(0)^c \subset \mathbb{R}^n \setminus \operatorname{spt}(\mu) = \{x \in \mathbb{R}^n \mid \mu(B_r(x)) = 0 \text{ for some } r > 0\}.$$

Let $\eta > 0$ so that

$$\eta^k \mu(B_\eta(0)^c) < \int_{B_\eta(0)^c} |x|^k \, d\mu \leq \int_{\mathbb{R}^n} |x|^k \, d\mu < R^k$$

and

$$\mu(B_\eta(0)^c) < \frac{R^k}{\eta^k}.$$

Hence, for all $\eta > R$ we see that

$$\mu(B_\eta(0)^c) < \epsilon^k \to 0$$

for some $1 > \epsilon > 0$. In other words, for all $\eta > R$

$$\mu(B_\eta(0)^c) = 0.$$

Now let $x \in B_R(0)^c$. Then $|x| > R$ and by choosing r small enough we have $B_r(x) \subset B_\eta(0)^c$ for some $\eta > R$. By monotonicity, $\mu(B_r(x)) = 0$ and so $B_R(0)^c \subset \mathbb{R}^n \setminus \operatorname{spt}(\mu)$.

Now consider the $0 < r < 1$ case. Here, we instead get

$$\mu(B_\eta(0))^c < \frac{R^{k^r}}{\eta^k}$$

which tends to zero as for any $\eta > 1$. By the same logic, we get that $B_1(0)^c \subset \mathbb{R}^n \setminus \operatorname{spt}(\mu)$. Note that we did not use the condition μ a finite measure. The above estimates show that in either case, the measure of the whole space is the measure of a ball; so we need only locally finite.

Problem 7 (Fall 2012). Let μ be a measure in the plane for which all open squares are measurable, with the property that there exists $\alpha \geq 1$, such that if two open squares Q and Q' are translates of each other and their closures $\text{Cl}(Q)$ and $\text{Cl}(Q')$ have a non-empty intersection, then

$$\mu(\text{Cl}(Q)) \leq \alpha\mu(Q') < \infty.$$

(For Lebesgue $\alpha = 1$, in general $\alpha \geq 1$.) Show that horizontal lines have zero measure.

Solution (courtesy of Joe Miller): Let L be a horizontal line with length 1. Let $\{Q_k\}_{k=1}^{2^n}$ be a collection of open cubes of side length 2^{-n} whose lower edges cover L. Since each cube Q_k is a translate of another one, $\mu(\overline{Q_k}) \leq \alpha\mu(Q_K)$. So:

$$\mu(L) \leq \mu(\cup_{k=1}^{2^n}\overline{Q_k}) \leq \alpha\sum_{k=1}^{2^n}\mu(Q_k) = \alpha\mu(\cup_{k=1}^{2^n}Q_k)$$

Since $R_n := \cup_{k=1}^{2^n}Q_k \to \emptyset$, we have by continuity of measure:

$$\mu(L) \leq \lim_{n\to\infty}\mu(R_n) = \mu(\emptyset) = 0$$

Problem 8. Show that the following notions of measurability are equivalent. Here, we let $\lambda : 2^{\mathbb{R}} \to [0, \infty]$ be the Lebesgue outer measure.

a) $E \subset \mathbb{R}$ is measurable iff for every $\epsilon > 0$ there exists an open set $O \supset E$ such that $\lambda(O\backslash E) < \epsilon$.

b) $E \subset \mathbb{R}$ is measurable iff for every set $A \subset \mathbb{R}$ (measurable or not) we have

$$\lambda(A \cap E) + \lambda(A \cap E^c) = \lambda(A).$$

Solution: By definition, $E \subset \mathbb{R}$ is measurable iff for every $\epsilon > 0$ there exists a collection of open intervals $\{I_k\}_{k=1}^{\infty}$ covering E such that

$$\sum_{k=1}^{\infty}|I_k| < |E| + \epsilon.$$

Now consider $O = \bigcup_{k=1}^{\infty} I_k$. It follows that

$$|O \setminus E| \leq \sum_{k=1}^{\infty}|I_k \setminus E| = \sum_{k=1}^{\infty}|I_k| - \sum_{k=1}^{\infty}|I_k \cap E| < \epsilon + |E| - \sum_{k=1}^{\infty}|I_k \cap E|$$

where we have assumed b). But, by monotonicity and the fact that $E \subset O$,

$$|E| = |E \cap O| = \left|\bigcup_{k=1}^{\infty} I_k \cap E\right| \leq \sum_{k=1}^{\infty}|I_k \cap E|.$$

Hence, the difference above is negative and

$$|O \setminus E| < \epsilon + \left[|E| - \sum_{k=1}^{\infty}|I_k \cap E|\right] < \epsilon$$

as desired. Now assume a). Let $A \subset \mathbb{R}$ and $\epsilon > 0$. By subadditivity,

$$|A| = |(A \cap E) \cup (A \cap E^c)| \leq |A \cap E| + |A \cap E^c|$$

so we need only show the other direction. As before, we can find a collection of open intervals $\{I_k\}_{k=1}^{\infty}$ covering A such that

$$\sum_{k=1}^{\infty}|I_k| < |A| + \epsilon.$$

Now, since $E \cap I_k$ and $E^c \cap I_k$ are measurable and disjoint we have

$$|I_k \cap E| + |I_k \cap E^c| = |I_k|.$$

As the I_k cover A, we have

$$|A \cap E| + |A \cap E^c| \leq \sum_{k=1}^{\infty}[|I_k \cap E| + |I_k \cap E^c|] = \sum_{k=1}^{\infty}|I_k| < |A| + \epsilon.$$

Taking $\epsilon \to 0$ gives the result.

Problem 9 (Fall 2020). Let μ be a finite measure on a σ-algebra $\mathcal{M}$, and let $\{E_t\}_{t>0}$ be a family of elements of $\mathcal{M}$ indexed over $(0, \infty)$. Show that if:

$$\mu(\cup_{t>0} E_t) < \infty$$

then $\mu(E_t) = 0$ for all but countably many values of t.

Solution: This is not true. Consider μ as Lebesgue measure on $[0, 1]$. Then, let $E_t = [0, 1 - \frac{1}{1+t}]$. Then, $\mu(E_t) > 0$ for all t, and $\mu(\cup_{t>0} E_t) \leq 1 < \infty$.

NB: Perhaps they wanted to say that the sets are all disjoint. Then it is (probably?) true.

Integration and Limits.

Problem 1 (Spring 2019). Show that $C_c(\mathbb{R}^n) := \{f \in C(\mathbb{R}^n) \mid f \text{ has compact support}\}$ is dense in $L^1(\mathbb{R}^n)$.

Solution: We know that simple functions are dense in $L^1(\mathbb{R}^n)$, so it suffices to show that $C_c(\mathbb{R}^n)$ is dense in the set of simple functions. Since a simple function is just a finite linear combination of indicator functions, we just need to approximate an arbitrary indicator function by a function in $C_c(\mathbb{R}^n)$. So, let E be measurable with $0 < |E| < \infty$. Consider now the case when $n = 1$. By Littlewood's first principle, there exists a finite collection of disjoint open intervals $\{I_k\}_{k=1}^K$ such that $|E \Delta \bigcup_{k=1}^K I_k| < \epsilon/2$. Now let $\eta = \epsilon/(2K)$ and consider the continuous function

$$g_k(x) = \begin{cases} 1 & x \in (a_k, b_k) \\ -1/\eta(x - b_k) + 1 & x \in [b_k, b_k + \eta) \\ 1/\eta(x - a_k) + 1 & x \in (a_k - \eta, a_k] \\ 0 & \text{else} \end{cases}$$

which is continuous and

$$\int_{\mathbb{R}} |g_k - \chi_{I_k}| = \frac{\eta}{2} + \frac{\eta}{2} = |I_k| + \eta.$$

Defining $g = g_1 + ... + g_K$ we then have

$$\int_{\mathbb{R}} |g - \chi_{\cup_k I_k}| = K\eta = \frac{\epsilon}{2}$$

(here we use disjointness of the I_k). Finally, observe that

$$\|\chi_E - \chi_{\cup_k I_k}\|_1 = \|\chi_{E \Delta \cup_k I_k}\|_1 < \frac{\epsilon}{2}$$

so

$$\|g - \chi_E\|_1 \leq \|g - \chi_{\cup_k I_k}\|_1 + \|\chi_E - \chi_{\cup_k I_k}\|_1 < \epsilon.$$

The higher dimension case is similar, except we approximate boxes rather than intervals.

Problem 2 (Spring 2019). Find an uncountable family of measurable functions $\mathcal{F} \subset \{f : \mathbb{R} \to \mathbb{R} \text{ measurable}\}$ that satisfies the following two conditions:

a) For all $f \in \mathcal{F}$, $\|f\|_\infty = 1$.
b) For all $f, g \in \mathcal{F}$, we have $\|f - g\|_\infty = 1$.

(Bonus: Show that this implies L^∞ is not separable.)

Solution: Consider the collection of open intervals $(-r/2, r/2)$. Note that each interval has measure $r > 0$ and if $(-R/2, R/2)$ is another open interval then

$$|(-r/2, r/2) \Delta (-R/2, R/2)| > |R - r| > 0.$$

By taking $\mathcal{F}$ to be the collection of indicator functions of these intervals, the above two statements show the two necessary conditions. It is clearly an uncountable family.

Suppose now that L^∞ is separable. Then there exists a countable dense family $\{g_k\}_{k=1}^\infty$. Consider the balls $B_1(f)$ (in the L^∞ norm) with $f \in \mathcal{F}$.

Problem 3 (Spring 2017, Fall 2014, Spring 2022). Let $1 \le p, q \le \infty$ with $1/p + 1/q = 1$. Show that if $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ then $f * g$ is bounded and continuous on $\mathbb{R}^n$.

Solution: We show first $f * g$ is bounded. An easy estimate gives

$$\begin{aligned}|f*g|(x) &= \left|\int_{\mathbb{R}^n} f(x-y)g(y)\ dy\right| \le \int_{\mathbb{R}^n} |f(x-y)||g(y)|\ dy \le \left[\int_{\mathbb{R}^n} |f(x-y)|^p\ dy\right]^{1/p}\left[\int_{\mathbb{R}^n} |g(y)|^q\ dy\right]^{1/q} \\ &= \|f\|_p\|g\|_q < \infty\end{aligned}$$

by Hölder's inequality and translation invariance. As for continuity, we show that if $x_n \to x$ then $(f*g)(x_n) \to (f*g)(x)$. Another estimate gives

$$\begin{aligned}|(f*g)(x_n) - (f*g)(x)| &= \left|\int_{\mathbb{R}^n} f(x_n-y)g(y)\ dy - \int_{\mathbb{R}^n} f(x-y)g(y)\ dy\right| \\ &= \left|\int_{\mathbb{R}^n} [f(x_n-y) - f(x-y)]g(y)\ dy\right| \le \int_{\mathbb{R}^n} |f(x_n-y) - f(x-y)|g(y)\ dy \\ &\le \left[\int_{\mathbb{R}^n} |f(x_n-y) - f(x-y)|^p\ dy\right]^{1/p}\|g\|_q\end{aligned}$$

by Hölder's inequality (justified since translations of f are in $L^p(\mathbb{R}^n)$ as well, and $L^p(\mathbb{R}^n)$ is a vector space). Now, since $f \in L^p(\mathbb{R}^n)$ there exists a sequence $\{h_k\}_{k=1}^\infty$ of compactly supported continuous functions such that $\|f - h_k\|_p \to 0$. Let $\epsilon > 0$. Then there exists a $K \in \mathbb{N}$ such that if $k \ge K$ then $\|f - h_k\|_p < \epsilon$. Moreover, since each h_k is continuous and $x_n - y \to x - y$, $h_k(x_n - y) \to h_k(x - y)$. Thus for fixed k, there exists an $N_k \in \mathbb{N}$ such that if $n \ge N_k$ then $|h_k(x_n - y) - h_k(x - y)| < \epsilon$. Putting these together, we see that

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x_n-y) - f(x-y)|^p\ dy &\le \int_{\mathbb{R}^n} |f(x_n-y) - h_K(x_n-y)|^p\ dy + \int_{\mathbb{R}^n} |h_K(x_n-y) - h_K(x-y)|^p\ dy \\ &\quad + \int_{\mathbb{R}^n} |h_K(x-y) - f(x-y)|^p\ dy \\ &\le 2\epsilon^p + \int_{(x-S)\cup(x_n-S)} \epsilon^p\ dy = [2 + |(x-S)\cup(x_n-S)|]\epsilon^p \\ &\le 2[1+|S|]\epsilon^p\end{aligned}$$

where $S = \text{spt}(h_K)$ is compact, and thus has finite measure. This estimate holds for all $n \ge N_K$, and thus

$$|(f*g)(x_n) - (f*g)(x)| \le 2^{1/p}[1+|S|]^{1/p}\epsilon$$

establishing continuity.

Problem 4 (Spring 2017). Let B be the closed unit ball in $\mathbb{R}^n$, and let f_1, f_2, f_3,... be nonnegative integrable functions on B. Assume that

i) $f_k \to f$ almost everywhere.
ii) For every $\epsilon > 0$ there exists $M > 0$ such that

$$\int_{\{x\in B\ \mid\ f_k(x)>M\}} f_k(x)\ dx < \epsilon, \qquad k = 1, 2, 3, ...$$

Show that $f_k \to f$ in $L^1(B)$.

Solution: Let's first show that $f \in L^1(B)$. let $\epsilon > 0$. Then there exists an $M > 0$ such that

$$\int_B f_k(x)\ dx = \int_{\{x\in B\ \mid\ f_k(x)\le M\}} f_k(x)\ dx + \int_{\{x\in B\ \mid\ f_k(x)>M\}} f_k(x)\ dx \le M|B| + \epsilon.$$

By Fatou's lemma, since $f_k \to f$ almost everywhere

$$\int_B f(x)\ dx = \int_B \liminf_{k\to\infty} f_k(x)\ dx \le \liminf_{k\to\infty} \int_B f_k(x)\ dx \le M|B| + \epsilon.$$

Now, since f is integrable, given our $\epsilon > 0$ there exists a $\delta > 0$ such that whenever A is measurable with $|A| < \delta$,

$$\int_A f(x)\ dx < \epsilon.$$

Markov's inequality states that

$$|\{f_k > \lambda\}| \le \frac{\|f_k\|_{L^1(B)}}{\lambda}.$$

Now, we have proven that the f_k are uniformly bounded in $L^1(B)$, say by C. Hence, by choosing λ large enough we can guarantee that

$$|\{f_k > \lambda\}| < \delta$$

for all $k \in \mathbb{N}$. Now, since

$$\int_{\{f_k > M\}} f_k(x)\ dx$$

is nonincreasing with M, we can choose $M \ge \lambda$ so that simultaneously

$$A_k := |\{f_k > M\}| < \delta, \qquad \int_{A_k} f_k(x)\ dx < \epsilon$$

for all $k \in \mathbb{N}$. Thus, we have that

$$\begin{aligned}\int_B |f - f_k|(x)\ dx &= \int_{\{f_k \le M\}} |f - f_k|(x)\ dx + \int_{\{f_k > M\}} |f - f_k|(x)\ dx \\ &< \int_{\{f_k \le M\}} |f - f_k|(x)\ dx + \int_{A_k} f(x)\ dx + \int_{A_k} f_k(x)\ dx \\ &< \int_{\{f_k \le M\}} |f - f_k|(x)\ dx + 2\epsilon.\end{aligned}$$

Finally, define $g_k := |f - f_k|\chi_{\{f_k \le M\}}$. Then clearly $|g_k| \le M + |f| \in L^1(B)$ since B has finite measure. Since $f_k \to f$ a.e. on B, we also get $g_k \to 0$. Hence by dominated convergence

$$\lim_{k\to\infty} \int_{\{f_k \le M\}} |f - f_k|(x)\ dx = 0.$$

Problem 5 (Fall 2016, Fall 2022). Let $\{f_k\}_{k=1}^\infty \subset L^p$ with $1 \le p < \infty$. If $f_k \to f$ pointwise a.e. and $\|f_k\|_p \to \|f\|_p$, show that $\|f - f_k\|_p \to 0$.

Solution: Recall the generalized dominated convergence theorem: If $\{g_k\}_{k=1}^\infty$ is a sequence of measurable functions such that $g_k \to g$ pointwise a.e., and there is a sequence of integrable functions $\{h_k\}_{k=1}^\infty$ such that $|g_k| \le h_k$ for all k then $\lim_{k\to\infty} \int h_k = \int h$ implies $\lim_{k\to\infty} \int g_k = \int g$. Here, let $g_k = |f_k - f|^p$, $g = 0$, $h_k = 2^p(|f_k|^p + |f|^p)$, and $h = 2^{p+1}|f|^p$. Note that

$$|g_k| \le (|f_k| + |f|)^p \le 2^p \max\{|f_k|, |f|\}^p \le 2^p(|f_k|^p + |f|^p) = h_k.$$

So, to apply generalized dominated convergence we need only show

$$\lim_{k\to\infty} \int h_k \to \int h$$

or, alternatively,

$$\lim_{k\to\infty} \int |f_k|^p = \int f$$

but this is assumed. Hence, we get

$$\lim_{k\to\infty} \int |f_k - f|^p = \lim_{k\to\infty} \int g_k = \int g = 0$$

as desired.

Here's another way to do it without generalized dominated convergence directly. Define g_k by $g_k := 2^p(|f_k|^p + |f|^p) - |f_k - f|^p$. By the above inequality, each $g_k \geq 0$ and $g_k \to 2^{p+1}|f|^p$ a.e. Hence by Fatou and the hypothesis $\|f_k\|_p \to \|f\|_p$,

$$\begin{aligned} 2^{p+1}\|f\|_p^p &= \int 2^{p+1}|f|^p = \int \liminf_{k\to\infty} g_k \leq \liminf_{k\to\infty} \int g_k = 2^p \left[\liminf_{k\to\infty} \|f_k\|_p^p + \|f\|_p^p\right] - \limsup_{k\to\infty} \int |f - f_k|^p \\ &= 2^{p+1}\|f\|_p^p - \limsup_{k\to\infty} \|f - f_k\|_p^p \end{aligned}$$

Rearranging this then gives

$$\limsup_{k\to\infty} \|f - f_k\|_p^p \leq 0$$

which completes the proof.

Problem 6 (Fall 2015, Spring 2023). Let $f \in L^1(\mathbb{R})$ and φ_ϵ be a mollifier. This means that $\varphi_\epsilon(x) = \epsilon^{-1}\varphi(x/\epsilon)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is a function that satisfies: $\varphi \geq 0$, φ is compactly supported, and $\int \varphi = 1$. Let $f_\epsilon := f * \varphi_\epsilon$. Show that

$$\int_{\mathbb{R}} \liminf_{\epsilon\to 0} |f_\epsilon| \leq \int_{\mathbb{R}} |f|.$$

Solution: First by Fubini-Tonelli,

$$\begin{aligned} \int_{\mathbb{R}} |f_\epsilon|(x)\ dx &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)|\varphi_\epsilon(y)\ dydx = \int_{\mathbb{R}} \varphi_\epsilon(y) \left[\int_{\mathbb{R}} |f(x-y)|\ dx\right] dy \\ &= \|f\|_1 \int_{\mathbb{R}} \varphi_\epsilon(y)\ dy = \|f\|_1 \end{aligned}$$

Then, Fatou's inequality implies that

$$\int_{\mathbb{R}} \liminf_{\epsilon\to 0} |f_\epsilon| \leq \liminf_{\epsilon\to 0} \int_{\mathbb{R}} |f_\epsilon| \leq \|f\|_1$$

as desired.

Problem 7 (Fall 2014, Spring 2021). Let $f \in L^1(X, \mu)$. Prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\int_A f\ d\mu\right| < \epsilon$$

for all measurable $A \subset X$ such that $\mu(A) < \delta$.

Solution: Suppose not. Then there exists an $\epsilon > 0$ such that whenever $\delta > 0$ there exists an $A \subset X$ measurable with $\mu(A_\delta) < \delta$ and

$$\int_A f\ d\mu \geq \epsilon.$$

Consider $\delta = 1/n$ and set $g_n = \chi_{A_{1/n}} f$. All the g_n are dominated by f, which is integrable, and $g_n \to 0$ since $\mu(A_{1/n}) < 1/n \to 0$. Then by dominated convergence

$$\epsilon \leq \lim_{n\to\infty} \int_{A_{1/n}} f\ d\mu = \lim_{n\to\infty} \int_X g_n\ d\mu = \int_X \lim_{n\to\infty} g_n\ d\mu = 0$$

a contradiction.

Problem 8 (Fall 2014, Fall 2022). Let $p \in [1, \infty)$ and suppose $\{f_n\}_{n=1}^{\infty} \subset L^p(\mathbb{R})$ is a sequence that converges to 0 in $L^p(\mathbb{R})$. Prove that one can find a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to 0$ almost everywhere.

Solution: Firstly note that L^p convergence implies convergence in measure by Chebyshev. So, f_n converges to 0 in measure. So, it suffices to show the general fact that a sequence that converges in measure has a subsequence that converges pointwise .a.e.. Indeed, f_n converges to 0 in measure, so for all k, $\exists$ an index n_k such that if we define $A_k := \{x \in \mathbb{R} : |f_{n_k}(x) - 0| \geq \frac{1}{k}\}$, then $\lambda(A_k) < 2^{-k}$ (note that we can choose $n_k > n_{k-1}$). Construct the subsequence f_{n_k} in this way. Then, f_{n_k} converges pointwise to 0 outside of $\limsup_{k\to\infty} A_k$, which has measure zero by Borel-Cantelli.

Problem 9 (Fall 2014, Fall 2021). Show that, if $f \in L^4(\mathbb{R})$, then

$$\int |f(\lambda x) - f(x)|^4 \, dx \to 0$$

as $\lambda \to 1$.

Solution: Consider $g = \chi_E$ where $E \subset \mathbb{R}$ is measurable and $|E| < \infty$. Let's first show that

$$\int |g(\lambda x) - g(x)|^4 \, dx \to 0.$$

To do this, we'll first analyze the symmetric difference of intervals. Let $I = [a, b]$ and $\lambda > 0$ so that $\lambda I = [\lambda a, \lambda b]$. There are two cases. First, if $0 \in I$ then for all $0 < \lambda < 1$ we have $\lambda I \subset I$ and thus

$$|I\Delta(\lambda I)| = (b - \lambda b) + (\lambda a - a) = (1 - \lambda)(b - a).$$

On the other hand, if $\lambda > 1$ then

$$|I\Delta(\lambda I)| = (\lambda b - b) + (a - \lambda a) = (\lambda - 1)(b - a).$$

Either way, if $0 \in I$ then

$$|I\Delta(\lambda I)| = |1 - \lambda|(b - a).$$

Now suppose $0 \notin I$. Assume first $a > 0$. Let $\lambda_1 = a/b$ and $\lambda_2 = b/a$. For all $0 < \lambda < \lambda_1$ and $\lambda > \lambda_2$ we have that I and λI are disjoint. These cases are irrelevant since we take $\lambda \to 1$, and $\lambda_1 < 1 < \lambda_2$. For $a/b \leq \lambda \leq b/a$, λI translates to the right and increases in size, filling in more and more of I. Eventually, it becomes all of I. Then, while still increasing in size, it continues to translate rightwards and empty I. Thus, for $a/b < \lambda < 1$

$$|I\Delta(\lambda I)| = b - \lambda b + a - \lambda a = (1 - \lambda)(b + a)$$

for $1 < \lambda < b/a$ we have

$$|I\Delta(\lambda I)| = \lambda b - b + \lambda a - a = (\lambda - 1)(b + a)$$

Similar analysis holds when $b < 0$. In all cases, we end up getting

$$|I\Delta(\lambda I)| = |\lambda - 1|(|b| + |a|).$$

It is clear then that as $\lambda \to 1$, $|I\Delta(\lambda I)| \to 0$.

Now, if $E \subset \mathbb{R}$ is measurable with $|E| < \infty$, then by Littlewood's first principle of analysis for $\epsilon > 0$ there exists a disjoint finite collection of intervals $I_k = [a_k, b_k]$, $k = 1, ..., K$ such that

$$\left|\bigcup_{k=1}^{K} (E\Delta I_k)\right| = \left|E\Delta\left(\bigcup_{k=1}^{K} I_k\right)\right| < \epsilon.$$

By dilation properties of the Lebesgue measure, we also have that

$$\left|\bigcup_{k=1}^{K} \lambda(E\Delta I_k)\right| = \left|\bigcup_{k=1}^{K} ((\lambda E)\Delta(\lambda I_k))\right| < \lambda\epsilon$$

when $\lambda > 0$. Now, as previously seen

$$|I_k\Delta(\lambda_k I_k)| = |\lambda_k - 1|(|b_k| + |a_k|)$$

for λ_k small. Since we have finitely many intervals, we can choose λ small so that

$$|I_k\Delta(\lambda I_k)| = |\lambda - 1|(|b_k| + |a_k|) < \frac{\epsilon}{K}$$

for $k = 1, ..., K$. Finally, with this λ,

$$|E\Delta(\lambda E)| \leq \left|\bigcup_{k=1}^{K}(E\Delta I_k)\right| + \sum_{k=1}^{K}|I_k\Delta(\lambda I_k)| + \left|\bigcup_{k=1}^{K}((\lambda E)\Delta(\lambda I_k))\right| < (2+\lambda)\epsilon < C\epsilon$$

where $C > 0$ is a constant independent of λ (we chose λ small, so it is bounded by some constant). It follows too that $|E\Delta(\lambda E)| \to 0$ as $\lambda \to 1$. Since $|g(\lambda x) - g(x)| = |\chi_{\lambda E} - \chi_E| = \chi_{E\Delta(\lambda E)}$ this completes the first part of the proof.

Finally, let $f \in L^4(\mathbb{R})$. Then for $\epsilon > 0$ there exists a simple function $g \in L^4(\mathbb{R})$ such that

$$\int_{\mathbb{R}} |f(x) - g(x)|^4 \, dx < \epsilon.$$

By a change of variables, we see that

$$\int_{\mathbb{R}} |f(\lambda x) - g(\lambda x)|^4 \, dx = \frac{1}{\lambda}\int_{\mathbb{R}} |f(x) - g(x)|^4 \, dx < \frac{\epsilon}{\lambda}.$$

By taking, say, $\lambda > 1/2$ we get that

$$\int_{\mathbb{R}} |f(\lambda x) - g(\lambda x)|^4 \, dx < 2\epsilon.$$

We already saw that $\int_{\mathbb{R}} |g(\lambda x) - g(x)|^4 dx \to 0$ by the previous step. Hence, for λ close to 1 we get

$$\int_{\mathbb{R}} |f(\lambda x) - f(x)|^4 \, dx \leq \int_{\mathbb{R}} |f(\lambda x) - g(\lambda x)|^4 \, dx + \int_{\mathbb{R}} |g(\lambda x) - g(x)|^4 \, dx + \int_{\mathbb{R}} |g(x) - f(x)|^4 \, dx < (3 + C)\epsilon$$

Solution (2): By the triangle inequality & density of $C_c^\infty(\mathbb{R})$ in $L^4(\mathbb{R})$, it STS the result for $g \in C_c^\infty(\mathbb{R})$. So, let $g \in C_c^\infty(\mathbb{R})$. We show the result via the generalized DCT: let $f_n(x) := |g([1 - \frac{1}{n}]x) - g(x)|^4$, $h_n(x) := (|g([1 - \frac{1}{n}]x)| + |g(x)|)^4$. Then, we have the following:

$$|f_n(x)| \leq |h_n(x)|$$

$$f_n \to 0 \text{ \& } h_n \to (2|g(x)|)^4 \text{ pointwise}$$

$$\int h_n(x)dx \to \int (2|g(x)|)^4 dx = \int h(x)dx \text{ by the regular DCT with dominating function } 2^4||g||_\infty^4\chi_{supp(g)}$$

So, by the generalized DCT, $\int f_n dx \to 0$ as $n \to \infty$.

Problem 10 (Spring 2014). Let f, g be bounded measurable functions on $\mathbb{R}^n$. Assume that g is integrable and satisfies $\int g = 0$. Define $g_k(x) = k^n g(kx)$ for $k \in \mathbb{N}$. Show that $f * g_k \to 0$ pointwise a.e. as $k \to \infty$.

Solution: First note that

$$\int_{\mathbb{R}^n} g_k(x) \, dx = \int_{\mathbb{R}^n} k^n g(kx) \, dx = \int_{\mathbb{R}^n} g(x) \, dx = 0.$$

We then have that

$$\int_{\mathbb{R}^n} f(x)g_k(y) \, dy = f(x)\int_{\mathbb{R}^n} g_k(y) \, dy = 0$$

and so

$$f * g_k(x) = \int_{\mathbb{R}^n} f(x-y)g_k(y) \, dy = \int_{\mathbb{R}^n} [f(x-y) - f(x)]g_k(y) \, dy.$$

Now let $\delta > 0$ and consider the following splitting:

$$f * g_k(x) = \int_{|y|\leq\delta} [f(x-y) - f(x)]g_k(y) \, dy + \int_{|y|>\delta} [f(x-y) - f(x)]g_k(y) \, dy.$$

For the first integral, we have

$$\left| \int_{|y|<\delta} [f(x-y) - f(x)] g_k(y)\ dy \right| \le \int_{|y|<\delta} |f(x-y) - f(x)| g_k(y)\ dy \le \|g\|_\infty k^n \int_{|y|<\delta} |f(x-y) - f(x)|\ dy$$
$$= \|g\|_\infty k^n \int_{|y-x|<\delta} |f(y) - f(x)|\ dy = \|g\|_\infty k^n \int_{B_\delta(x)} |f(y) - f(x)|\ dy.$$

We now recognize the integral from the Lebesgue differentiation theorem. Recall that it states

$$\lim_{\delta \to 0} \frac{1}{|B_\delta|} \int_{B_\delta(x)} f(y)\ dy = f(x)$$

for almost every x. For Lebesgue points (which also occur almost everywhere), we have the stronger statement that

$$\lim_{\delta \to 0} \frac{1}{|B_\delta|} \int_{B_\delta(x)} |f(y) - f(x)|\ dy$$

So, we need to introduce a factor of $1/|B_\delta|$. Observe that we already have a factor of k^n, so we are inclined to use $\delta = C/k$, where C is a constant to be chosen. We will see the importance of C later. Regardless, we have

$$\left| \int_{|y|<\delta} [f(x-y) - f(x)] g_k(y)\ dy \right| \le \frac{\|g\|_\infty C^n |B_1|}{|B_{C/k}|} \int_{B_{C/k}(x)} |f(y) - f(x)|\ dy \to 0$$

for k large enough. For the second integral, we have

$$\left| \int_{|y|>\delta} [f(x-y) - f(x)] g_k(y)\ dy \right| \le 2\|f\|_\infty \int_{|y|>k\delta} |g(y)|\ dy = 2\|f\|_\infty \int_{|y|>C} |g(y)|\ dy$$

where we have applied the fact that f is bounded and a change of variable $ky \mapsto y$. Notice if we did not have control over C (i.e., if we just carelessly chose $\delta = 1/k$ previously) we would not be able to proceed. But, as $C \to \infty$ the sets $|y| > C$ decrease to the empty set. It follows by dominated convergence that

$$\lim_{k \to \infty} \int_{|y|>C} |g(y)|\ dy = 0.$$

Problem 11 (Fall 2013). Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of integrable functions on $[0,1]$ such that $\|f_n\|_{L^1([0,1])} \le n^{-2}$ for all $n \in \mathbb{N}$. Show that $f_n \to 0$ pointwise a.e.

Solution: Define $f := |f_1| + |f_2| + ...$ (which is well defined in the extended reals). Now, by the triangle inequality we have

$$\|f\|_{L^1([0,1])} \le \sum_{n=1}^\infty \|f_n\|_{L^1([0,1])} \le \sum_{n=1}^\infty \frac{1}{n^2} < \infty.$$

This tells us that f is integrable, and hence is finite almost everywhere. Now, consider the series

$$|f(x)| = \sum_{n=1}^\infty |f_n(x)| < \infty$$

for almost every x. It follows for these x that $|f_n(x)| \to 0$, otherwise the series would diverge.

Problem 12 (Spring 2013). Let $f \in L^\infty(\mu)$ be a nonnegative bounded μ-measurable function. Consider the set R_f consisting of all positive real numbers w such that $\mu(\{x \mid |f(x) - w| \le \epsilon\}) > 0$ for every $\epsilon > 0$.

a) Prove that R_f is compact.
b) Prove that $\|f\|_\infty = \sup R_f$.

Solution:

a) Clearly R_f is bounded. We show now it is closed. Let $w_n \to w \in [0,\infty)$ such that w is a limit point of R_f. Let $\epsilon > 0$; then there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|w_n - w| < \epsilon/2$. By definition of w_n, we have for all n that

$$\mu(\{x \mid |f(x) - w_n| \leq \epsilon/2\}) > 0$$

Now, by the triangle inequality, if $|f(x) - w_n| \leq \epsilon/2$ then

$$|f(x) - w| \leq |f(x) - w_n| + |w_n - w| < \epsilon$$

for all $n \geq N$. Hence

$$\{x \mid |f(x) - w_n| \leq \epsilon/2\} \subset \{x \mid |f(x) - w| < \epsilon\}$$

for $n \geq N$, and by monotonicity we find that $w \in R_f$.

b) Clearly if $f \equiv 0$ there is nothing to do. By definition,

$$\begin{aligned}\|f\|_\infty &:= \inf\{M \geq 0 \mid |f(x)| \leq M \text{ for almost every } x\} \\ &= \inf\{M \geq 0 \mid \mu(\{x \mid |f(x)| \geq M\}) = 0\}.\end{aligned}$$

We show that, equivalently,

$$\|f\|_\infty = \sup\{w \geq 0 \mid \mu(\{x \mid |f(x) - w| \leq \epsilon\}) > 0\}$$

for all $\epsilon > 0$. Denote the above sup by S. Suppose that $\|f\|_\infty > S$. Then there exists an $\epsilon > 0$ such that $\mu(\{x \mid |f(x)| \geq \|f\|_\infty - \epsilon\}) = 0$. But, this contradicts the definition of $\|f\|_\infty$ since we would have $\|f\|_\infty - \epsilon$ as an admissible M in the inf definition. Hence $\|f\|_\infty \leq S$. On the other hand, suppose $\|f\|_\infty < S$. Then there exists an $\epsilon > 0$ such that $\|f\|_\infty < \|f\|_\infty + 3\epsilon/2 < S$. By definition of $\|f\|_\infty$ we have $\mu(\{x \mid |f(x)| \geq \|f\|_\infty + \epsilon/2\}) = 0$. This implies, in particular, that

$$\mu(\{x \mid |f(x) - (\|f\|_\infty + \epsilon)| \leq \epsilon/2\}) = 0$$

by monotonicity. It follows that $S < \|f\|_\infty + \epsilon$, a contradiction. Hence $\|f\|_\infty = S$.

Problem 13 (Spring 2013). Let f, f_1, f_2, ... be functions in $L^1([0,1])$ such that $f_k \to f$ pointwise almost everywhere. Show that $\|f - f_k\|_1 \to 0$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$, such that $|\int_A f_k \, dx| < \epsilon$ for all k and all measurable sets $A \subset [0,1]$ with measure $|A| < \delta$.

Solution: That $\|f - f_k\|_1 \to 0$ implies that

$$\int_{[0,1]} |f - f_k| \, dx \to 0$$

In particular, on any measurable subset $A \subset [0,1]$ we have

$$\int_A |f - f_k| \, dx \leq \int_{[0,1]} |f - f_k| \, dx \to 0.$$

Now since f is integrable, if $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\int_A |f| \, dx < \frac{\epsilon}{2}$$

for all measurable $A \subset [0,1]$ with $|A| < \delta$ (see Problem 7). Consequently,

$$\left| \int_A f_k \, dx \right| \leq \int_A |f - f_k| \, dx + \int_A |f| \, dx.$$

Now, choose K large enough so that for all $k \geq K$ we have

$$\int_A |f - f_k| \, dx < \frac{\epsilon}{2}$$

from which we immediately deduce

$$\left| \int_A f_k \, dx \right| < \epsilon$$

for $k \geq K$. However, we need this statement for all k. So, we reapply Problem 7 with $f_1, ..., f_{k-1}$ and extract $\delta_1, ..., \delta_{k-1}$ such that

$$\left| \int_A f_i \, dx \right| < \epsilon$$

for all measurable $A \subset [0,1]$ with $|A| < \delta_i$. Hence, taking $\delta' = \min\{\delta, \delta_1, ..., \delta_{k-1}\}$ we get

$$\left| \int_A f_k \, dx \right| < \epsilon$$

for all k whenever $A \subset [0,1]$ is measurable with $|A| < \delta'$.

Suppose the latter and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that

$$\left| \int_A f_k \, dx \right| < \frac{\epsilon}{2}$$

for all k and measurable $A \subset [0,1]$ with measure $|A| < \delta$. Define $A_k^+ := \{f_k \geq 0\}$ and $A_k^- := \{f_k \leq 0\}$. Then,

$$\begin{aligned} \int_A |f_k| \, dx &= \int_{A_k^+} |f_k| \, dx + \int_{A_k^-} |f_k| \, dx = \int_{A_k^+} f_k \, dx + \int_{A_k^-} (-f_k) \, dx \\ &\leq \left| \int_{A_k^+} f_k \, dx \right| + \left| \int_{A_k^-} (-f_k) \, dx \right| = \left| \int_{A_k^+} f_k \, dx \right| + \left| \int_{A_k^-} f_k \, dx \right| < \epsilon \end{aligned}$$

since monotonicity implies that $|A_k^+| < \delta$ and $|A_k^-| < \delta$. Now, we apply Problem 7 once more and, by taking a minimum if necessary, find a $\delta > 0$ such that whenever $|A| < \delta$ then

$$\int_A |f| \, dx < \epsilon, \qquad \int_A |f_k| \, dx < \epsilon \qquad \forall k \in \mathbb{N}.$$

Since $[0,1]$ is compact, we can cover it with finitely many balls $B_\delta(x_n)$, $n = 1, ..., N$. Then,

$$\int_{[0,1]} |f_k - f| \, dx \leq \sum_{n=1}^{N} \int_{[0,1] \cap B_\delta(x_n)} |f_k - f| \, dx \leq \sum_{n=1}^{N} \int_{A_n} |f_k| \, dx + \sum_{n=1}^{N} \int_{A_n} |f| \, dx < 2N\epsilon$$

where $A_n = [0,1] \cap B_\delta(x_n)$ is a measurable subset of $[0,1]$ with $|A_n| < \delta$. XXX

Problem 14 (Spring 2012, Spring 2021). Let $f_k \to f$ a.e. on $\mathbb{R}$. Show that given $\epsilon > 0$, there exists E, with $|E| < \epsilon$, so that $f_k \to f$ uniformly on $I \setminus E$, for any given finite interval I.

Solution: This is just Egorov's theorem. Let I be a finite interval. Let $\epsilon > 0$. Then, for n fixed, define the set $A_{k,n} := \{x \in I : |f_j(x) - f(x)| > \frac{1}{n} \forall j \geq k\}$. The sequence of sets $\{A_{k,n}\}_{k=1}^{\infty}$ is an increasing sequence, and as the convergence of f_k to f holds pointwise a.e., $\cup_{k=1}^{\infty} A_{k_n} = I \setminus N$, where N is a set of measure zero. So, by continuity of measure, $\exists$ an index k_n such that $\lambda(I \setminus A_{k_n,n}) < \frac{\epsilon}{2^n}$. In this way, define sets $\{A_{k_n,n}\}_{n=1}^{\infty}$. Then, taking $E = \cap_{n=1}^{\infty} A_{k_n,n}$ is the desired set.

NB: The counterexample in the infinite measure space case is $f_n(x) = \chi_{[n,n+1]}$, of $f_n(x) = \chi_{-\infty,n]}$.

Problem 15 (Fall 2012). Let (X, A, μ) be a measure space with $\mu(X) < \infty$. Show that a measurable function $f : X \to [0, \infty)$ is integrable if and only if $\sum_{n=0}^{\infty} \mu(\{x \in X \mid f(x) \geq n\})$ converges.

Solution: Suppose first that the series converges. Construct the function

$$g(x) = \sum_{n=0}^{\infty} \chi_{\{f \geq n\}}(x).$$

Observe that $g(x) < f(x)$. Suppose that $N \leq f(x_0) < N+1$ for some $N \in \mathbb{N}$. Then $x_0 \in \{f \geq n\}$ for $0 \leq n \leq N$ but $x_0 \notin \{f \geq n\}$ for $n > N$. Hence,

$$g(x_0) = \sum_{n=0}^{\infty} \chi_{\{f \geq n\}}(x_0) = \sum_{n=0}^{N} 1 = N + 1 > f(x_0).$$

Consequently,

$$\int_X f(x)\ d\mu(x) < \int_X g(x)\ d\mu(x) = \sum_{n=0}^{\infty} \mu(\{f \geq n\}) < \infty.$$

Now suppose f is integrable. Construct the function

$$h(x) = \sum_{n=1}^{\infty} \chi_{\{f\geq n\}}(x).$$

Once more, if $N \leq f(x_0) < N+1$ then

$$h(x_0) = \sum_{n=1}^{N} 1 = N \leq f(x_0)$$

and so

$$\sum_{n=1}^{\infty} \mu(\{f \geq n\}) = \int_X h(x)\ d\mu(x) \leq \int_X f(x)\ d\mu(x) < \infty.$$

But, since $\mu(X) < \infty$ we know also $\mu(\{f \geq 0\}) < \infty$. In total, the entire series converges.

Problem 16 (Spring 2012). Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $f \in L^1(\Omega)$. Prove that

$$\lim_{p\to 0} \left[\int_\Omega |f|^p\ d\mu\right]^{1/p} = \exp\left[\int_\Omega \log|f|\ d\mu\right],$$

where $\exp[-\infty] = 0$. To simplify the problem, you may assume $\log|f| \in L^1(\Omega)$.

Solution: If $f = 0$ a.e. we have equality, so assume that $f \neq 0$ on a set of positive measure. Then

$$\int_\Omega |f|^p\ d\mu > 0$$

for all p. Define a_p by

$$a(p) := \left[\int_\Omega |f|^p\ d\mu\right]^{1/p}$$

so that $a(p) > 0$ for all p. Then by continuity of the logarithm,

$$\log\left(\lim_{p\to 0} a(p)\right) = \lim_{p\to 0} \log a(p) = \lim_{p\to 0}\left(\frac{1}{p}\log\left(\int_\Omega |f|^p\ d\mu\right)\right).$$

As $p \to 0$, $|f|^p \to 1$, and since μ is a probability measure the integral tends to 1. Hence, the logarithm tends to zero while the denominator does too. Applying L'hopital's rule gives

$$\lim_{p\to 0}\left(\frac{1}{p}\log\left(\int_\Omega |f|^p\ d\mu\right)\right) = \lim_{p\to 0}\left(\frac{d}{dp}\log\left(\int_\Omega |f|^p d\mu\right)\right) = \lim_{p\to 0}\left(\frac{\int_\Omega |f|^p \log|f|\ d\mu}{\int_\Omega |f|^p\ d\mu}\right).$$

Once more, as $p \to 0$, we have $|f|^p \to 1$ and μ is a probability measure. Thus

$$\log\left(\lim_{p\to 0}\left[\int_\Omega |f|^p\ d\mu\right]^{1/p}\right) = \int_\Omega \log|f|\ d\mu.$$

Problem 17 (Spring 2012). Let h be a bounded, measurable function, such that, for any interval I

$$\left|\int_I h\right| \leq |I|^{1/2}.$$

Let $h_\epsilon = h(x/\epsilon)$. Show that for any A with $|A| < \infty$,

$$\int_A h_\epsilon(x)\ dx \to 0, \text{ as } \epsilon \to 0.$$

Solution: Since A is measurable with $|A| < \infty$ for $\delta > 0$ there exist a collection of finite intervals $\{I_k\}_{k=1}^{\infty}$ which cover A and

$$\sum_{k=1}^{\infty} |I_k| < |A| + \delta.$$

It suffices to show that

$$\left| \int_A h_\epsilon(x)\ dx \right| \to 0$$

as $\epsilon \to 0$. To this end, note that

$$\left| \int_A h_\epsilon(x)\ dx \right| \leq \sum_{k=1}^{\infty} \left| \int_{I_k} h\left(\frac{x}{\epsilon}\right) \right| = \epsilon \sum_{k=1}^{\infty} \left| \int_{I_k/\epsilon} h \right| \leq \frac{\epsilon}{\sqrt{\epsilon}} \sum_{k=1}^{\infty} |I_k| < \sqrt{\epsilon}(|A| + \delta).$$

Since $|A| + \delta < \infty$, taking $\epsilon \to 0$ gives the result. XXX

Problem 18 (Fall 2011). For $1/p + 1/q = 1$, let $S = \{f \in L^p(\mathbb{R}) \mid \text{spt}(f) \subset [-1,1], \text{ and } \|f\|_p \leq 1\}$, and let g be a fixed but arbitrary function in $L^1(\mathbb{R})$, with $\text{spt}(g) \subset [-1,1]$. Show that the image of S under the map $f \mapsto f * g$ is a compact set in $C^0([-2,2])$.

Solution: XXX

Problem 19 (Fall 2011). Let $f_0, f_1, f_2, ...$ be nonnegative Lebesgue-integrable functions on $\mathbb{R}^n$, such that

$$\sum_{k=1}^{\infty} \int (f_k - f_{k-1})^+ < \infty, \qquad \lim_{k\to\infty} \int f_k = 0.$$

Show that $\limsup_{k\to\infty} f_k \equiv 0$ almost everywhere.

Solution: Define g_n by

$$g_n = \sum_{k=1}^{n} (f_k - f_{k-1})^+$$

so that $g_1 \leq g_2 \leq ....$ Then, by monotone convergence

$$\int \sum_{k=1}^{\infty} (f_k - f_{k-1})^+ = \int \lim_{n\to\infty} g_n = \lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \sum_{k=1}^{n} \int (f_k - f_{k-1})^+ < \infty.$$

Next, observe that for each k

$$f_k - f_{k-1} \leq (f_k - f_{k-1})^+$$

and thus

$$f_n - f_0 = \sum_{k=1}^{n} f_k - f_{k-1} \leq \sum_{k=1}^{n} (f_k - f_{k-1})^+ = g_n \leq g.$$

Hence, the f_n are dominated by $g + f_0$ and $g + f_0 \in L^1(\mathbb{R}^n)$. It follows that

$$0 = \lim_{n\to\infty} \int f_n = \int \limsup_{n\to\infty} f_n$$

from which we discover $\limsup_{n\to\infty} f_n = 0$.

Problem 20 (Fall 2020). Given $f : \mathbb{R}^n \to \mathbb{R}$, let $\tau_M(f) = \chi_{B_M(0)} min\{M, max\{f, -M\}\}$ for $M > 0$. Show that $\tau_M(f) \to f$ in $L^p(\mathbb{R}^n, \mu)$ as $M \to \infty$ whenever $p \in [1, \infty)$, $f \in L^p(\mathbb{R}^n, \mu)$, and μ is a locally finite Borel measure on $\mathbb{R}^n$. Does the result hold if $p = \infty$?.

Solution: If f non-negative, then result follows from monotone convergence. Otherwise, split f into f^+, f^-. The result does not hold if $p = \infty$: consider $f = 1$ on $\mathbb{R}^n$ with Lebesgue measure.

Problem 21 (Fall 2020). Let L be a bounded, contractive ($||L|| < 1$) linear map from a Banach space to itself. Define the sequence $\{x_k\}$ by the recursive relation $x_{k+1} = L(x_k)$. Show that $\{x_k\}$ is a Cauchy sequence, and deduce the existence of a fixed point of L.

Solution: The sequence is Cauchy: $||x_m - x_n|| = ||L^m(x_0) - L^n(x_0)|| = ||L^n[(L^{m-n}(x_0)) - x_0]|| \leq ||L||^n||(L^{m-n}(x_0)) - x_0|| \leq 2||L||^n||x_0|| \to 0$ as $m, n \to \infty$ (assuming WLOG that $m > n$). So, by completeness, $\exists$ a limit x. x is a fixed point of L:

$$x = lim_{k\to\infty}x_k = lim_{k\to\infty}L(x_{k-1}) = L(lim_{k\to\infty}x_{k-1}) = L(x)$$

Problem 22 (Fall 2020). Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $f : X \times (-1, 1) \to \mathbb{R}$ be a function $f(x, t)$ such that for each $t \in (-1, 1)$, $f(\cdot, t) : X \to \mathbb{R}$ is $\mathcal{M}-$measurable and for $\mu-$a.e. $x \in X$, $f(x, \cdot)$ has a classical derivative in the following sense:

$$\frac{\partial f}{\partial t}(x, 0) = \lim_{h\to 0^+} \frac{f(x, h) - f(x, 0)}{h}$$

which exists for $\mu-$a.e. $x \in X$. Show that if there exists M such that:

$$|f(x, t) - f(x, 0)| \leq M|t| \text{ for } \mu\text{-a.e. } x \in X$$

then the function:

$$g(t) = \int_X f(x, t)d\mu(x)$$

is differentiable at $t = 0$ with:

$$g'(0) = \int_X \frac{\partial f}{\partial t}(x, 0)d\mu(x)$$

Solution: Dominated convergence theorem on the sequence $f_k(x) = \frac{f(x,\frac{1}{k})-f(x,0)}{\frac{1}{k}}$. The sequence is dominated by $g = M$, which is integrable as X is a finite measure space.

Problem 23 (Spring 2022). Let $f : \mathbb{R}^n \to [0, \infty]$ be a measurable function. Show:

(1) $|\{x \in \mathbb{R}^n : f(x) \geq k\} \leq \frac{1}{k} \int f$
(2) If f is integrable, then $|\{x \in \mathbb{R}^n : f(x) = \infty\} = 0$.

Solution:

(a) Chebyshev inequality (pf. is the same).
(b) Use part (a) and see that as $k \to \infty$, $\frac{1}{k} \int f$ goes to zero. $\int f < \infty$ as f is integrable.

Problem 24 (Fall 2021). Let Σ be a compact set of functions in $L^p([0, 1])$. Show that the subset of Σ $\Sigma^+ := \{f^+ : f \in \Sigma\}$ is also compact.

Solution: It STS that every sequence in Σ^+ has a convergent subsequence. Let f_n^+ be a sequence in Σ^+. Then, f_n is a sequence in Σ, so there exists a convergent subsequence $f_{n_k} \to f$ in L^p. So, there exists a further subsequence $\{f_{n_{k_j}}\}$ that converges to f pointwise a.e.. It follows that $\{f^+_{n_{k_j}}\}$ converges to f^+ pointwise a.e.. Finally, $\{f^+_{n_{k_j}}\}$ converges to f^+ in L^p by the generalized dominated convergence theorem.

Convergence in Measure.

Problem 1 (Spring 2019). Let the sequence of measurable functions $f_k(x)$ converge in measure to zero in $B_1(\mathbb{R}^n)$ and satisfy $\|f_k\|_{L^2}$ less or equal than M for all k. Show that f_k converges to zero in L^1.

See Problem 4

Problem 2 (Fall 2016). Prove that, on a finite measure space, if $f_k \to f$ in measure and $g_k \to g$ in measure, then $f_k g_k \to fg$ in measure.

Solution: It suffices to show that all subsequences of $f_k g_k$ have a further subsequence that converges to fg in measure. So, let $f_{k_j} g_{k_j}$ be a subsequence. Then, f_{k_j} converges to f in measure, so there exists a subsequence $f_{k_{j_\ell}}$ that converges to f pointwise a.e.. Now, $g_{k_{j_\ell}}$ converges to g in measure, so $\exists$ a subsequence $g_{k_{j_{\ell_n}}}$ that converges to g pointwise a.e..

In total, we can find a subsequence of $f_{k_j} g_{k_j}$ that converges to fg pointwise a.e.. On a finite measure space, convergence pointwise a.e. implies convergence in measure (Egorov), so we are done.

Problem 3 (Fall 2014). Recall that a sequence $\{f_i\}_{i=1}^\infty$ of real-valued measurable functions on the real line is said to *converge in measure* to a function f if

$$\lim_{i\to\infty} \lambda(\{x \in \mathbb{R} \mid |f_i(x) - f(x)| \geq \epsilon\}) = 0, \qquad \forall \epsilon > 0$$

where λ denotes Lebesgue measure on $\mathbb{R}$. Suppose that in addition to this, there exists an integrable function g such that $|f_i| \leq g$ for all i. Prove that $\{f_i\}_{i=1}^\infty$ converges to f in $L^1(\mathbb{R})$.

Solution: Recall that if a sequence of real numbers is such that every subsequence has a further subsequence which converges to the same limit, then the original sequence does too. To this end, since $f_i \to f$ in measure, all of its subsequences do, and there exists a subsubsequence $\{f_{i_{n_k}}\}$ which converges to f almost everywhere. Hence, by dominated convergence $f_{i_{n_k}} \to f$ in $L^1(\mathbb{R})$. By the above observation, $f_i \to f$ in $L^1(\mathbb{R})$.

Problem 4 (Spring 2014). Let (X, Σ, μ) be a finite measure space and $1 \leq q < p < \infty$. Let f_1, $f_2, ... \in L^p(X, \mu)$ with $\|f_k\|_p \leq 1$ for all k. Assuming $f_k \to f$ in measure, show that $f \in L^p(X, \mu)$, and that $\|f_k - f\|_q \to 0$.

Solution: First, since $f_k \to f$ in measure there exists a subsequence f_{k_n} which converges to f μ-almost everywhere in X. In particular, $|f_{k_n}| \to |f|$ μ-almost everywhere. It follows by Fatou's lemma that

$$\int_X |f|^p \, d\mu = \int_X \liminf_{n\to\infty} |f_{k_n}|^p \, d\mu \leq \liminf_{n\to\infty} \int_X |f_{k_n}|^p \, d\mu = \liminf_{n\to\infty} \|f_{k_n}\|_p^p \leq 1.$$

It follows $f \in L^p(X, \mu)$.

Now, to show that $f_k \to f$ in L^q, it suffices to show that all subsequences have a further subsequence that converges to f in L^q. So, let f_{k_j} be a subsequence. Then, it converges to f in measure, so there is a subsequence $f_{k_{j_n}}$ that converges to f pointwise a.e.. Now, let $\epsilon > 0$. Then, by Egorov, $\exists$ a set A such that $\mu(X \setminus A) < \epsilon$, and the converges to f is uniform on A. So:

$$||f_k - f||_q^q = \int_X |f_k - f|^q = \int_A |f_k - f|^q + \int_{X\setminus A} |f_k - f|^q$$

The first term, we bound by $\mu(X)\epsilon^q$ for k sufficiently large. The second term, use Holder to bound it by $\epsilon^{1-\frac{q}{p}}||f_k - f||_p^q \leq \epsilon^{1-\frac{q}{p}}(2)^q$ by the uniform bounds on the L^p norm.

<u>NB:</u> It is tempting to try to show that $f_k \to f$ in L^p and thus converges in L^q by Holder as we are on a finite measure space, but this is actually not true: consider the sequence $f_k = \sqrt[p]{k}\chi_{[0,\frac{1}{k}]}$ defined on $[0, 1]$ with the Lebesgue measure. Then, $||f_k||_p = 1$ and f_k converges to 0 in measure, but not in L^p.

Weak L^p and Fubini.

Problem 1 (Spring 2019). Let H be a monotone function of $f(x)$, a non-negative measurable function. Write

$$\int H(f(x)) \, dx$$

in terms of $g(\lambda) = |\{f > \lambda\}|$.

Solution: Since H is monotone, it has a derivative almost everywhere. We may also assume that $H(0) = 0$. By the fundamental theorem of calculus we have that

$$H(f(x)) = \int_0^{f(x)} H'(t)\ dt = \int_{-\infty}^{\infty} \chi_{[0,f(x)]}(t)H'(t)\ dt.$$

Then, applying Fubini's theorem

$$\int H(f(x))\ dx = \int \int_{-\infty}^{\infty} \chi_{[0,f(x)]}(t)H'(t)\ dtdx = \int_{-\infty}^{\infty} H'(t) \left[\int \chi_{[0,f(x)]}(t)\ dx\right] dt = \int_{-\infty}^{\infty} H'(t)g(t)\ dt.$$

Problem 2 (Spring 2016). Show that if $p > 1$ and $f \in L^p([0,\infty), m)$ then the 'mean functional' of f,

$$F(y) := \frac{1}{y}\int_0^y f(t)\ dt = \int_0^1 f(xy)\ dx$$

is also in $L^p([0,\infty), m)$ and moreover

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p.$$

Hint: consider $f(xy)$ as a function of two variables on $[0,1]\times[0,\infty)$ and use the generalized Minkowski inequality (which states that if $g : X \times Y \to \mathbb{R}$ is any measurable function on the direct product of two sigma-finite measure spaces $(X,\mu), (Y,\nu)$ then

$$\|\|g\|_{L^1(X,\mu)}\|_{L^p(Y,\nu)} \leq \|\|g\|_{L^p(Y,\nu)}\|_{L^1(X,\mu)}).$$

Solution: Using the hint, let's define $g(x,y) = f(xy)$ on $X \times Y = [0,1] \times [0,\infty)$. Both (X,m) and (Y,m) are sigma-finite measure spaces so that we can apply generalized Minkowski:

$$\left[\int_0^\infty \left[\int_0^1 |g(x,y)|\ dx\right]^p dy\right]^{1/p} \leq \int_0^1 \left[\int_0^\infty |g(x,y)|^p\ dy\right]^{1/p} dx.$$

Note that

$$|F(y)| \leq \int_0^1 |f(xy)|\ dx = \int_0^1 |g(x,y)|\ dx$$

so the left hand side is bounded below by

$$\left[\int_0^\infty \left[\int_0^1 |g(x,y)|\ dx\right]^p dy\right]^{1/p} \geq \left[\int_0^\infty F(y)^p dy\right]^{1/p} = \|F\|_p.$$

It suffices now to bound the right-hand side in terms of $p/(p-1)\|f\|_p$. We have that

$$\begin{aligned}\int_0^1 \left[\int_0^\infty |g(x,y)|^p\ dy\right]^{1/p} dx = \int_0^1 \left[\int_0^\infty |f(xy)|^p\ dy\right]^{1/p} dx = \int_0^1 \left[\int_0^\infty \frac{1}{x}|f(y)|^p\ dy\right]^{1/p} dx \\ = \int_0^1 \frac{1}{x^{1/p}} \left[\int_0^\infty |f(y)|^p\ dy\right]^{1/p} dx = \|f\|_p \int_0^1 \frac{1}{x^{1/p}}\ dx \\ = \frac{\|f\|_p}{1-1/p} x^{1-1/p}\Big|_0^1 = \frac{p}{p-1}\|f\|_p\end{aligned}$$

by a change of variables $xy \mapsto y$. Note that $p > 1$ is vital, since we need $1 - 1/p > 0$ in order for the lower limit to be defined.

Problem 3 (Fall 2016). Let f be a locally integrable function on $\mathbb{R}^2$. Assume that, for any given real numbers a and b outside some set of measure zero, $f(x,a) = f(x,b)$ for almost every $x \in \mathbb{R}$ and $f(a,y) = f(b,y)$ for almost every $y \in \mathbb{R}$. Show that f is constant almost everywhere on $\mathbb{R}^2$.

Solution: Let $E \subset \mathbb{R}$ be such that $|E^c| = 0$ and for all $a, b \in E$ we have $f(x,a) = f(x,b)$ for almost every $x \in \mathbb{R}$ and $f(a,y) = f(b,y)$ for almost every $y \in \mathbb{R}$. Choose $a, b \in E$ such that $f(a,y) = f(b,y)$ for almost every $y \in \mathbb{R}$. Now, since E has full measure there exist $c, d \in E$ such that $f(x,c) = f(x,d)$ for almost every $x \in \mathbb{R}$ and both $f(a,c) = f(a,d)$ and $f(b,c) = f(b,d)$. Consider now the following difference of integrals:

$$\int_c^d \int_a^b f(x,y)\ dxdy - \int_c^d \int_{a+\delta}^{b+\delta} f(x,y)\ dxdy = \int_c^d \int_a^b [f(x,y) - f(x,y+\delta)]\ dxdy$$

Let $a,b,c,d \in \mathbb{R}$, $\delta > 0$ and consider the following difference of integrals:

$$\int_c^d \int_a^b f(x,y)\ dxdy - \int_c^d \int_{a+\delta}^{b+\delta} f(x,y)\ dxdy = \int_c^d \int_a^b [f(x,y) - f(x,y+\delta)]\ dxdy.$$

Define $g_y(x) = f(x,y) - f(x,y+\delta)$. If y is such that y and $y+\delta$ are in E then $g_y(x) = 0$ for almost every x. But, E has full measure, so $E+\delta$ does too. Hence $E \cap (E+\delta)$ has full measure, and in particular for every $y \in [c,d]$ we have y and $y+\delta$ are in E. It follows that $g_y(x) = 0$ a.e. for almost every $y \in [c,d]$. Hence, the above difference is zero and

$$\int_c^d \int_a^b f(x,y)\ dxdy = \int_c^d \int_{a+\delta}^{b+\delta} f(x,y)\ dxdy.$$

A similar conclusion holds by translating the y coordinate instead. Hence, we see that $\int_Q f(x,y)dxdy$ depends only on $|Q|$. Let $I(Q) := \int_Q f(x,y)\ dxdy$. Lebesgue differentiation says that for almost every $(x_0,y_0) \in \mathbb{R}^2$,

$$f(x_0,y_0) = \lim_{r\to 0} \frac{1}{|Q_r|} \int_{Q_r(x_0,y_0)} f(x,y)\ dxdy = \lim_{r\to 0} \frac{I(Q_r)}{Q_r} = c.$$

Where $Q_r(x_0,y_0)$ is a square of side length r centered at (x_0,y_0).

Problem 4 (Fall 2015). Let f and g be real valued measurable integrable functions on a measure space (X,μ) and let

$$F_t = \{x \in X \mid f(x) > t\}, \qquad G_t = \{x \in X \mid g(x) > t\}.$$

Prove that

$$\|f-g\|_1 = \int_{-\infty}^{\infty} \mu(F_t \Delta G_t)\ dt$$

where

$$F_t \Delta G_t = (F_t \setminus G_t) \cup (G_t \setminus F_t).$$

Solution: Note the resemblance to the layer-cake formula. We use this as our inspiration for solving the problem. First, break up the integral as follows

$$\|f-g\|_1 = \int_X |f(x)_g(x)|\ d\mu(x) = \int_{\{f>g\}} [f(x)-g(x)]\ d\mu(x) + \int_{\{g>f\}} [g(x)-f(x)]\ d\mu(x).$$

We compute the first integral and note the second will be the same, except with f replaced by g (and vice verse). If x is such that $f(x) > g(x)$ then

$$f(x) - g(x) = \int_{g(x)}^{f(x)} 1\ dt = \int_{-\infty}^{\infty} \chi_{[g(x),f(x)]}(t)\ dt = \int_{-\infty}^{\infty} \chi_{\{g<t\}}(x)\chi_{\{f>t\}}(x)\ dt.$$

Observe that if $g(x) > f(x)$ then for almost every $t \in \mathbb{R}$ we never have that $x \in \{t > g\} \cap \{f > t\}$. Hence we can actually conclude that

$$\chi_{\{f>g\}}(x)[f(x)-g(x)] = \int_{-\infty}^{\infty} \chi_{\{g<t\}}(x)\chi_{\{f>t\}}(x)\ dt = \int_{-\infty}^{\infty} \chi_{\{g<t\}}(x)\chi_{F_t\setminus G_t}(x)\ dt$$

Next, by Fubini's theorem

$$\begin{aligned}\int_{\{f>g\}} [f(x)-g(x)]\ d\mu(x) &= \int_X \chi_{\{f>g\}}(x)[f(x)-g(x)]\ d\mu(x) = \int_X \left[\int_{-\infty}^{\infty} \chi_{F_t\setminus G_t}(x)\ dt\right] d\mu(x) \\ &= \int_{-\infty}^{\infty} \left[\int_X \chi_{F_t\setminus G_t}(x)\ d\mu(x)\right] dt = \int_{-\infty}^{\infty} \mu(F_t\setminus G_t)\ dt.\end{aligned}$$

Using our previous symmetry observation,

$$\int_{\{g>f\}} [g(x) - f(x)]\ d\mu(x) = \int_{-\infty}^{\infty} \mu(G_t \setminus F_t)\ dt.$$

Finally, note that $F_t \setminus G_t$ and $G_t \setminus F_t$ are disjoint for all t, so that

$$\|f-g\|_1 = \int_{-\infty}^{\infty} \mu(F_t \setminus G_t)\ dt + \int_{-\infty}^{\infty} \mu(G_t \setminus F_t)\ dt = \int_{-\infty}^{\infty} \mu([F_t \setminus G_t] \cup [G_t \setminus F_t])\ dt = \int_{-\infty}^{\infty} \mu(F_t \Delta G_t)\ dt.$$

Problem 5 (Spring 2014, Fall 2022). Let $0 < q < p < \infty$. Let $E \subset \mathbb{R}^n$ be measurable with measure $|E| < \infty$. Let f be a measurable function on $\mathbb{R}^n$ such that $N := \sup_{\lambda > 0} \lambda^p |\{x \in \mathbb{R}^n \mid |f(x)| > \lambda\}|$ is finite.

a) Prove that $\int_E |f|^q$ is finite.
b) Refine the argument of a) to prove that

$$\int_E |f|^q \leq C N^{q/p} |E|^{1-q/p},$$

where C is a constant that depends only on n, p, and q.

Solution:

a) Let $0 < q < p < \infty$ so there exists an $\epsilon > 0$ such that $p - q = \epsilon > 0$. Then by the layer-cake formula,

$$\begin{aligned}\int_E |f|^q \leq \int_{\mathbb{R}^n} |f|^q &= \int_0^{\infty} |\{|f|^q > \lambda\}|\ d\lambda = \int_0^{\infty} |\{|f| > \lambda^{1/q}\}|\ d\lambda \\ &= q \int_0^{\infty} \lambda^{q-1} |\{|f| > \lambda\}|\ d\lambda\end{aligned}$$

where we applied a change of variables $\lambda^{1/q} \mapsto \lambda$. Notice that the integrand is almost in the form of N, so we need to introduce a λ^p. We transform it as follows:

$$\begin{aligned}\int_E |f|^q &= q \int_0^{\delta} \lambda^{q-1} |\{|f| > \lambda\}|\ d\lambda + q \int_{\delta}^{\infty} \frac{\lambda^p |\{|f| > \lambda\}|}{\lambda^{p-q+1}}\ d\lambda \\ &\leq q \int_0^{\delta} \lambda^{q-1} |E|\ d\lambda + q \int_0^{\infty} \frac{N}{\lambda^{\epsilon+1}}\ d\lambda = |E|\lambda^q \Big|_0^{\delta} - \frac{qN}{\epsilon \lambda^{\epsilon}} \Big|_{\delta}^{\infty} = |E|\delta^q + \frac{qN}{(p-q)\delta^{p-q}} < \infty\end{aligned}$$

whenever $\delta > 0$. Note that we have to take $\delta > 0$; if not, it would be as if we took $\delta = 0$ in the above, which clearly diverges.

b) To refine this, notice that we can optimize in δ That is, let $g(\delta) = |E|\delta^q + qN/((p-q)\delta^{p-q})$. Then, the derivative of this is

$$g'(\delta) = q|E|\delta^{q-1} - \frac{qN}{\delta^{p-q+1}}$$

and this is zero if

$$q|E|\delta^{q-1} = \frac{qN\delta^{q-1}}{\delta^p} \quad \Leftrightarrow \quad \delta = \left(\frac{N}{|E|}\right)^{1/p}$$

This point is a local minimum of g, and thus is the best δ to use to bound $\int_E |f|^q$. We have that

$$\begin{aligned}g(\delta) &\geq |E| \left(\frac{N}{|E|}\right)^{q/p} + \frac{qN}{p-q} \left(\frac{N}{|E|}\right)^{(q-p)/p} \\ &= N^{q/p}|E|^{1-q/p} + \left(\frac{q}{p-q}\right) N^{1+q/p-1} |E|^{(p-q)/p} = \left(\frac{p}{p-q}\right) |N|^{q/p} |E|^{1-q/p}\end{aligned}$$

Hence,

$$\int_E |f|^q \leq \int_{\delta > 0} g(\delta) = \left(\frac{p}{p-q}\right) |N|^{q/p} |E|^{1-q/p}.$$

Problem 6 (Spring 2013). . Let $p > 0$, and denote by $L^p_{\text{weak}}(\mathbb{R})$ the space of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ for which

$$N_p(f) := \sup_{\alpha>0} \alpha^p |\{x \in \mathbb{R}^n \mid |f(x)| > \alpha\}|$$

is finite. Prove that the simple functions are not dense in $L^p_{\text{weak}}(\mathbb{R})$, in the sense that there exists a function $f \in L^p_{\text{weak}}(\mathbb{R})$ such that $N_p(f - h_k) \to 0$ fails to hold for every sequence of simple functions $h_1, h_2, ...$

Solution: XXX

Problem 7 (Fall 2011). Let $1 < p < \infty$ and $f(x) = |x|^{-n/p}$ for $x \in \mathbb{R}^n$. Prove that f is not the limit of a sequence $f_k \in C_0^\infty(\mathbb{R}^n)$ in the sense of convergence in $L^p_{\text{weak}}(\mathbb{R}^n)$. That is, $\limsup_{k\to\infty} \sup_{\lambda>0} \lambda^p |\{x \in \mathbb{R}^n \mid |f(x) - f_k(x) > \lambda\}| > 0$ for any such sequence.

Solution: XXX

Problem 8 (Fall 2020). Let μ_1 be counting measure on $\mathbb{R}$, and μ_2 be Lebesgue measure. Let $E = \{(x, y) \in \mathbb{R}^2 : 0 \le x = y \le 1\}$. Show that the integrals

$$\int d\mu_1(x) \int \chi_E d\mu_2(x)$$
$$\int d\mu_2(x) \int \chi_E d\mu_1(x)$$

are well-defined, but not equal. Explain why this does not contradiction Fubini/Tonelli's theorem.

Solution: First integrates to zero, second integrates to to 1. This does not contradict Tonelli because μ_1 is not σ-finite.

Problem 9 (Fall 2021). Show an example of a function $f(x)$ such that $f \in L^{p,w}(B_1^n(0), dx)$, but not in the classical $L^p(B_1^n(0))$.

Solution: $f(x) = |x|^{\frac{-n}{p}}$.

Maximal Functions.

Problem 1 (Spring 2017). For $f \in L^1(\mathbb{R})$ denote by Mf be the restricted maximal function defined by

$$(Mf)(x) = \sup_{0<t<1} \frac{1}{2t} \int_{x-t}^{x+t} |f(z)|\, dz.$$

Show that $M(f * g) \le (Mf) * (Mg)$ for all $f, g \in L^1(\mathbb{R})$.

Solution: By Fubini we have

$$\begin{aligned}
\sup_{0<t<1} \frac{1}{2t} \int_{x-t}^{x+t} \left| \int_{-\infty}^{\infty} f(z-y)g(y)\, dy \right| dz &\le \sup_{0<t<1} \frac{1}{2t} \int_{-\infty}^{\infty} |g(y)| \left[\int_{x-t}^{x+t} |f(z-y)|\, dz \right] dy \\
&= \int_{-\infty}^{\infty} |g(y)| \left[\sup_{0<t<1} \frac{1}{2t} \int_{x-t}^{x+t} |f(z-y)|\, dz \right] dy \\
&= \int_{-\infty}^{\infty} |g(y)| \left[\sup_{0<t<1} \frac{1}{2t} \int_{x-y-t}^{x-y+t} |f(z)|\, dz \right] dy \\
&= \int_{-\infty}^{\infty} |g(y)| Mf(x-y)\, dy
\end{aligned}$$

By Lebesgue differentiation, we have for almost every $x \in \mathbb{R}$ that

$$\lim_{r\to 0} \frac{1}{2r} \int_{x-r}^{x+r} |g(y)|\, dy = |g|(x)$$

In particular, for fixed $0 < r < 1$ we have

$$\frac{1}{2r}\int_{x-r}^{x+r} |g(y)|\ dy \le (Mg)(x)$$

and by taking $r \to 0$ we see $|g|(x) \le (Mg)(x)$ almost everywhere. Hence,

$$\int_{-\infty}^{\infty} |g(y)|Mf(x-y)\ dy \le \int_{-\infty}^{\infty} Mf(x-y)Mg(y) = (Mf) * (Mg)(x).$$

Problem 2 (Fall 2016, Fall 2022). For a function $f \in L^1(\mathbb{R}^2)$ let $\tilde{M}f$ be the unrestricted maximal function

$$\tilde{M}f(x_0, y_0) = \sup_Q \frac{1}{|Q|}\int_Q |f(x,y)|\ dxdy,$$

where the supremum is over all $Q = [x_0 - k, x_0 + k] \times [y_0 - l, y_0 + l]$ with $k, l > 0$.

a) Show that $\tilde{M}f(x_0, y_0) \le M_1 M_2 f(x_0, y_0)$, where

$$M_1 f(x_0, y) = \sup_{k>0} \frac{1}{2k}\int_{x_0-k}^{x_0+k} |f(x,y)|\ dx, \qquad M_2 f(x, y_0) = \sup_{l>0} \frac{1}{2l}\int_{y_0-l}^{y_0+l} |f(x,y)|\ dy.$$

b) Show that there exists $C > 0$ such that if $f \in L^2(\mathbb{R}^2)$ then

$$\|\tilde{M}f\|_{L^2(\mathbb{R}^2)} \le C\|f\|_{L^2(\mathbb{R}^2)}.$$

Solution:

a) Let $Q = [x_0 - k, x_0 + k] \times [y_0 - l, y_0 + l]$. Then clearly by Fubini,

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x,y)|\ dydx &= \frac{1}{4kl}\int_{x_0-k}^{x_0+k}\left[\int_{y_0-l}^{y_0+l} |f(x,y)|\ dy\right] dx \\ &= \frac{1}{2k}\int_{x_0-k}^{x_0+k}\left[\frac{1}{2l}\int_{y_0-l}^{y_0+l} |f(x,y)|\ dy\right] dx \le \frac{1}{2k}\int_{x_0-k}^{x_0+k} M_2 f(x, y_0) dx \\ &\le \frac{1}{2k}\int_{x_0-k}^{x_0+k} M_2 f(x, y_0)\ dx \le M_1 M_2 f(x_0, y_0).\end{aligned}$$

It follows from this that $\tilde{M}f(x_0, y_0) \le M_1 M_2 f(x_0, y_0)$.

b) I suspect there is a more direct way to do this (likely with part a...), but I'm not sure how. Rather, we know that $\tilde{M}$ is a bounded operator from $L^1(\mathbb{R}^2)$ to $L^1_{\text{weak}}(\mathbb{R}^2)$ – this is the well known Hardy-Littlewood maximal theorem. We can also show that $\tilde{M}$ is a bounded operator from $L^\infty(\mathbb{R}^2)$ to $L^\infty(\mathbb{R}^2)$. Indeed, if $f \in L^\infty(\mathbb{R}^2)$ then,

$$\tilde{M}f(x,y) = \sup_Q \frac{1}{|Q|}\int_Q |f(u,v)|\ dudv \le \sup_Q \frac{1}{|Q|}\|f\|_{L^\infty(\mathbb{R}^2)}|Q| = \|f\|_{L^\infty(\mathbb{R}^2)}.$$

It follows by the Marcinkiewicz interpolation theorem that $\tilde{M}$ is a bounded operator from $L^p(\mathbb{R})^2$ to $L^p(\mathbb{R}^2)$ for any $1 < p < \infty$.

Problem 3 (Spring 2014). Consider the Hardy-Littlewood maximal function (for balls)

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|}\int_B |f|, \qquad f(x) := \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1, \end{cases} \qquad x \in \mathbb{R}^n,$$

Prove that Mf belongs to $L^1_{\text{weak}}(\mathbb{R}^n)$.

Solution: Recall the Vitali covering lemma, which says if we have a collection of open balls $\mathcal{B}$ in $\mathbb{R}^n$ then there exist disjoint $B_1, ..., B_k \in \mathcal{B}$ such that

$$\left|\bigcup_{B\in\mathcal{B}} B\right| \le 3^n \sum_{i=1}^k |B_i|.$$

The proof proceeds by using a compact subset which approximates the union, extracting a finite subcover, then applying a greedy algorithm. Now let $E_t = \{Mf > t\}$. For each $x \in E_t$ we can choose an $r_x > 0$ and c_x such that $B_x := B_{r_x}(c_x)$ contains x and

$$\frac{1}{|B_x|}\int_{B_x} |f| > t.$$

Applying the Vitali covering lemma to the collection $\mathcal{B} = \{B_x \mid x \in E_t\}$ yields a finite subcollection $B_x^1, ..., B_x^k$ such that

$$|E_t| \leq \left|\bigcup_{B\in\mathcal{B}} B\right| \leq 3^n \sum_{i=1}^k |B_i| \leq 3^n \sum_{i=1}^k \frac{1}{t}\int_{B_x^i} |f| = \frac{3^n}{t}\int_{\cup_i B_x^i} |f| \leq \frac{3^n}{t}\|f\|_1$$

where we used the disjointness of the B_x^i to combine the integrals. The above says that

$$|\{Mf > t\}| \leq \frac{3^n}{t}\|f\|_1$$

so that $Mf \in L^1_{\text{weak}}(\mathbb{R}^n)$.

Weak Derivatives and Absolute Continuity.

Problem 1 (Spring 2016). Let $1 < p < \infty$. Assume $f \in L^p(\mathbb{R})$ satisfies

$$\sup_{0<|h|<1} \int \left|\frac{f(x+h) - f(x)}{h}\right|^p dx < \infty.$$

Show that f has a weak derivative $g \in L^p$, which by definition satisfies $\int \psi g = -\int \psi' f$ for every C^∞ function ψ on $\mathbb{R}$ with compact support.

Solution: Let $f_k(x) = \frac{f(x+\frac{1}{k})-f(x)}{\frac{1}{k}}$. What the assumption tells us is that $||f_k||_p$ is uniformly bounded. So, by the theorem of Banach-Alaoglu, $\exists$ a weak-* convergent subsequence, that converges to a limit function $g \in L^p$.

Now, we just need to show that g satisfies the definition of the weak derivative. Let $\psi \in C_c^\infty(\mathbb{R})$. Then:

$$\int \psi g dx = \lim_{k\to\infty} \int \psi(x) f_k(x) dx$$

Now:

$$\int \psi(x)\frac{f(x+\frac{1}{k}) - f(x)}{\frac{1}{k}}dx = \int \frac{\psi(x)f(x+\frac{1}{k})}{\frac{1}{k}}dx - \int \frac{\psi(x)f(x)}{\frac{1}{k}} =$$

$$\int \frac{\psi(x-\frac{1}{k})f(x)}{\frac{1}{k}}dx - \int \frac{\psi(x)f(x)}{\frac{1}{k}} = \int f(x)\frac{\psi(x-\frac{1}{k}) - f(x)}{\frac{1}{k}}dx$$

Now, via the mean value theorem, for every x, $\exists$ a point $c_n(x)$ such that $\frac{\psi(x-\frac{1}{k})-f(x)}{\frac{1}{k}} = \psi'(c_n(x))$. So, we can bound $|f(x)\frac{\psi(x-\frac{1}{k})-f(x)}{\frac{1}{k}}|$ by $|f|||\psi'||_\infty \chi_{supp(\psi')}$, which is integrable on because $f \in L^p(supp(\psi'))$, which has finite measure, so $f \in L^1(supp(\psi'))$. So, using DCT, we can pass to the limit again and see:

$$\int \psi g dx = \lim_{k\to\infty} \int \psi(x) f_k(x) dx = -\int f(x)\psi'(x)dx$$

Problem 2 (Spring 2016, Fall 2021). Assuming $f : [0,1] \to \mathbb{R}$ is absolutely continuous, prove that f is Lipschitz if and only if f' belongs to $L^\infty([0,1])$.

Solution:

($\Longrightarrow$) Let f be Lipschitz. Then, let $x \in [0,1]$. Then:

$$lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \leq \frac{|f(x+h) - f(x)|}{|(x+h) - x|} \leq lim_{h\to 0}C = C$$

where C is is Lipschitz constant for f.

($\Longleftarrow$) Let $f' \in L^\infty([0,1])$. Then, for $x, y \in [0,1]$, $|f(y) - f(x)| = |\int_x^y f'(x)dx| \leq C|x-y|$, where $C = ||f||_\infty$.

Problem 3 (Fall 2015, Spring 2017). Let f be a nondecreasing function on $[0,1]$. You may assume that f is differentiable almost everywhere.

a) Prove that
$$\int_0^1 f'(t)\ dt \leq f(1) - f(0).$$

b) Let $\{f_n\}$ be a sequence of non-decreasing functions on $[0,1]$ such that $F(x) = \sum_{n=1}^\infty f_n(x)$ converges for $x \in [0,1]$. Prove that $F'(x) = \sum_{n=1}^\infty f_n'(x)$ almost-everywhere.

Solution:
(a) Firstly, extend f to $[0,2]$ by just saying $f(x) = f(1)$ for $x \in [1,2]$. Then, by Fatou's lemma and a change of variables:
$$\int_0^1 f'(t)dt \leq \liminf_{h\to\infty} \int_0^1 \frac{f(t+\frac{1}{h}) - f(t)}{(\frac{1}{h})}dt = \liminf_{h\to\infty} h \int_{\frac{1}{h}}^{1+\frac{1}{h}} f(t)dt - h\int_0^1 f(t)dt$$
Now, using the fact that f is non-decreasing:
$$\liminf_{h\to\infty} h \int_{\frac{1}{h}}^{1+\frac{1}{h}} f(t)dt - h\int_0^1 f(t)dt = \liminf_{h\to\infty} \int_1^{1+\frac{1}{h}} f(t)dt - h\int_0^{\frac{1}{h}} f(t)dt \leq \liminf_{n\to\infty} f(1) - f(0) = f(1) - f(0)$$
(b) Now, let f_n be a sequence of non-decreasing functions on $[0,1]$ such that $F(x) = \sum_{n=1}^\infty f_n(x)$. Then, we have:
$$F'(x) = S_n'(x) + h_n'(x)$$
where $S_n(x) = \sum_{k=1}^n f_n(x)$. and $h_n(x) = \sum_{k=n+1}^\infty f_n(x)$. So, to show that $F'(x) = \lim_{n\to\infty} S_n'(x)$ almost everywhere, it STS that $h_n'(x)$ converges to zero almost everywhere.

Firstly, we show that $h_n'(x)$ goes to zero in $L^1([0,1])$. By **(a)**, $\int_0^1 |h_n'(t)|dt = \int_0^1 h_n'(t)dt \leq h_n(1) - h_n(0) \to 0$ as S_n converges at $0,1$. So, convergence in L^1 is established. Now, by passing to a subsequence, we get a subsequence $h_{n_k}'(x)$ that converges to 0 pointwise a.e.. However, as h_n is a monotone decreasing sequence (as all the terms are positive), it follows that the full sequence converges to 0 pointwise a.e..

Problem 4 (Spring 2014). Is the function $f : [0,1] \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases}$$
absolutely continuous on $[0,1]$? Explain fully.

Solution: Recall that absolutely continuous functions are of bounded variation, so it suffices to show that f is not of bounded variation. Recall that f is of bounded variation on $[a,b]$ if
$$V(f) := \sup_{P\in\mathcal{P}} \sum_{i=0}^{n_P - 1} |f(x_{i+1}) - f(x_i)| < \infty$$
where $\mathcal{P}$ is the set of partitions $P = \{x_0, ..., x_{n_P}\}$ of $[a,b]$ (that is, $x_i \leq x_{i+1}$ for all $0 \leq i < n_P$ and the partition is formed by $[x_0, x_1]$, $[x_1, x_2]$,...,$[x_{n_P - 1}, 1]$).

Let $n \geq 0$ be even and choose the partition $P = \{x_0, x_1, ..., x_n, x_{n/2+1}\}$ with
$$x_i = \frac{2}{(n - 2i + 1)\pi}$$
for $i = 1, ..., n/2$ and $x_0 = 0$, $x_{n/2+1} = 1$. XXX

Problem 5 (Spring 2013). Let $f : \mathbb{R} \to \mathbb{R}$ be absolutely continuous with compact support, and let $g \in L^1(\mathbb{R})$. Prove that $f * g$ is absolutely continuous on $\mathbb{R}$.

Solution: XXX

Explicit Computations and Counterexamples.

Problem 1 (Fall 2015). Find a non-empty closed set in $L^2([0,1])$ which does not contain an element of minimal norm.

Solution: An example is the set C that is the union of the sequence:

$$f_n(x) = \frac{(1+\frac{1}{n})}{\sqrt{\frac{1}{n}}}\chi_{[0,\frac{1}{n}]}$$

Firstly, note that straightfoward calculation shows $||f_n||_2 = 1 + \frac{1}{n}$. Further C is closed: indeed, assume that there is a sequence $\{f_k\}$ in C that converges to $g \notin C$ in L^2. Then, by passing to a subsequence if necessary, we can assume that $\{f_k\}$ converges pointwise a.e. to g. However, it is clear that g must equal 0 then, as the original sequence f_n converges pointwise *a.e.* to 0. However, f_k cannot converge to 0 in L^2 as $||f_k|| > 1$ for all k, a contradiction. So, C is closed.

Problem 2 (Fall 2015). Give an example of a sequence $\{f_h\}_{h\in\mathbb{N}} \subset L^1(\mathbb{R})$ such that $f_h \to 0$ a.e. on $\mathbb{R}$ but f_h does not converge to 0 in $L^1_{\text{loc}}(\mathbb{R})$.

Solution: We let $f_h(x) = h\chi_{[0,1/h]}(x)$ so that $f_h(x) \to 0$ a.e. but $\|f_h\|_{L^1(0,1)} = 1$ for all h. If $f_h \to 0$ in $L^1_{\text{loc}}(\mathbb{R})$, then $f_h \to 0$ in $L^1(\Omega)$ for each $\Omega \subset\subset \mathbb{R}$. With $\Omega = (0,1)$, we see that f_h cannot converge to 0 in $L^1_{\text{loc}}(\mathbb{R})$.

Problem 3 (Spring 2015). For any natural number n construct a function $f \in L^1(\mathbb{R}^n)$ such that for any ball $B \subset \mathbb{R}^n$, f is not essentially bounded on B.

Solution: First define $g : \mathbb{R}^n \to (0,\infty)$ by

$$g(x) = \begin{cases} 1/|x|^{n-1/2} & |x| \le 1, \\ 1/|x|^{n+1} & \text{else} \end{cases}.$$

Then,

$$\int_{\mathbb{R}^n} |g(x)|\,dx = \int_0^\infty \left[\int_{S^{n-1}} g(r)r^{n-1}\,dS^{n-1}\right] dr = |S^{n-1}|\int_0^1 r^{n-1}g(r)\,dr + |S^{n-1}|\int_1^\infty r^{n-1}g(r)\,dr$$
$$= |S^{n-1}|\int_0^1 \frac{1}{r^{1/2}}\,dr + |S^{n-1}|\int_1^\infty \frac{1}{r^2}\,dr = 3|S^{n-1}|.$$

So, $g \in L^1(\mathbb{R}^n)$ but is not essentially bounded for any ball B containing the origin. Now let $\{q_k\}_{k=1}^\infty$ be an enumeration of $\mathbb{Q}^n$. Define f by

$$f(x) := \sum_{k=1} 2^{-k}g(x-q_k).$$

Note that

$$\int_{\mathbb{R}^n} |f(x)|\,dx \le \sum_{k=1}^\infty \frac{1}{2^k}\int_{\mathbb{R}^n} |g(x-q_k)|\,dx = \sum_{k=1}^\infty \frac{1}{2^k}\int_{\mathbb{R}^n} |g(x)|\,dx = 3|S^{n-1}|\sum_{k=1}^\infty 2^{-k} = 3|S^{n-1}| < \infty.$$

So, $f \in L^1(\mathbb{R}^n)$ too. Yet, for any ball $B \subset \mathbb{R}^n$ surely there exists a $q_k \in B$. Now, all the $g(x-q_i)$ are non-negative, and in particular $g(x-q_k)$ is not essentially bounded on B. Hence, f is not essentially bounded on B either.

Problem 4 (Spring 2015). Let $g \in L^1(\mathbb{R}^n)$, $\|g\|_{L^1(\mathbb{R}^n)} < 1$. Prove that there is a unique $f \in L^1(\mathbb{R}^n)$ such that

$$f(x) + (f * g)(x) = e^{-|x|^2}, \qquad x \in \mathbb{R}^n \text{ a.e.}$$

Solution: Suppose that such an f exists. Taking the Fourier transform of both sides gives

$$\mathscr{F}[f(x)](t) + (2\pi)^{n/2}\mathscr{F}[f(x)](t)\mathscr{F}[g(x)](t) = \mathscr{F}[e^{-|x|^2}](t).$$

Recall that

$$\mathscr{F}[e^{-|x|^2/2}](t) = e^{-|t|^2/2}, \qquad \mathscr{F}[f(rx)](t) = \frac{1}{r^n}\mathscr{F}[f](t/r).$$

Putting the two together, we see that

$$\mathscr{F}[e^{-|x|^2}](t) = \mathscr{F}[e^{-|\sqrt{2}x|^2/2}](t) = \frac{1}{2^{n/2}}\mathscr{F}[e^{-|x|^2/2}](t/\sqrt{2}) = \frac{1}{2^{n/2}}e^{-|t|^2/4}.$$

Hence,

$$\mathscr{F}[f(x)](t) = \frac{e^{-|t|^2/4}/2^{n/2}}{1 + (2\pi)^{n/2}\mathscr{F}[g(x)](t)} = \frac{e^{-|t|^2/4}}{2^{n/2} + 2^n\pi^{n/2}\mathscr{F}[g(x)](t)}.$$

Thus, if such an f exists it is unique. We can also use this to show existence. Since $\|g\|_{L^1(\mathbb{R}^n)} < 1$ we have

$$|\mathscr{F}[g(x)](t)| \leq \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} |g(x)|\ dx < \frac{1}{(2\pi)^{n/2}}.$$

It follows that

$$|\mathscr{F}[g(x)](t) \leq \frac{1}{(2\pi)^{n/2}} - \epsilon$$

for some $\epsilon > 0$ and thus

$$\frac{1}{2^n\pi^{n/2}\epsilon} \geq \frac{1}{2^{n/2} + 2^n\pi^{n/2}\mathscr{F}[g(x)](t)}.$$

Consequently,

$$|\mathscr{F}[f(x)](t)]| \leq \frac{e^{-|t|^2/4}}{2^n\pi^{n/2}\epsilon}$$

and thus $\mathscr{F}[f(x)](t) \in L^1(\mathbb{R}^n)$. By L^1 inversion we conclude that such an f exists.

Problem 5 (Fall 2013). Provide an example of a sequence of measurable functions on $[0,1]$ which converges in L^1 to the zero function but does not converge pointwise a.e.

Solution: Consider the sequence $\{f_n\}_{n=1}^\infty$ defined by $f_n = \chi_{[(n-2^k)/2^k,(n-2^k+1)/2^k]}$ for $k \geq 0$ and $2^k \leq n < 2^{k+1}$. What this effectively does is produce an interval of size $1/2^k$, starting at $[0, 1/2^k]$, translate it rightward in steps of $1/2^k$ until it gets to $[1 - 1/2^k, 1]$, then increase k by 1 and repeat. Hence for any $x \in [0,1]$ there exist infinitely many n such that $f_n(x) = 0$ and infinitely many n where $f_n(x) = 1$. It follows that f_n does not converge pointwise for any x. However, for every $2^k \leq n < 2^{k+1}$ we obviously have $\|f_n\|_{L^1} = 1/2^k$ which tends to zero. So, $f_n \to 0$ in L^1. This sequence is commonly called the typewriter sequence.

Problem 6 (Fall 2013). Let $(x_1, x_2, ...)$ be an arbitrary sequence of real numbers in $[0,1]$ (possibly dense). Show that the series

$$\sum_k k^{-3/2}|x - x_k|^{-1/2}$$

converges for almost every $x \in [0,1]$.

Solution: XXX

Problem 7 (Fall 2013). Let f be a continuous function on $[0, 1]$. Find

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\ dx.$$

Justify your answer.

Solution: We first make the change of variables $x^n \mapsto x$ to find

$$n \int_0^1 x^n f(x)\ dx = \int_0^1 x^{1/n} f(x^{1/n})\ dx.$$

Define $g_n(x) := x^{1/n} f(x^{1/n})$. We have that $g_n(0) = f(0)$ for all n, but for $0 < x \leq 1$ notice that $x^{1/n} \to 0$ as $n \to \infty$. Hence $g_n(x) \to f(1)$ on $(0, 1]$. Since f is continuous, it is bounded on $[0, 1]$, say by M. Then, note that

$$|g_n(x)| = |x|^{1/n} |f(x^{1/n})| \leq |f(x^{1/n})| \leq M$$

since $x^{1/n}$ maps $[0, 1]$ to $[0, 1]$. But M is integrable over $[0, 1]$, so by dominated convergence

$$\lim_{n\to\infty} n \int_0^1 x^n f(x)\ dx = \lim_{n\to\infty} g_n(x)\ dx = \int_0^1 \lim_{n\to\infty} g_n(x)\ dx = \int_0^1 f(1)\ dx = f(1).$$

Problem 8 (Fall 2012). If $f(x, y) \in L^2(\mathbb{R}^2)$, show that $f(x + x^3, y + y^3) \in L^1(\mathbb{R}^2)$.

Solution: XXX

Problem 9 (Spring 2021). Show that if X is a complete metric space and X is the countable union of closed sets X_j, then at least one X_j has non-empty interior.

Solution: If all X_j had empty interior, this would contradict Baire Category Theorem.

Problem 10 (Fall 2021, Spring 2021). Give an example of a sequence that weakly converges in $L^2(\mathbb{R})$ but admits no pointwise a.e. convergent subsequence.

Solution: The sequence is $f_n = cos(nx)\chi_{[0,\pi]]}$. You can easily check that it converges to zero weakly by approximating by step functions. However, no subsequence converges to zero pointwise a.e.: indeed, assume a subsequence f_{n_k} converged to zero pointwise a.e.. (We know that the pointwise limit of any subsequence, if it exists, must be zero because $||f_n||$ is bounded). Then, DCT with $\chi_{[0,\pi]]}$ would imply that $f_n \to f$ in L^2, a contradiction as $||f_n|| \not\to 0$ (just calculate this).