APPLIED I PRELIM SOLUTIONS

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1. January 2014

Problem 1.

(1) Intuition about why this should be correct: power series for $f(x) = \frac{1}{1-x}$ about $x = 0$. Series is convergent because $||A|| < 1$, so sequence of partial sums is Cauchy, so series converges.
(2) $(A - B)^{-1} = [A(I - A^{-1}B)]^{-1} = (I - A^{-1}B)^{-1}A^{-1}$ is invertible if $||B|| < \frac{1}{||A||}$.
(3) Let $B : C^0 \to L^2$ be injective. Then, define $A_\epsilon : C^0 \to L^2$ as in the problem. Then, BA_ϵ is not injective (where B is defined on the image of A_ϵ in L^2), and the operator norm $||BA_\epsilon - B|| < \epsilon$.

Problem 2.

(1) $f_k \to f \iff$ any subsequence has a further subsequence that converges to f. Use Banach Alaoglu on a subsequence to get weak convergence to some limit g, then the pointwise convergence to f enforces that $g = f$ (localize to balls B_r and use DCT (thanks to L^∞ bound) and weak convergence to show $g = f$ on B_r, and take $r \to \infty$).
(2) Expand $||f_n - f||_2^2$.

Problem 3.

(1) Integrate by parts (boundary term goes away), use Holder, divide like term back across.
(2) f_n converges a.s. to 0, but the L^2 norms are constantly 1, so no subsequence may converge strongly (if it did, then it would also converge in measure, so would have a further subseq. converging pointwise a.e., but then it must converge to 0).

2. August 2014

Problem 1.

(1) Pf. by contradiction.
(2) Let $y_n = T(x_n), y_n \to y$. Then, STS there is x with $T(x) = y$. Using bd. below, $\{x_n\}$ is a Cauchy seq, so there is a limit x. $T(x) = y$ by continuity.
(3) Right shift operator on ℓ^∞.
(4) Bijective by definition, $R(T)$ is a Banach space b/c closed subspace of a Banach space. In fact bounded inverse thm shows that T is an isomorphism, although the problem does not ask for this.

Date: May 2024.

Problem 2.

(1) Use closed graph thm, and extract subsequences that converge a.e. (ignore the hint).
(2) Riesz rep. thm for first part, and then choose $u = 1$ for the second.

Problem 3.

(1) Open mapping theorem
(2) Do the induction. Base case follows from previous, inductive step is straightfoward, also uses previous step.
(3) If $\alpha < \sigma$, one can bound the norm of the difference by a geometric series and see the convergence. Checking that $B(\sum x_n) = f$ is straightfoward as well.

3. August 2015

Problem 1. Firstly, note that this must be an NLS over a complete field to be true. Just take a Cauchy sequence, show that it converges using the equivalence of norms on finite dimensional spaces, then use that complete spaces are closed. For the projection, extend the coefficient operators by Hahn-Banach. In the case of a 1-D subspace, you can use Cor. 2.28 in the book of Arbogast.

Problem 2.

(1) https://math.stackexchange.com/questions/216858/positive-operator-is-bounded
(2) Use closed graph theorem again.

Problem 3.

(1) Holder's inequality
(2) Firstly, note that A maps not only into L^p, but actually $C([a, b])$. So, we can use Arzela-Ascoli to establish convergence of a subsequence in $C([a, b])$, which will grant convergence in L^p. So it suffices to show that the image of a bounded sequence $\{u_n\}$ is bounded and equicontinuous. The bounded part follows from Holder's inequality, while the equicontinuity follows from using Young's inequality, then Holder 2x.

Problem 4.

(1) Firstly, note that A is self-adjoint (Prop 4.20 in Arbogast). Then, follow proof of Prop. 8.4 in Arbogast (you should be able to do this without the book).
(2) This follows from the spectrum of a positive operator being contained in the positive real axis (Prop 4.24 in Arbogast).
(3) Check that the limiting operator is 0 on the range of A (has norm 0, one may check (https://math.stackexchange.com/questions/1007191/norm-of-the-resolvent-of-a-self-adjoint-operator)), and is the identity on the null space.

4. January 2016

Problem 1.

(1) Consider the map $f \mapsto ||f||_\infty - f$.
(2) Use part 1.

Problem 2.

(1) Definition is standard. Use PUB on ϕ_{x_n}.
(2) Weakest topology generated by linear functionals, pull back open intervals.
(3) Use B-A and the fact that T preserves weak convergence.

Problem 3.

(1) ($\implies$) inner product is continuous. ($\impliedby$) double orthogonal complement is closure of original, so closure of original is whole space (on the exam, you probably should prove $M^{\perp\perp} = \overline{M}$).
(2) $1-1$ by simple contradiction argument, to show range is equal let $Tx = y$, let $x = n + n^{\perp}$, then $S(n^{\perp}) = y$.

5. August 2016

Problem 1. Define $f : Y + \mathbb{R}w \to \mathbb{R}$ by $f(y + aw) = ad$. One can check this is linear, and satisfies the desired properties. It is also bounded:

$$\frac{|f(y+aw)|}{||y+aw||}\frac{ad}{a||w-(\frac{-y}{a})||} < \frac{d}{d} = 1$$

Then, use Hahn-Banach to extend.

Problem 2.

(1) A W-weak open set is a union of finite intersections of pullbacks of open ball around zero via functionals in W.
(2) Let $\mathcal{O}$ be the $W-$weak topology. Consider $L^{-1}(B_1(0))$. This is open in the topology $\mathcal{O}$, so it contains a basic open set $U = \cap_{i=1}^{n} w_i^{-1}(B_{\epsilon_i}(0))$. Now, let $w_i(x) = 0$ for all $i = 1, ..., n$. Then, for any $\epsilon > 0$, $x \in w_i^{-1}(B_{\epsilon\epsilon_i}(0)$, so $x \in \epsilon U \implies x \in L^{-1}(B_{\epsilon}(0))$. Taking $\epsilon \to 0$, we obtain that $L(x) = 0$. Apply the hint to get the result.
(3) Part b implies that any weak-* continuous linear functional on X^* is itself an evaluation map.

NB: Nowhere is the fact that W separates points used. In fact, it is not needed.

Problem 3.

(1) Let $\phi_n \to \phi$ in the test function topology. Then, $\phi_n \to \phi$ in $C^k(\Omega)$ for any Ω compact subset of $(-1,1)^2$. In particular, $\phi_n(\cdot, 0) \to \phi(\cdot, 0)$ in $C^k(\Omega)$ for any Ω compact subset of $(-1, 1)$.
(2) $T'(\delta_0) = \delta_{(0,0)}$. $T'(\delta_0') = \partial_x\delta_{(0,0)}$.

6. January 2017

Problem 1.

(1) Let $x = y + y^{\perp}$, where $y \in M_j$, $y^{\perp} \in M_j^{\perp}$. Then, for any $x \neq 0$, calculate:

$$\frac{||P_j(x)||}{||x||} = \frac{||y||}{||y+y^{\perp}||} \leq \frac{||y||}{||y||} \leq 1$$

Also, $\langle P_j x, x\rangle = \langle y, y + y^{\perp}\rangle = ||y||^2 \geq 0$.

(2) $(i) \implies (ii)$: $||P_1x|| = ||P_1P_2x|| \leq ||P_1||||P_2x|| \leq ||P_2x||$.
$(ii) \implies (iii)$: If $x \in M_2$, then we check: $\langle P_1x, x\rangle \leq ||P_2x||||x|| \implies \langle P_2x, x\rangle - \langle P_1x, x\rangle \geq \langle P_2x, x\rangle - ||P_2x||||x|| = 0$ by the equality case of Cauchy Schwarz. If $x \in M_2^{\perp}$, both terms in the previous step are 0. For general x, decompose x into the orthogonal decomp. and apply the previous steps.
$(iii) \implies (iv)$: Let $P_2x = 0$. Then, $\langle P_1x, x\rangle \leq 0$, which implies $P_1x = 0$ or $x = 0$ (because $P_1 \geq 0$).
$(iv) \implies (v)$: $N_1 = M_1^{\perp}, N_2 = M_2^{\perp}$. So, $N_1 \supset N_2 \implies N_1^{\perp} \subset N_2^{\perp} \implies M_1 \subset M_2$.
$(v) \implies (i)$: Let $M_1 \subset M_2$. Then, let $x \in H$. There are decompositions $x = y_1 + y_1^{\perp}$, and $x = y_2 + y_2^{\perp}$, where these are both unique decomps. The inclusion $M_1 \subset M_2$ implies $y_1 = y_2$, which then implies $y_1^{\perp} = y_2^{\perp}$. The properties follow easily from this.

Problem 2.

(1) Formally, $B^* : X^{**} \to Y^*$. So, B^*x for $x \in X$ is really, $B^*\phi_x$, where ϕ_x is the evaluation operator corresponding to x.
(2) Cauchy-Schwarz shows A is bounded below. Further, one can check $||A|| \geq \alpha$, which implies $||A^{-1}|| \leq \frac{1}{\alpha}$ (note A^{-1}) exists and is bounded by bounded inverse theorem, where surjectivity of A is given and injectivity follows from bounded below).
(3) To show the hint, evaluate first eq. at x, second at y, then add (the B terms cancel).

Problem 3.

(1) Self-adjointness follows from Fubini's theorem. Check that it may be applied as the integrand is integrable. The compactness is found using Arzela-Ascoli. Firstly, note that A maps L^2 into $C(I)$. Then, note that it suffices to use A-A, as convergence in $C(I)$ is stronger than in L^2. Then, check that $A - A$ may be applied (use Holder a bunch and sin is uniformly cts. on $[0, 1]$).
(2) Let $f \in L^2$ with $||f||_2 \leq 1$, then just calculate $||Af||_2$. Use Holder and the fact that $||\sin||_\infty = 1$.
(3) By the spectral theorem for compact self-adjoint operators, everything in the spectrum is guaranteed to be an eigenvalue. Further, the infimum of the Rayleigh quotient of a self-adjoint operator is guaranteed to be in the spectrum. So, calculate with $f(x) = \frac{1}{2} - x$ to see that $\langle Af, f\rangle < 0$, so the infimum must be less than 0 (how I came up with this example: I noticed that $\sin(\frac{x+y}{2})$ is always positive and is monotone increasing in both x, y, so I just chose a function f that is equal parts negative and positive, but the negative part is where the kernel is larger.

7. January 2019

Problem 1.

(1) $(\implies)$ Let T be self-adjoint. Then, $\langle x, Tx\rangle = \langle Tx, x\rangle = \overline{\langle x, Tx\rangle} \implies \langle x, Tx\rangle \in \mathbb{R}$.
$(\impliedby)$ The hint gives that T is self-adjoint on the diagonal. Then, it is standard to extend this to the whole space (use the linearity of the inner

product. Hopefully if someone asks me about it I remember how to do it on the spot).
(2) Read the book of Arbogast and memorize the spectral theorem for compact self-adjoint oeprators.
(3) Decompose the operator into its eigenvalue decomp. Then P is the operator corresponding to the positive eigenvalues, and N is the one corresponding to the negatives.

Problem 2.

(1) Use corollary 2.28 in Arbogast (you should know how to prove this also).
(2) Read Arbogast book and memorize B-A thm.
(3) Let $f_n \in D$ such that $Tx_n = f_n$, $f_n \to f$ in Y. Then, f_n is Cauchy. The bd. below gives that x_n is Cauchy, so it converges to some limit x. Then, $Tx = f$. Uniqueness is given by injectivity of T as a consequence of it being bd below.

Problem 3.

(1) ($\Longrightarrow$) Let K exist. Then, $L_y(f) = f(y) = \langle f, K(\cdot, y)\rangle) \leq ||K(\cdot, y)||_H ||f||$. ($\Longleftarrow$) RRT.
(2) Uniqueness is a simple consequence of (ii). For the second, test the equality (ii) with $f(y) = K(\cdot, y)$.
(3) $z(y) = \overline{LK(\cdot, y)} = \overline{\langle K(\cdot, y), f_L\rangle} = \langle f_L, K(\cdot, y)\rangle = f_L(y)$, where $f_L \in H$ represents the action of L by RRT. Thus, $z = f_l \in H$. Finally, $Lz = Lf_L = \langle f_L, f_L\rangle = \langle z, z\rangle = ||z||^2$.

August 2019

Problem 1.

(1) Memorize the spectral theorem.
(2) Apply the spectral theorem, then test with all f test functions, apply the fundamental lemma of COV.
(3) Use Tonelli's theorem

Problem 2.

(1) Do the algebra.
(2) See the book of Arbogast (Theorem 3.7)

Problem 3.

(1) Assume that S is not bounded. Then, there exists a sequence $\{f_n\}$ in X^* such that $||f_n|| = 1$, but $||S(f_n)|| > n$. Then, by B-A, there exists a weakly convergent subsequence $\{f_{n_k}\}$. The hypothesis gives that $S(f_{n_k})$ is weak-$*$ convergent and thus bounded in norm, contradicting $||S(f_n)|| > n$.
(2) $T^*(f_n)(x) = f_n(T(x)) \to f(T(x)) = T^*(f(x))$.
(3) Apply the last part.

8. January 2020

Problem 1. See August 2015

Problem 2.

(1) Firstly, it does not converge to 0 at $x = 0$, so it is only a.e. convergence. Write the Taylor series for e^x, bound it by one term, then take the polynomial limit in n.
(2) Just compute the L^p norm and see that it does not converge to zero.
(3) For $p > 1$, use density of simple functions in L^q. For $p = 1$, make a COV $y = nx$ and use DCT to see that this fucntion squence approximates a dirac delta.

Problem 3. This is a repeat of August 2016 Problem 1.

9. August 2020

Problem 1. PUB+triangle inequality

Problem 2. https://math.stackexchange.com/questions/2275429/if-x-ast-separable-there-exists-x-n-subset-s-e-such-that-x-n-rightha?rq=1

Problem 3. ($\implies$) Let e_n be an ONB for H. Then, we have $Tx = \sum_{n=1}^{\infty} \langle Tx, e_n \rangle e_n$. Define $T^k(x)$ as the cutoff sum of the first k terms.
($\impliedby$) The operators A_n are closed, and the set of compact operators is a closed subspace of the operators (diagonalization argument, see Arbogast thm 4.12).

Problem 4. ($\implies$) Let P be continuous. Then, null space is always closed. For the range, let y_n be in the range of P, and $y_n \to y$. Then, $P(y_n) = y_n$ (because $P^2 = P$), so $P(y) = y$.
($\impliedby$) Let $x_n \to x$, $P(x_n \to y)$. By CGT, sts $Px = y$. Range is closed, so $Px_n \to Pz$. Assume $Px \neq P(z)$. Then, $x - z \notin N$. However, $x_n - Px_n \to x - z$, contradiction the fact that N is closed.

January 2021

Problem 1. Use the closed graph theorem. Let $x_n \to x$ in X, $Ax_n \to y$. STS $Ax = y$. Then, $BAx_n \to By$ by bdness of B, but also $BAx_n \to BAx$ by bdness of BA. B 1-1 implies $Ax = y$.

Problem 2. ($\implies$) PUB
($\impliedby$) triangle inequality

Problem 3. Image of compact operator is separable (https://math.stackexchange.com/questions/654965/why-is-the-image-of-a-compact-operator-separable). For the second part, algebra.

Problem 4. (Don't worry about this one, it is extremely unlikely you need to know the Fourier transform). Just use that the fourier transform is an isomorphism on L^2 and compute it.

August 2021

Problem 1.

(1) use the definition
(2) use the sequence given
(3) Define $y_i = \overline{f(e_i)}$. Prove by contradiction that $(y_i) \in \ell^1$.

Problem 2.

(1) Prove the statement separately for $x \in M, M^\perp$, then use triangle inequality (to prove the statement for $x \in M, M^\perp$, use triangle inequality with $P_M(x)$).
(2) ($\implies$) let P be orthogonal projection onto M. Then, let $y \in N, ||y|| = 1$. Then, $\inf x \in M||y - x|| = ||y - Py|| = ||y|| = 1$.
($\impliedby$) $P_M(y) = 0$ via the hint. So, $y \in M^\perp$.

Problem 3.

(1) state PUB
(2) triangle inequality+PUB
(3) $L_n = x_n = e_n$, $L_n(e_n) = 1$ always.

January 2022

Problem 1.

(1) easy
(2) Dual separates points (use Hahn banach to show that for any x, there is a $\phi \in X^*$ with $\phi(x) = ||x||, ||\phi|| = 1$.
(3) use PUB on the evaluation maps $\phi_{x_n} \in X^{**}$.
(4) Use the same corollary as in part (b): $||x|| = \phi(x) = \liminf_n \phi(x_n) \leq \liminf_n ||x_n||$.

Problem 2.

(1) Know BCT
(2) Once one shows that X_n has empty interior, then $X = \cup X_n$, contradiction to BCT. Given $x \in X_n$, just take $y = x + \epsilon e_{n+1}$.

9.1. **Problem 3.**

(1) Standard argument
(2) $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$.
(3) See thm 4.22 in Arbogast

August 2022

Problem 1.

(1) give the definition
(2) easy
(3) give the thm
(4) use PUB

Problem 2.

(1) Show the hint (for the second inequality, test with zero vector). Then, if $||y|| > 3||x_0||$, reverse triangle inequality shows that $||x_0 - y|| \geq 2||x_0||$.
(2) $\mathcal{B}$ is compact b/c heine-borel. Use compactness to get convergence along a subsequence
(3) use the example they give.

Problem 3.

(1) easy
(2) give the spectral theorem for compact operators
(3) $S + T = S[I + S^{-1}T]$, where $S^{-1}T$ is compact by the first part. Now, the injectivity of $S + T$ gives that $[I + S^{-1}T]$ is injective, so by the spectral theorem, it must be invertible (because only things in spectrum are eigenvalues).

10. January 2023

Problem 1.

(1) $T^*(g)(x) = g(T(x))$. Check that $T^*(g) \in X^*$ via the definition.
(2) Use the definition,
(3) test with the functional given by H-B.

Problem 2.

(1) inner product is continuous
(2) easy

Problem 3.

(1) Convergence in C^m for any m.
(2) Theorem 5.6 arbogast

11. August 2023

Problem 1. For the first part, assume that f is not linear, then there exists x, y with $f(x) + f(y) = f(x + y) + z$, $z \neq 0$. Then, use ϕ such that $\phi(z) = ||z||$ to gain a contradiction. For the second part, use PUB (note that one applies the PUB to the family of operators $\phi_{T(x)}$, for $||x|| \leq 1$).

Problem 2. See August 2020.

Problem 3. Let $x_n \rightharpoonup x$ Then, $f(x_n) \rightharpoonup f(x)$. Then, consider $f(x_n)$. To show $f(x_n) \to f(x)$, it STS that every subsequence $f(x_{n_k})$ has a further subsquence that converges to $f(x)$. So, let $f(x_{n_k})$ be a subsequence. Then, by compactness (as x_n bounded), it has a strongly convergent subsequence. As the sequence weakly converges to $f(x)$, the strong limit must be $f(x)$ also.

Problem 4. https://math.stackexchange.com/questions/2390715/sum-of-unitary-operators-converges-to-projection-operator

Problem 5. Change variables, use dominated convergence and things, use the fact that sin is odd.