SMC 2023: HILBERT'S 19^{TH} PROBLEM – Day I

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1 Introduction

This summer course will be a presentation of a solution to Hilbert's 19^{th} problem. Essentially, it is a course on elliptic regularity theory. For the required background, I assume you are comfortable with say, the content of the applied math II prelim course. Everything in this notes is a mixture of content from the book "Regularity Theory for Elliptic PDE" by Fernandez-Real/Ros-Oton, or the notes of Vasseur: THE DE GIORGI METHOD FOR ELLIPTIC AND PARABOLIC EQUATIONS AND SOME APPLICATIONS.

In 1900, David Hilbert posed a list of 23 major unsolved problems in mathematics. We will focus on problem 19, which was solved circa 1957 by Ennio de Giorgi & John Nash, using fairly different methods. In this course, we will focus on the method of de Giorgi. Problem 19 is as follows:

PROBLEM: Consider any local minimizer u of an energy functional of the form:

$$\mathscr{E}(w) = \int_\Omega L(\nabla w) dx$$

where:

1. $\mathscr{E} : H^1(\Omega) \to \mathbb{R}$
2. $L : \mathbb{R}^n \to \mathbb{R}$ is smooth & uniformly convex (Matrix of second order partial derivatives is uniformly positive-definite & bounded)
3. $\Omega \subset \mathbb{R}^n$ is open, bounded

Then, is it true that $u \in C^\infty(\Omega)$?

A couple notes:

1. Energy functions of this form are generalizations of the Dirichlet energy $D(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 dx$. In this case, local minimizers u verify $\Delta u = 0$ weakly. Any weak solution can be shown to actually be a strong solution $u \in C^2(\Omega)$, using a combination of difference quotients & Sobolev embedding. Then, harmonic functions are known to be smooth (in even dimensions, can be seen as the real part of a holomorphic function. In general, granted by the mean-value property). This suggests that Hilbert's problem might be a reasonable conjecture.
2. If L is not convex, then we can cook up counterexamples even in dimension 1. Say, for example, L reaches a minimum at two points p_1 and p_2. Then, any function with constant slope either p_1 or p_2 is a minimizer, but it is merely Lipschitz (for example, take $p_1 = -\frac{\sqrt{2}}{2}, p_2 = \frac{\sqrt{2}}{2}, L(x) = x^4 - x^2, u(x) = \frac{\sqrt{2}}{2}|x|$, which is only Lipschitz if $0 \in \Omega$).

2 Outline of Proof

Today, we will give a brief outline of the proof (black-boxing many things), then spend the next couple weeks filling in the holes. Firstly, we need to reduce the problem to an elliptic regularity problem. Let's firstly specify some things we said above:

Def. *$u \in H^1(\Omega)$ is a local minimizer of $\mathscr{E}$ if $\mathscr{E}(u) \leq \mathscr{E}(u+\phi)\ \forall\ \phi \in C_0^\infty(\Omega)$.*

Def. *We say $L : \mathbb{R}^n \to \mathbb{R}$ is uniformly convex if $\exists\ \lambda, \Lambda > 0$ such that:*

$$0 < \lambda Id \leq D^2 L(p) \leq \Lambda Id\ \forall\ p \in \mathbb{R}^n$$

(for matrices, $A \geq B$ if A-B is PSD)

Note that the uniform convexity of L translates into a uniform ellipticity condition on D^2L.

Now, let u minimize $\mathscr{E}$. Then, for any $\eta > 0, \phi \in C_0^\infty(\Omega)$:

$$\int_\Omega L(\nabla u + \eta \nabla \phi)dx \geq \int_\Omega L(\nabla u)dx$$

Taylor expanding L around ∇u, and using the uniform boundedness:

$$\eta \int_\Omega DL(\nabla u) \cdot \nabla \phi + \int_\Omega o(||\eta \nabla \phi||^2)dx \geq -\frac{\eta^2 \Lambda}{2} \int_\Omega |\nabla \phi|^2 dx$$

Since $\eta > 0$, dividing by η doesn't change the sign of anything, so we can do so. Then, take the limit $\eta \to 0$ to obtain:

$$\int_\Omega DL(\nabla u) \cdot \nabla \phi \geq 0\ \forall\ \phi \in C_0^\infty$$

Doing the same procedure with $-\phi$ yields:

$$\int_\Omega DL(\nabla u) \cdot \nabla \phi = 0\ \forall\ \phi \in C_0^\infty$$

This is exactly the weak formulation of $\text{div}(DL(\nabla u)) = 0$.

Upshot: Any local minimizer is a weak solution to $\text{div}(DL(\nabla u)) = 0$.

Right now, we only know $u \in H^1$. Assume we know that $u \in H^2$.

■ BLACKBOX NUMBER 1

Theorem 1. *Let $u \in H^1(\Omega)$ solve $div(DL(\nabla u)) = 0$ weakly. Then, $u \in H^2(\Omega)$.*

With this, we can expand the equation and see that we have a weak solution to:

$$\sum_{i,j=1} (\partial_{ij} L)(\nabla u) \partial_{ij} u = 0$$

Forgetting about the dependence of the coefficients on u, we can view this as a linear second order equation:

$$\sum_{i,j=1} a_{ij}(x)\partial_{ij}u = 0$$

where $a_{ij}(x) = (\partial_{ij}L)(\nabla u)$. Note in particular that a_{ij} is as regular as ∇u is. So, if $\nabla u \in C^{0,\alpha}$, then $a_{ij} \in C^{0,\alpha}$. So, we need to show that $\nabla u \in C^{0,\alpha}$. This particular step was the breakthrough made by de Giorgi & Nash in the 50's.

Going back to the above, we have a weak solution to:

$$\text{div}(DL(\nabla u)) = 0$$

Taking the derivative w.r.t. x_i, we get:

$$\text{div}(D^2L(\nabla u)\nabla w) = 0$$

where $w = \partial_i u$. This is an equation for which we want some regularity on w, but we have no information on $D^2L(\nabla u)$ besides the uniform ellipticity. The regularity is given by the theorem of de Giorgi-Nash-Moser.

■ BLACKBOX NUMBER 2

Theorem 2. *Let $v \in H^1(\Omega)$ be a weak solution to $div(A(x)\nabla v)$, where A is uniformly elliptic. Then, $\exists\ \alpha > 0$ such that $v \in C^{0,\alpha}(\Omega)$.*

Going back to our expansion, we now see that we have a solution u to:

$$\sum_{i,j=1} a_{ij}(x)\partial_{ij}u = 0$$

where $a_{ij} \in C^{0,\alpha}$. From here, we bootstrap using the Schauder estimates (in non-divergence form), attributed to Schauder circa 1935:

■ BLACKBOX NUMBER 3

Theorem 3. *Let $u \in C^{k,\alpha}$ solve:*

$$\sum_{i,j=1} a_{ij}(x)\partial_{ij}u = 0$$

in Ω. Then, $u \in C^{k+2,\alpha}(\Omega)$.

Applying this with $k = 0$, we see that $u \in C^{2,\alpha}$. However, then this gives that the coefficients a_{ij} are actually in $C^{1,\alpha}$. So, we can apply the Schauder estimates with $k = 1$ to see that $u \in C^{3,\alpha}$. Continuing, we see $u \in C^\infty$.

$$a_{ij} \in C^{0,\alpha} \implies u \in C^{2,\alpha} \implies a_{ij} \in C^{1,\alpha} \implies u \in C^{3,\alpha} \implies \ldots \implies u \in C^\infty$$

So, we have a complete solution to the problem, modulo these three very non-trivial developments.