SMC 2023: HILBERT'S 19^{TH} PROBLEM – Day II

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1 Schauder Estimates for the Laplacian – I

We will start by talking about the Schauder estimates. Although our final goal will be Schauder estimates for elliptic operators in non-divergence form, we will start by talking about Schauder estimates for the Laplacian. The goal is to show the following *a priori* Schauder estimate:

Theorem 1. *Let $\alpha \in (0, 1)$, $u \in C^{2,\alpha}(B_1)$ satisfy:*

$$\Delta u = f \in B_1$$

where $f \in C^{0,\alpha}(B_1)$. Then:

$$||u||_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C(\alpha, n)(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)})$$

A couple remarks:

1. There is a simple argument to go from the ball to any compactly contained subset of an open, bounded region Ω (I probably won't talk about this, but there is something like this in the exercises).

2. One also needs to make a mollification argument to go from the a priori estimate to an actual theorem for weak solutions (we will talk about this in a couple days).

Firstly, we need to know a bunch of stuff about harmonic functions.

2 A Crash Course on harmonic functions

2.1 Maximum Principle

Firstly, we want a maximum principle:

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be bounded, open. Let u satisfy, weakly:*

$$\begin{cases} -\Delta u \geq 0 & \text{in } \Omega \\ u \geq 0 & \text{on } \partial\Omega \end{cases}$$

then, $u \geq 0$ in Ω.

Proof. By the definition of $\Delta u \geq 0$ in Ω:

$$\int_\Omega \nabla u \cdot \nabla v dx \geq 0 \ \forall v \geq 0, v \in H_0^1(\Omega)$$

Define:

$$\begin{cases} u^-(x) = \max\{-u(x), 0\} \\ u^+(x) = \max\{u(x), 0\} \end{cases}$$

Note that $u = u^+ - u^-$. Now, taking $v = u^- \geq 0$ as an admissible test function:

$$0 \leq \int_\Omega \nabla u \cdot \nabla u^- dx = \int_\Omega \nabla u^+ \cdot \nabla u^- dx - \int_\Omega |\nabla u^-|^2 dx = -\int_\Omega |\nabla u^-|^2 dx$$

As $u^-|_{\partial\Omega} \equiv 0$, this implies $u^- \equiv 0$ in Ω, i.e. $u \geq 0$ in Ω (you can mollify to see that the gradient being equal to 0 a.e. implies that the function is constant). $\square$

As a consequence of the maximum principle, we have the following quantitative bound:

Theorem 3. *Let u be a weak solution of:*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

Then, $||u||_{L^\infty(\Omega)} \leq C(diam(\Omega))(||f||_{L^\infty(\Omega)} + ||g||_{L^\infty(\Omega)})$

Proof. Define $\tilde{u}(x) := \frac{u(x)}{||f||_{L^\infty(\Omega)} + ||g||_{L^\infty(\Omega)}}$. It STS that $|\tilde{u}| \leq C$ in Ω. Note that $\tilde{u}$ solves:

$$\begin{cases} -\Delta u = \tilde{f} & \text{in } \Omega \\ u = \tilde{g} & \text{on } \partial\Omega \end{cases}$$

with $|\tilde{f}|, |\tilde{g}| \leq 1$. Now, choose R large such that $\Omega \subset B_R$. In B_R, consider the function:

$$w(x) = \frac{R^2 - x_1^2}{2} + 1$$

Then, w satisfies:

$$\begin{cases} -\Delta w = -1 & \text{in } \Omega \\ w \geq 1 & \text{on } \partial\Omega \end{cases}$$

So, via the maximum principle, $\tilde{u} \leq w$ in Ω. Now, w is bounded, say $w \leq C(R) \implies \tilde{u} \leq C(R)$ in Ω. Repeat the argument with $-\tilde{u}$ to finish. $\square$

2.2 Poisson Kernel & Mean-value property

The weak solution to the Dirichlet problem in a ball is explicitly given by the Poisson Kernel:

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \\ u = g & \text{on } \partial B_1 \end{cases} \implies u(x) = c_n \int_{\partial B_1} \frac{1 - |x|^2 g(\sigma)}{|x - \sigma|^n} d\sigma$$

This is easily checked to be rescaleable (just look at $\tilde{u}(x) = u(rx)$):

$$\begin{cases} \Delta u = 0 & \text{in } B_r \\ u = g & \text{on } \partial B_r \end{cases} \implies u(x) = \frac{c_n}{r} \int_{\partial B_1} \frac{r^2 - |x|^2 g(\sigma)}{|x - \sigma|^n} d\sigma$$

Taking $x = 0$, this grants the mean-value property for harmonic functions. In other words, if $\Delta u = 0$ weakly in Ω, and $B_r \subset \Omega$, we have:

$$u(0) = \frac{c_n}{r} \int_{\partial B_r} \frac{r^2 u(y)}{|y|^n} dy = \fint_{\partial B_r} u(y) dy = \fint_{B_r} u(x) dx$$

Of course, this can be translated around in space as well.

Now, we already know that weak solutions to Laplace's equation are smooth (although one can prove it using this representation by differentiation under the integral sign), but an important corollary of this representation is quantitative bounds on the derivatives:

Theorem 4. *Let u satisfy $\Delta u = 0$ weakly in $H^1(B_1)$. Then:*

$$||u||_{C^k(B_{\frac{1}{2}})} \leq C(k,n)||u||_{L^\infty(B_1)}$$

2.3 Estimates for Harmonic functions

Now, we condense all of this into three estimates we will use for the proof of the Schauder estimates (the proofs of all of these are given as exercises):

Theorem 5. *Let $\Delta w = 0$ in B_r. Then*

$$||D^k w||_{L^\infty(B_{\frac{r}{2}})} \leq C(n,k)r^{-k}||w||_{L^\infty(B_r)}$$

Theorem 6. *Let $\Delta w = \lambda$ in B_r. Then*

$$||D^2 w||_{L^\infty(B_{\frac{r}{2}})} \leq C(n)r^{-2}(||w||_{L^\infty(B_r)} + |\lambda|)$$

Theorem 7. *Let u be a weak solution of:*

$$\begin{cases} \Delta u = f & \text{in } B_1 \\ u = g & \text{on } \partial B_1 \end{cases}$$

Then, for any $k \in \mathbb{N}$:

$$||u||_{L^\infty(B_{2^{-k}})} \leq C(r^{-2}||f||_{L^\infty(B_{2^{-k}})} + ||g||_{L^\infty(\partial B_{2^{-k}})})$$

3 Schauder Estimates for the Laplacian – II

Finally, we can show the proof of the a priori Schauder estimates for the Laplacian.

Idea: The power of the approach is enabled by the Holder regularity of f. Zooming in closer and closer around a fixed point, $\Delta u = f$ looks closer and closer to $\Delta u = c$ (and we can give a quantitative statement for this). So, after subtracting a paraboloid, we get closer and closer to a harmonic function as we zoom in (as in the proof of 6). So, we can employ our estimates for harmonic functions (really, we are using this idea whenever we use 6).

Let's state the theorem again:

Theorem 8. *Let $\alpha \in (0,1)$, $u \in C^{2,\alpha}(B_1)$ satisfy:*

$$\Delta u = f \in B_1$$

where $f \in C^{0,\alpha}(B_1)$. Then:

$$||u||_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C(\alpha,n)(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)})$$

Proof. It STS the following:

$$|D^2u(z) - D^2(u(y)| \leq C|z-y|^\alpha(||u||_{L^\infty(B_1)} + [f]_{C^{0,\alpha}(B_1)} \ \forall\ y,z \in B_{\frac{1}{32}})$$

because the L^∞ norms of the derivatives of u are already covered by 5. A couple of notes:

1. We restrict to showing bound on the Holder constant in the ball of radius $\frac{1}{16}$. Using the covering argument, this can be expanded back out to the ball of radius $\frac{1}{2}$.

2. WLOG, we can assume $y = 0$, so that $z \in B(\frac{1}{32})$.

3. After dividing by $||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)}$, we can assume $||u||_{L^\infty} \leq 1, [f]_{C^{0,\alpha}(B_1)} \leq 1$, and then we need to show:
$$|D^2u(z) - D^2u(y)| \leq C|z|^\alpha \ \forall \ z \in B_{\frac{1}{16}}$$

Now, $\forall \ k \in \mathbb{N}$, let u_k be the solution to:

$$\begin{cases} \Delta u_k = f(0) & \text{in } B_{2^{-k}} \\ u_k = u & \text{on } \partial B_{2^{-k}} \end{cases}$$

<u>**Idea:**</u> Zooming in close to 0 (i.e. only looking in the ball $B_{2^{-k}}$), u_k is a very good approximation of u for $k \gg 1$ (because f is Holder continuous). Further, $u_k - u_{k+1}$ is harmonic in the ball $B_{2^{-(k+1)}}$.

Now, we have:

$$|D^2u(z) - D^2u(0)| \leq \underbrace{|D^2u(z) - D^2u_k(z)|}_{I_2} + \underbrace{|D^2u_k(z) - D^2u_k(0)|}_{I_3} + \underbrace{|D^2u_k(0) - D^2u(0)|}_{I_1}$$

It STS $I_j \leq C|z|^\alpha$ for $j = 1, 2, 3$, $z \in B_{\frac{1}{16}}$. $\square$