SMC 2023: HILBERT'S 19$^{\text{TH}}$ PROBLEM – Day III

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1 Estimates for Harmonic functions (again)

Theorem 1. *Let $\Delta w = 0$ in B_r. Then*

$$||D^k w||_{L^\infty(B_{\frac{r}{2}})} \leq C(n,k)r^{-k}||w||_{L^\infty(B_r)}$$

Theorem 2. *Let $\Delta w = \lambda$ in B_r. Then*

$$||D^2 w||_{L^\infty(B_{\frac{r}{2}})} \leq C(n)r^{-2}(||w||_{L^\infty(B_r)} + |\lambda|)$$

Theorem 3. *Let u be a weak solution of:*

$$\begin{cases} \Delta u = f & \text{in } B_1 \\ u = g & \text{on } \partial B_1 \end{cases}$$

Then, for any $k \in \mathbb{N}$:

$$||u||_{L^\infty(B_{2^{-k}})} \leq C(r^{-2}||f||_{L^\infty(B_{2^{-k}})} + ||g||_{L^\infty(\partial B_{2^{-k}})})$$

2 Schauder Estimates for the Laplacian – II (cont.)

Proof. Recall that we have:

$$|D^2u(z) - D^2u(0)| \leq \underbrace{|D^2u(z) - D^2u_k(z)|}_{I_2} + \underbrace{|D^2u_k(z) - D^2u_k(0)|}_{I_3} + \underbrace{|D^2u_k(0) - D^2u(0)|}_{I_1}$$

It STS $I_j \leq C|z|^\alpha$ for $j = 1,2,3$, $z \in B_{\frac{1}{16}}$.

<u>I_1</u>**:** Note that $u_k - u$ solves:

$$\begin{cases} \Delta(u_k - u) = f(0) - f & \text{in } B_{2^{-k}} \\ u_k - u = 0 & \text{on } \partial B_{2^{-k}} \end{cases}$$

Now, using 3 and the fact that $[f]_{C^{0,\alpha}(B_1)} \leq 1$:

$$||u_k - u||_{L^\infty(B_{2^{-k}})} \leq C(2^{-k})^2||f(0) - f||_{L^\infty(B_{2^{-k}})} \leq C2^{-k(2+\alpha)}$$

<u>**Note:**</u> This is a quantitative justification to my previous assertion that "u_k is a good approximation to u zooming in close to 0".
So, via the triangle inequality:

$$||u_{k+1} - u_k||_{L^\infty(B_{2^{-k-1}})} \leq C2^{-k(2+\alpha)}$$

Further, $u_{k+1} - u_k$ is harmonic in $B_{2^{-k-1}}$, so by using 1:

$$||D^2(u_{k+1} - u_k)||_{L^\infty(B_{2^{-k-2}})} \leq C(2^{(k+1)})^2 ||u_{k+1} - u_k||_{L^\infty(B_{2^{-k-1}})} \leq C2^{2(k+1)}2^{-2k-k\alpha} = C2^{-k\alpha}$$

Next, we need to know that $D^2u(0) = \lim_{k\to\infty} D^2u_k(0)$. Indeed, let $\tilde{u}(x) := u(0) + x \cdot \nabla u(0) + \frac{1}{2}x^T D^2u(0)x$ be the second order Taylor expansion of u at 0. A couple notes:

1. $||\tilde{u} - u||_{L^\infty(B_r)} = o(r^2)$ as $r \to 0$ (Taylor's theorem)
2. $D^2\tilde{u}(0) = D^2u(0)$

So:

$$|D^2u_k(0) - D^2u(0)| \leq ||D^2(u_k - \tilde{u})||_{L^\infty(B_{2^{-k-1}})} \underbrace{\leq}_{1} C2^{2k}||u_k - \tilde{u}||_{L^\infty(B_{2^{-k}})} \leq$$

$$C2^{2k}||u_k - u||_{L^\infty(B_{2^{-k}})} + C2^{2k}||u - \tilde{u}||_{L^\infty(B_{2^{-k}})} \to 0 \text{ as } k \to \infty$$

(the first term goes to zero using the bound $||u_k - u||_{L^\infty(B_{2^{-k}})} \leq C2^{-k(2+\alpha)}$ and the second one goes to zero by Taylor's theorem).

Finally, putting it all together: choose $k \in \mathbb{N}$ such that $2^{k-4} \leq |z| \leq 2^{-k-3}$ (we don't need this for I_3, but it will become needed in I_2). Then:

$$|D^2u_k(0) - D^2u(0)| = \lim_{j\to\infty} |D^2u_k(0) - D^2u_k(0)| \leq \lim_{j\to\infty} \sum_{n=k}^{j} |D^2u_n(0) - D^2u_{n+1}(0)| =$$

$$\sum_{n=k}^{\infty} |D^2u_n(0) - D^2u_{n+1}(0)| \leq C\sum_{n=k}^{\infty} 2^{-n\alpha} = C\sum_{n=k}^{\infty}(2^{-\alpha})^n = C2^{-k\alpha}$$

$$\leq C2^{-k\alpha}2^{-4\alpha} \leq C|z|^\alpha$$

$\underline{I_2}$: We can employ the same strategy as before, but this time zooming in around z instead of 0. Let v_k solve:

$$\begin{cases} \Delta v_k = f(z) & \text{in } B_{2^{-k}(z)} \\ v_k = u & \text{on } \partial B_{2^{-k}(z)} \end{cases}$$

Then:

$$|D^2u_k(z) - D^2u(z)| \leq |D^2u_k(z) - D^2v_k(z)| + |D^2v_k(z) - D^2u(z)|$$

The second term is seen to be $\leq C2^{-k\alpha}$ using the same proof as I_1 (translated), so we just need to deal with the first term. Choosing k sufficiently large ($|z| \leq 2^{-k-3}$) such that $B_{2^{-k-1}(z)} \subset B_{2^{-k}} \cap B_{2^{-k}(z)}$. Then, in $B_{2^{-k-2}}$, we have:

$$|D^2u_k(z) - D^2v_k(z)| \leq ||D^2(u_k - v_k)||_{L^\infty(B_{2^{-k-2}}(z))} \underbrace{\leq}_{2} C2^{2k}||u_k - v_k||_{L^\infty(B_{2^{-k-1}}(z))} + C|f(z) - f(0)| \leq$$

$$C2^{2k}||u_k - v_k||_{L^\infty(B_{2^{-k-1}}(z))} + C2^{-k\alpha}$$

(the last term is because $|z| \leq 2^{-k-3}$ and $[f]_{C^{0,\alpha}(B_1)} \leq 1$). Finally, k has been chosen sufficiently large such that $B_{2^{-k-1}}(z))$ is a region for which both $v_k, u_k \sim u$. So, we derive the final quantitative estimate using 3 as before:

$$||u_k - u||_{L^\infty(B_{2^{-k-1}}(z))} \leq ||u_k - u||_{L^\infty(B_{2^{-k}}} \leq C2^{-k(2+\alpha)}$$

$$||v_k - u||_{L^\infty(B_{2^{-k-1}}(z))} \le ||v_k - u||_{L^\infty(B_{2^{-k}}(z))} \le C2^{-k(2+\alpha)}$$

Putting this all together, we see $|D^2u_k(z) - D^2v_k(z)| \le C2^{-k\alpha}$.

$\underline{I_3}$**:** This result is significantly different in spirit than the first two. Denote $h_j := u_j - u_{j-1}$ for $j = 1, ..., k$. Note that h_j are harmonic in $B_{2^{-j}}$. So:

$$\left|\frac{D^2h_j(z) - D^2h_j(0)}{|z|}\right| \underbrace{\le}_{\text{MVT}} ||D^3h_j||_{L^\infty(B_{2^{-k-3}})} \le ||D^3h_j||_{L^\infty(B_{2^{-k-1}})} \underbrace{\le}_{1}$$
$$C2^{3j}||h_j||_{L^\infty(B_{2^{-j}})} \underbrace{\le}_{\text{Max. Princ.}} C2^{3j}||h_j||_{L^\infty(\partial B_{2^{-j}})} = C2^{3j}||u - u_{j-1}||_{L^\infty(\partial B_{2^{-j}})} \le$$
$$C2^{3j}||u - u_{j-1}||_{L^\infty(B_{2^{-j}})} \le C2^{3j}||u - u_{j-1}||_{L^\infty(B_{2^{-j+1}})} \underbrace{\le}_{3} C2^{j(1-\alpha)}$$

So:

$$|D^2u_k(0) - D^2u_k(z) \le |D^2u_0(z) - D^2u_0(0)| + \sum_{j=1}^{k} |D^2h_j(z) - D^2h_j(0)| \le |D^2u_0(z) - D^2u_0(0)| + C|z|\sum_{j=1}^{k} 2^{j(1-\alpha)}$$

Unfortunately, we need to deal with the $j = 0$ term separately. Define $w = u_0 - \frac{f(0)}{2n}|x|^2 + \frac{f(0)}{2n}$. Then, we have the following properties:

1. w is harmonic in B_1
2. $D^3w = D^3u_0$
3. $w = u_0$ on ∂B_1

So:

$$\left|\frac{D^2u_0(z) - D^2u_0(0)}{z}\right| \underbrace{\le}_{\text{MVT}} ||D^3u_0||_{L^\infty(B_{\frac{1}{2}})} = ||D^3w||_{L^\infty(B_{\frac{1}{2}})} \underbrace{\le}_{1} C||w||_{L^\infty(B_1)} \underbrace{\le}_{\text{Max. Princ.}}$$
$$C||w||_{L^\infty(\partial B_1)} = C||u_0||_{L^\infty(\partial B_1)}$$

So:

$$|D^2u_k(0) - D^2u_k(z) \le C|z|||u_0||_{L^\infty(B_1)} + C|z|\sum_{j=1}^{k} 2^{j(1-\alpha)}$$

Now, the first term here is $\le C|z|$, because $||u_0 - u||_{L^\infty(B_1)} \underbrace{\le}_{3} C \implies ||u_0||_{L^\infty(B_1)} \le 1 + C = C$. The second term is bounded by $C|z|2^{k(1-\alpha)}$, because $\sum_{j=1}^{k} 2^{j(1-\alpha)} = 2^{k(1-\alpha)}\sum_{j=1}^{k}(2^{\alpha-1})^j \le 2^{k(1-\alpha)}\sum_{j=1}^{\infty}(2^{\alpha-1})^j = C2^{k(1-\alpha)}$. So, we have:

$$|D^2u_k(0) - D^2u_k(z) \le C|z| + C|z|2^{k(1-\alpha)} \le C2^{-k-3} + C2^{-k\alpha} \le$$
$$C(2^{-k-3})^\alpha + C2^{-k\alpha} \le C2^{-k\alpha} \le C|z|^\alpha$$

□

Note: You can get higher order Schauder estimates by induction (in general, if $f \in C^{k,\alpha}$, then $u \in C^{k+2,\alpha}$.