SMC 2023: HILBERT'S 19^{TH} PROBLEM – Day IV

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1 Schauder Estimates for Elliptic Operators in Non-Divergence Form – I

Now, we move to the result we actually want to prove. Consider the following elliptic PDE in non-divergence form:

$$\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u(x) = f(x) \text{ in } B_1$$

where $A(x) = [a_{ij}(x)]$ is uniformly elliptic ($0 < \lambda Id \leq A(x) \leq \Lambda Id \ \forall \ x \in B_1$). Then, we have an analogous a priori Schauder estimate (note that now, we require some regularity on the coefficients as well).

Theorem 1. *Let $u \in C^{2,\alpha}$ solve:*

$$\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u(x) = f(x) \text{ in } B_1$$

for A uniformly elliptic, $a_{ij}, f \in C^{0,\alpha}$. Then:

$$||u||_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C(||a_{ij}||_{C^{0,\alpha}}, \alpha, n, \lambda, \Lambda)(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)})$$

Note: WLOG, we can always assume that $A(x)$ is symmetric (let $a_{ij} = a_{ji} = \frac{a_{ij}+a_{ji}}{2}$).

Of course, we are going to need analogs of some of the results for harmonic functions from before.

2 Maximum Principle for Elliptic Operators in Non-Divergence Form

We firstly need a maximum principle:

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be bounded, open. Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfy:*

$$\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}u(x) = f(x) \text{ in } \Omega$$

for A uniformly elliptic. Then:

$$\sup_{\Omega} u = \sup_{\partial\Omega} u$$

The proof is not that interesting and I don't want to do it. You can find it in the book of Evans or F-R/R-O.

Of course, we get a quantitative L^∞ bound from the maximum principle in the same way we did for the Laplacian:

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be bounded, open. Let $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfy:*

$$\begin{cases} \sum_{i,j=1}^n a_{ij}(x)\partial_{ij}u(x) = f(x) \text{ in } \Omega\Omega \\ u = \tilde{g} \qquad\qquad\qquad\qquad \text{on } \partial\Omega \end{cases}$$

for A uniformly elliptic. Then:

$$||u||_{L^\infty(\Omega)} \leq C(||f||_{L^\infty(\Omega)} + ||g||_{L^\infty(\partial\Omega)})$$

The proof is exactly as the same as it was before, using the new maximum principle.

3 Schauder Estimates for Elliptic Operators in Non-Divergence Form – II

Now, we can show the proof of the Schauder estimate. We need an interpolation inequality here that I will not prove:

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Theorem 4. *Let $u \in C^{2,\alpha}(B_1)$. Then, $\forall\ \epsilon > 0$, $\exists\ C_\epsilon$ such that:*

$$||u||_{C^2(\overline{B_1})} \leq \epsilon[D^2u]_{C^{0,\alpha}(B_1)} + C_\epsilon ||u||_{L^\infty(B_1)}$$

Proof. F-R/R-O refer the reader to Gilbarg-Trudinger for the proof. It's really ugly. □

Here is the Schauder estimate again:

Theorem 5. *Let $u \in C^{2,\alpha}$ solve:*

$$\sum_{i,j=1}^n a_{ij}(x)\partial_{ij}u(x) = f(x) \text{ in } B_1$$

for A uniformly elliptic, $a_{ij}, f \in C^{0,\alpha}$. Then:

$$||u||_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C(||a_{ij}||_{C^{0,\alpha}}, \alpha, n, \lambda, \Lambda)(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)})$$

Proof. We make the same reductions as before: $||f||_{C^{0,\alpha}(B_1)} \leq 1, ||u||_{L^\infty(B_1)} \leq 1$, $\underline{||D^2u||_{L^\infty(B_1)} \leq 1}$, and it STS that:

$$[D^2u]_{C^{0,\alpha}(B_{\frac{1}{2}})} \leq C(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)})$$

(we can use the interpolation inequality to cover the other terms in the $C^{2,\alpha}$ norm). Now, we try to proceed as before, but we are gonna get caught up a bit. Let u_k solve:

$$\begin{cases} \sum_{i,j=1}^n a_{ij}(0)\partial_{ij}u_k(x) = f(0) & \text{in } B_{2^{-k}} \\ u_k = u & \text{on } \partial B_{2^{-k}} \end{cases}$$

Note that now, the coefficients are not constant, so we need to freeze these at 0 as well. Then, $v_k := u - u_k$ satisfies:

$$\begin{cases} \sum_{i,j=1}^n a_{ij}(0)\partial_{ij}v_k(x) = f(x) - f(0) + \sum_{i,j=1}^n (a_{ij}(0) - a_{ij}(x))\partial_{ij}u & \text{in } B_{2^{-k}} \\ u_k = 0 & \text{on } \partial B_{2^{-k}} \end{cases}$$

Now, as before, we can use the estimate 3 (rescaled) to translate this into a quantitative statement:

$$||u - u_k||_{L^\infty(B_{2^{-k}})} \leq C2^{-2k}(||f(0) - f||_{L^\infty(B_{2^{-k}})} + ||\sum_{i,j=1}^n [a_{ij}(x) - a_{ij}(0)]\partial_{ij}u||_{L^\infty(B_{2^{-k}})}) \leq$$
$$C2^{-2k}(2^{-\alpha k} + n^2 2^{-\alpha k}) \leq C2^{-k(2+\alpha)}$$

So, using the exact same proof as the Laplacian case, we can show:

$$[D^2u]_{C^{0,\alpha}(B_{\frac{1}{2}})} \leq C(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)} + \underbrace{||D^2u||_{L^\infty(B_1)}}_{\text{need to kill}})$$

(we gain the third bad term in the RHS because of the fact that we have the extra term on the RHS of the PDE that v_k solves). So we just need to get rid of that term. We will do so using the interpolation inequality. So, for any $\delta > 0$, we have the following:

$$[D^2u]_{C^{0,\alpha}(B_{\frac{1}{2}})} \leq C(C_\epsilon||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)} + \epsilon[D^2u]_{C^{0,\alpha}(B_1)}) \underbrace{\leq}_{\text{choose } \epsilon = \frac{\delta}{C}}$$
$$C_\delta(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)}) + \delta[D^2u]_{C^{0,\alpha}(B_1)}$$

Finally, we need to be able to compare the Holder constant in $B_{\frac{1}{2}}$ versus B_1 in order to absorb it.

Lemma 1. *Let $k \in \mathbb{R}, \gamma > 0$. Let S be a non-neg. function on the class of open convex sets of B_1 that is sub-additive. Then, $\exists\ \delta(n,k) > 0$ such that if:*

$$\rho^k S(B_{\frac{\rho}{2}}(x_0) \leq \delta\rho^k S(B_\rho(x_0)) + \gamma \ \forall\ B_{\rho(x_0)} \subset B_1$$

then:

$$S(B_{\frac{1}{2}}) \leq C(n,k)\gamma$$

Proof. Let $Q := \sup_{B_\rho(x_0)\subset B_1} \rho^k S(B_{\frac{\rho}{2}}(x_0))$. By assumption, we have:

$$(\frac{\rho}{2})^k S(B_{\frac{\rho}{4}}(x_0)) \leq \delta(\frac{\rho}{2})^k S(B_{\frac{\rho}{2}}(x_0)) + \gamma \leq \delta Q + \gamma \ \forall\ B_\rho(x_0) \subset B_1$$

Then, taking the sup over $B_\rho(x_0) \subset B_1$:

$$\tilde{Q} := \sup_{B_\rho(x_0)\subset B_1} (\frac{\rho}{2})^k S(B_{\frac{\rho}{4}}(x_0)) \leq \delta Q + \gamma$$

<u>Claim:</u> $Q \leq C\tilde{Q}, C = C(n,k)$. It so:

$$\frac{1}{C}Q \leq \tilde{Q} \leq \delta Q + \gamma \implies Q \leq \tilde{C}(\delta, n, k)\gamma$$

if δ is small enough. So, it STS this claim.

Let $B_\rho(x_0) \subset B_1$. Then, cover $B_{\frac{\rho}{2}}(x_0)$ with N smaller balls $B_{\frac{\rho}{8}}(z_j)$, with $z_j \in B_{\frac{\rho}{2}}(x_0)$. Note that this can be uniformly bounded in terms of the dimension: $N \leq C(n)$. $B_\rho(x_0) \subset B_1$, so, we have:

$$(\frac{\rho}{4})^k S(B_{\frac{\rho}{8}}(x_0)) \leq \tilde{Q}$$

This implies:

$$\rho^k S(B_{\frac{\rho}{2}}(x_0)) \leq \sum_{j=1}^{N} \rho^k S(B_{\frac{\rho}{8}}(x_0)) \leq N 3^k \tilde{Q} = C\tilde{Q}$$

□

Finally, we use this lemma to get our final result. We take:

1. $S(A) = [D^2 u]_{C^{0,\alpha}(A)}$
2. $k = 2 + \alpha$
3. γ to be determined

It STS that $\exists\ \delta$ small enough such that:

$$\rho^\alpha [D^2 u]_{C^{0,\alpha}(B_{\frac{\rho}{2}}(x_0))} \leq \delta \rho^\alpha [D^2 u]_{C^{0,\alpha}(B_\rho(x_0))} + \gamma\ \forall\ B_\rho(x_0) \subset B_1$$

We have:

$$[D^2 u]_{C^{0,\alpha}(B_{\frac{1}{2}})} \leq C_\delta(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)}) + \delta [D^2 u]_{C^{0,\alpha}(B_1)}$$

Rescaled to $B_\rho(x_0)$ (just look at the function $u_\rho(x) = u(x_0 + \rho x)$):

$$\rho^{2+\alpha}[D^2 u]_{C^{0,\alpha}(B_{\frac{\rho}{2}}(x_0))} \leq \delta \rho^{2+\alpha}[D^2 u]_{C^{0,\alpha}(B_\rho(x_0))} + C_\delta(||u||_{L^\infty(B_1)} + \rho^2 ||f||_{L^\infty(B_1)} + \rho^{2+\alpha}[f]_{C^{0,\alpha}(B_1)}) \leq$$
$$\delta \rho^{2+\alpha}[D^2 u]_{C^{0,\alpha}(B_\rho(x_0))} + \gamma$$

with $\gamma = C_\delta(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)})$. So, using the lemma, we see:

$$[D^2 u]_{C^{0,\alpha}(B_{\frac{1}{2}})} \leq C_\delta(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)})$$

Using the interpolation inequality to cover the other components of the $C^{2,\alpha}$ norm gives the result.

□