SMC 2023: HILBERT'S 19$^{\text{TH}}$ PROBLEM – Day V

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1 Extending A Priori Estimates

Today, we will talk about extending a priori estimates (in the case of the Laplacian. The case of non-divergence form operators is exactly the same). What we have right now is:

Theorem 1. *Let $\alpha \in (0,1)$, $u \in C^{2,\alpha}(B_1)$ satisfy:*

$$\Delta u = f \in B_1$$

where $f \in C^{0,\alpha}(B_1)$. Then:

$$||u||_{C^{2,\alpha}(B_{\frac{1}{2}})} \leq C(\alpha, n)(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)})$$

However, what we really want is the following:

Theorem 2. *Let $\alpha \in B_1$, $u \in H^1(B_1) \cap L^\infty(B_1)$ be a weak solution to:*

$$\Delta u = f \quad \text{in } B_1$$

where $f \in C^{0,\alpha}(B_1)$. Then, $\forall\ r \in (0,1), u \in C^{2,\alpha}(B_r)$, and:

$$||u||_{C^{2,\alpha}(B_r)} \leq C(r)(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)}$$

<u>**Note:**</u> As we will see, $C(r) \uparrow +\infty$ as $r \uparrow 1$.

The proof is a simple use of mollification and the theorem of Arzela-Ascoli. We show a (special case of it) here:

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$, $\{u_k\}$ a sequence such that:*

$$||u_k||_{C^{0,\alpha}(\Omega)} \leq C$$

Then, $\exists$ a subsequence $\{u_{j_k}\}$ that converges uniformly to u.

Proof. Use the classical Arzela-Ascoli theorem, noting that equicontinuity is given by the uniform bounds on the Holder constant. □

The way we will use this theorem is in the following theorem:

Theorem 4. *Let $\{u_j\} \to u$ uniformly in Ω, $||u_j||_{C^{k,\alpha}(\Omega)} \leq C$. Then, $u \in C^{k,\alpha}(\Omega)$, and:*

$$||u||_{C^{k,\alpha}(\Omega)} \leq C$$

Proof. <u>$k=0$:</u> u is continuous as a uniform limit of continuous functions. Further, for any $x \neq y \in \Omega$:

$$||u_j||_{L^\infty(\Omega)} + \frac{|u_j(x) - u_j(y)|}{|x-y|^\alpha} \leq C$$

Take the limit, then the sup over $x \neq y$ to obtain the result.

<u>$k \geq 1$:</u> Note that we have $C^{m,\alpha}(\Omega) \subset C^{k,\alpha}(\Omega)\ \forall\ m \leq k$. Further, $||u_j||_{C^{m,\alpha}(\Omega)} \leq ||u_j||_{C^{k,\alpha}(\Omega)} \leq C$. So, for any differential operator D^β, $|\beta| \leq k$, $\{D^\beta u_j\} \subset C^{k-|\beta|,\alpha}(\Omega)$, and $||D^\beta u_j||_{C^{0,\alpha}(\Omega)} \leq C$. So, can pass to subsequences over and over using 3 until we have a sequence $\{u_j\}$ such that all partial derivatives converge uniformly (the limit of the derivatives is the derivative of the limit as all the convergences are uniform). So, for any $x \neq y \in \Omega$:

$$||u_j||_{C^k(\Omega)} + \frac{|D^k u_j(x) - D^k u_j(y)|}{|x-y|^\alpha} \leq C$$

take the limit, then the sup as before to obtain the result. ☐

Finally, we can show the proof of the Schauder theorem:

Proof. Let $r \in (0,1)$. Then, consider the standard mollifier:

$$\phi(x) = \begin{cases} C(n)e^{\frac{1}{|x|^2-1}} & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

where $C(n)$ is chosen such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Now, let $\epsilon_0 := \frac{1-r}{2}$. For any $\epsilon < \epsilon_0$, define:

$$\phi_\epsilon(x) := \frac{1}{\epsilon^n}\phi(\frac{x}{\epsilon})$$
$$u_\epsilon := u * \phi_\epsilon$$
$$f_\epsilon := f * \phi_\epsilon$$

Note that $u_\epsilon \in C^\infty$, as ϕ_ϵ is (derivatives get shoved onto smooth function). Further, these functions satisfy:

$$\Delta u_\epsilon = f_\epsilon \quad \text{in } B_{1-\epsilon}$$

So, applying our a priori Schauder estimate (rescaled):

$$||u_\epsilon||_{C^{2,\alpha}(B_{\frac{1-\epsilon_0}{2}})} \leq C(\epsilon_0)(||u_\epsilon||_{L^\infty(B_{1-\epsilon_0})} + ||f_\epsilon||_{C^{0,\alpha}(B_{1-\epsilon_0})})$$

Next, we employ the covering argument to move back to B_r:

$$||u_\epsilon||_{C^{2,\alpha}(B_r)} \leq C(\epsilon_0, r)(||u_\epsilon||_{L^\infty(B_{1-\epsilon_0})} + ||f_\epsilon||_{C^{0,\alpha}(B_{1-\epsilon_0})})$$

It is very crucial that this can be done in a uniform manner independent of ϵ. Moreover, <u>u is $C^{0,\alpha}$</u> (I cannot justify it to you now, but we will talk about it next week...even if we don't prove it fully). This implies that the convergence $u_\epsilon \to u$ is uniform (on compact subsets of $\mathbb{R}^n$). Finally, we need to get uniform bounds on the RHS in order to apply 4 and finish. We see:

1. $|u_\epsilon(x)| = |u * \phi_\epsilon(x)| = |\int_{\mathbb{R}^n} u(x)\phi_\epsilon(x-y)dy| \leq ||u||_\infty ||\phi_\epsilon||_1 = ||u||_\infty$
2. Same as above for $|f_\epsilon(x)|$
3. For $x, y \in B_{1-\epsilon_0}$, $|f_\epsilon(x) - f_\epsilon(y)| = |\int_{B_\epsilon}[f(x-z) - f(y-z)]\phi_\epsilon(z)dz| \leq$ $\int_{B_\epsilon} |f(x-z) - f(y-z)|\phi_\epsilon(z)dz \leq [f]_{C^{0,\alpha}(B_1)}|x-y|^\alpha$

Thus, we see $||u_\epsilon||_{L^\infty(B_{1-\epsilon_0})} \leq ||u||_{L^\infty(B_1)}$, and $||f_\epsilon||_{C^{0,\alpha}(B_{1-\epsilon_0})} \leq ||f||_{C^0,\alpha(B_1)}$. Apply 4 to finish. ☐