SMC 2023: HILBERT'S 19^{TH} PROBLEM – Day VI

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1 Reorienting Ourselves

Recall what we were actually trying to prove:

PROBLEM: Consider any local minimizer u of an energy functional of the form:

$$\mathscr{E}(w) = \int_{\Omega} L(\nabla w) dx$$

where:

1. $\mathscr{E} : H^1(\Omega) \to \mathbb{R}$
2. $L : \mathbb{R}^n \to \mathbb{R}$ is smooth & uniformly convex (Jacobian matrix is uniformly positive-definite & bounded)
3. $\Omega \subset \mathbb{R}^n$ is open, bounded

Then, is it true that $u \in C^\infty(\Omega)$?

Last week, I gave you the following outline:

$$u \in H^1 \underbrace{\implies}_{\text{difference quotients}} u \in H^2 \underbrace{\implies}_{\text{de Giorgi-Nash-Moser}} \nabla u \in C^{0,\alpha} \underbrace{\implies}_{\text{Schauder}} u \in C^\infty$$

We are now going to talk about the middle step. Recall that any local minimizer u is a weak solution to:

$$\operatorname{div}(DL(\nabla u)) = 0$$

Taking derivatives w.r.t. x_i:

$$\operatorname{div}(D^2L(\nabla u)\nabla w) = 0$$

where $w = \partial_i u$. This is an equation for which we want some regularity on w, but we have no information on $D^2L(\nabla u)$ besides the uniform ellipticity. This is called a uniformly elliptic equation in divergence form with bounded measurable coefficients. The regularity is given by the famous theorem of de Giorgi-Nash-Moser.

2 The de Giorgi $L^2 \to L^\infty$ Lemma – I

Theorem 1. *Let $v \in H^1(\Omega)$ be a weak solution to $div(A(x)\nabla v) = 0$, where A is uniformly elliptic. Then, $\exists\ \alpha > 0$ such that $v \in C^{0,\alpha}(\tilde{\Omega})\ \forall\ \tilde{\Omega} \Subset \Omega$. Further:*

$$||v||_{C^{0,\alpha}(\tilde{\Omega})} \leq C||v||_{L^2(\Omega)}$$

Note: It is vital that $u \in H^2$ so that $w \in H^1$ and 1 may be applied to w.

The proof proceeds in 2 steps:

1. $v \in L^2(\Omega) \implies v \in L^\infty(\tilde{\Omega})$ (This is where the technique of de Giorgi-iteration was introduced)
2. $v \in L^\infty(\tilde{\Omega}) \implies v \in C^{0,\alpha}(\tilde{\Omega})$

Today, we will start the first step and introduce the iteration. As before, we show the proof for $\Omega = B_1$. Here is the theorem:

Theorem 2. *Let $\mathcal{L}v = -div(A(x)\nabla v)$, where A is uniformly bounded & elliptic. Then, $\exists\ \delta = \delta(n, \lambda, \Lambda) \geq 0$ such that if $v \in H^1(B_1)$ solves:*

$$\mathcal{L}v \leq 0 \ \ in\ B_1, \int_{B_1} v_+^2 \leq \delta$$

then $v \leq 1$ in $B_{\frac{1}{2}}$.

This may not look like an $L^2 \to L^\infty$ result, but it is, as seen from the following corollary:

Corollary 1. *Let $v \in H^1(B_1)$ such that $\mathcal{L}v = 0$ in B_1. Then:*

$$||v||_{L^\infty(B_{\frac{1}{2}})} \leq C||v||_{L^2(B_1)}$$

Proof. Let $\tilde{v} = \frac{\sqrt{\delta}}{||v_+||_{L^2(B_1)}} v_+$. Then, applying 2 to $\tilde{v}$ we see that $||\tilde{v}||_{L^\infty(B_{\frac{1}{2}})} \leq 1 \implies ||v_+||_{L^\infty(B_1)} \leq \frac{||v_+||_{L^2(B_1)}}{\sqrt{\delta}}$. Doing the same with v_-, we get:

$$||v||_{L^\infty(B_{\frac{1}{2}})} \leq C||v_+||_{L^2(B_1)} + C||v_-||_{L^2(B_1)} \leq C||v||_{L^2(B_1)}$$

□

3 Inequalities

Now, we list the inequalities needed in the proof of the de Giorgi lemma. The first is the Sobolev embedding theorem.

Theorem 3. $||v||_{L^p(\Omega)} \leq C||\nabla v||_{L^2(\Omega)}\ \forall\ p \leq \frac{2d}{d-2}$

(the case $d = 1$ is a little different, this holds for any $1 \leq p \leq \infty$). The second is Chebyshev's inequality:

Theorem 4. *Let $a > 0$. Then:*

$$\lambda(\{x : |f| \geq a\}) \leq \frac{1}{a^p} \int_{|f| \geq a} |f|^p d\lambda \leq \frac{1}{a^p}||f||_p^p$$

The last is a sort of reverse Poincare inequality, called the Caccioppoli inequality:

Theorem 5. *Let $v \geq 0 \in H^1(B_1)$ be such that $\mathcal{L}v \leq 0$. Then, $\forall\ \phi \in C_0^\infty(B_1)$, we have:*

$$\int_{B_1} |\nabla(v\phi)|^2 dx \leq C||\nabla\phi||^2_{L^\infty(B_1)} \int_{B_1 \cap supp(\phi)} u^2 dx$$

Proof. The weak formulation of $\mathcal{L}v \leq 0$ is:

$$\langle A\nabla v, \nabla\eta\rangle \leq 0\ \forall\ \eta \in H_0^1(B_1), \eta \geq 0$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product. The energy estimate follows by choosing $\eta = \phi^2 v$:

$$\begin{gathered}\langle A\nabla v, \nabla(\phi^2 v)\rangle \leq 0 \\ \langle A\nabla v, \phi\nabla(\phi v)\rangle + \langle A\nabla v, \phi v\nabla(\phi)\rangle \leq 0 \\ \langle A\phi\nabla v, \nabla(\phi v)\rangle + \langle A\phi\nabla v, v\nabla\phi\rangle \leq 0 \\ \langle A[\nabla(\phi v) - v\nabla\phi], \nabla(\phi v) + v\nabla\phi\rangle \leq 0 \\ \langle A\nabla(\phi v), \nabla(\phi v)\rangle \leq \langle Av\nabla\phi, v\nabla\phi\rangle + \langle Av\nabla\phi, \nabla(\phi v)\rangle + |\langle A\nabla(\phi v), v\nabla\phi\rangle|\end{gathered}$$

Now, using the uniform ellipticity and boundedness:

$$\lambda\int_{B_1} |\nabla(\phi v)|^2 dx \leq \Lambda\int_{B_1} u^2|\nabla\phi|^2 dx + 2\Lambda\int_{B_1} |\nabla(\phi v)||v\nabla\phi| dx$$

Using Young's inequality with ϵ to split the last term and absorb the bad term into the LHS, then gathering constants gives the result. □

4 The de Giorgi $L^2 \to L^\infty$ Lemma – II

Finally, we can show the de Giorgi iteration.

Proof. Define a sequence of:

1. balls $B_k := B(0, \frac{1}{2} + \frac{1}{2^{k+1}})$
2. constants $c_k := 1 - \frac{1}{2^k}$
3. cutoffs $\phi_k \in C_0^\infty(B_{k-1})$, such that:

$$\begin{cases}\phi_k(x) \equiv 1 & x \in B_k \\ \phi_k(x) \equiv 0 & x \in B_{k-1}^c\end{cases}$$

4. $||\nabla\phi_k||_{L^\infty(B_1)} \leq 2^k$

Define $v_k := (v - c_k)_+$, $V_k := \int_{B_k} |\phi_k v_k(x)|^2 dx$. Roughly, this measures the energy above c_k in the ball B_k. The goal is to set up a de Giorgi iteration on the sequence $\{V_k\}$, that is to establish a non-linear relation:

$$V_k \leq C^k V_{k-1}^\beta \quad \text{for } \beta > 1, C \neq C(k)$$

Then, we will try to take the limit to see that $V_k \to 0$. Then, passing into the limit, this will say that:

$$\int_{B_{\frac{1}{2}}} |v-1|_+^2 dx = 0$$

So v must be ≤ 1 in $B_{\frac{1}{2}}$. Although the C^k looks scary, as long as V_0 is small enough (this is where the δ in the theorem statement comes from), $V_k \to 0$.

Let's set up the iteration. For $k \geq 1$:

$$V_k = \int_{B_1} (\phi_k v_k)^2 dx = \int_{B_1} (\phi_k v_k)^2 \chi_{\{v_k \geq 0\}} dx = \int_{B_1} (\phi_k v_k)^2 \chi_{\{v_{k-1} \geq \frac{1}{2^k}\}} dx$$

Now, applying Holder with $p = \frac{n}{n-2}, p' = \frac{n}{2}$:

$$\int_{B_k} (\phi_k v_k)^2 \chi_{\{v_{k-1} \geq \frac{1}{2^k}\}} dx \leq [\int_{B_k} (\phi_k v_k)^{\frac{2n}{n-2}} dx]^{\frac{n-2}{n}} [\int_{B_k} \chi_{\{v_{k-1} \geq \frac{1}{2^k}\}} dx]^{\frac{2}{n}} \leq$$
$$[\int_{B_k} (\phi_k v_k)^{\frac{2n}{n-2}} dx]^{\frac{n-2}{n}} [\int_{B_k} \chi_{\{\phi_{k-1} v_{k-1} \geq \frac{1}{2^k}\}} dx]^{\frac{2}{n}}$$

Now, using Chebyshev's inequality:

$$[\int_{B_k} (\phi_k v_k)^{\frac{2n}{n-2}} dx]^{\frac{n-2}{n}} [\int_{B_k} \chi_{\{\phi_{k-1} v_{k-1} \geq \frac{1}{2^k}\}} dx]^{\frac{2}{n}} \leq [\int_{B_k} (\phi_k v_k)^{\frac{2n}{n-2}} dx]^{\frac{n-2}{n}} [||\phi_{k-1} v_{k-1}||^2_{L^2(B_1)} 2^{2k}]^{\frac{2}{n}}$$

Note that the second term here is exactly $C^k U_{k-1}^{\frac{2}{n}}$ (note that $(2^{2k})^{\frac{2}{n}} \leq 2^{4k} = C^k$). We still need the first term to bump the $\frac{2}{n}$ up to bigger than 1. Let's deal with it:

$$[\int_{B_k} (\phi_k v_k)^{\frac{2n}{n-2}} dx]^{\frac{n-2}{n}} \underbrace{\leq}_{\text{Sobolev}} C \int_{B_1} |\nabla(\phi_k v_k)|^2 dx \underbrace{\leq}_{\text{Caccioppoli}} C||\nabla \phi_k||^2_{L^\infty(B_1)} \int_{B_{k-1}} v_k^2 dx \leq$$
$$C^k \int_{B_{k-1}} v_{k-1}^2 dx \leq C^k \int_{B_{k-1}} v_{k-1}^2 \phi_{k-1}^2 dx = C^k V_{k-1}$$

So, $V_k \leq C^k V_{k-1}^{1+\frac{2}{n}}$. □