SMC 2023: HILBERT'S 19^{TH} PROBLEM – Day VII

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1 The de Giorgi $L^2 \to L^\infty$ Lemma – II (cont.)

Let's finish the proof of the de Giorgi $L^2 \to L^\infty$ Lemma.

Proof. Remember what we had: we defined a sequence of:

1. balls $B_k := B(0, \frac{1}{2} + \frac{1}{2^{k+1}})$
2. constants $c_k := 1 - \frac{1}{2^k}$
3. cutoffs $\phi_k \in C_0^\infty(B_{k-1})$, such that:

$$\begin{cases} \phi_k(x) \equiv 1 & x \in B_k \\ \phi_k(x) \equiv 0 & x \in B_{k-1}^c \end{cases}$$

4. $||\nabla \phi_k||_{L^\infty(B_1)} \leq 2^k$

Then, we defined $v_k := (v - c_k)_+$, $V_k := \int_{B_k} |\phi_k v_k(x)|^2 dx$. We showed:

$$V_k \leq C^k V_{k-1}^{1+\frac{2}{n}}$$

<u>**Claim:**</u> If V_0 is small enough, $\{V_k\} \to 0$. We have:

$$V_k \leq C^k V_{k-1}^{1+\frac{2}{n}} \leq C^k C^{(k-1)(1+\frac{2}{n}} V_{k-2}^{1+\frac{2}{n}} \leq ... \leq C^{(\sum_{i=1}^k \frac{i}{(1+\frac{2}{n})^i})(1+\frac{2}{n})^k} V_0^{(1+\frac{2}{n})^k}$$

In other words, $\{V_k\}$ is bounded by the geometric series $\{C^{(\sum_{i=1}^k \frac{i}{(1+\frac{2}{n})^i})(1+\frac{2}{n})^k} V_0^{(1+\frac{2}{n})^k}\}$. Does this sum converge? YES, because $\tilde{C} = C^{(\sum_{i=1}^\infty \frac{i}{(1+\frac{2}{n})^i})}$ is well-defined and $< \infty$! (indeed, it suffices to note that $\frac{i}{(1+\frac{2}{n})^i} \leq \frac{1}{\sqrt{(1+\frac{2}{n})^i}}$ for $i \gg 1$). So, take V_0 such that $\tilde{C}V_0 < 1$, i.e. $\delta < \frac{1}{\tilde{C}}$. Then, $\sum_{k=0}^\infty V_k$ converges $\implies \{V_k\} \to 0$. Finally:

$$\lim_{k\to\infty} V_k = 0 \implies \lim_{k\to\infty} \int_{B(\frac{1}{2}+\frac{1}{2^{k+1}})} (v - 1 + \frac{1}{2^k})_+^2 = 0 \underbrace{\implies}_{\text{DCT}} \int_{B(\frac{1}{2})} (v-1)_+^2 dx = 0$$

In other words, $||v_+||_{L^\infty(B(\frac{1}{2})} \leq 1$. $\square$

2 A Case Study on the Holder Regularity of Harmonic Functions

Before we show the proof of the second step of the theorem of de Giorgi-Nash-Moser, let's step aside and talk about a general outline for showing Holder continuity of elliptic PDEs. This will be important when we return to de Giorgi-Nash-Moser tomorrow.

We already know that $\Delta u = 0 \implies u \in C^\infty$. However, by very different methods, we can show that $u \in C^{0,\alpha}$ also (in fact, this even holds with a RHS $f \in L^\infty$–this is what I was alluding to a couple of days ago, during the mollification–or for elliptic equations more generally). The first step is the Harnack inequality:

Theorem 1. *Let $u \in H^1(B_1)$ be a non-negative, harmonic function in B_1, i.e.:*

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \\ u \geq 0 & \text{in } B_1 \end{cases}$$

Then, $\sup_{B_{\frac{1}{2}}} u \leq C(n) \inf_{B_{\frac{1}{2}}} u$.

Proof. We use the mean value property. Let $x, y \in B_{\frac{1}{2}}$. Let $\eta = \frac{1}{2}, \epsilon = \frac{1}{4}$. Then, choose two points x_1, x_2 that lie on the straight line between them such that the distant between all the points is the same ($< \frac{1}{4}$). Then:

$$u(x) = \fint_{B_{\epsilon(x)}} u(z)dz = \frac{1}{|B_\epsilon|} \int_{B_{\epsilon(x)}} u(z)dz \leq \frac{1}{|B_\epsilon|} \int_{B_{\eta(x_1)}} u(z)dz = \frac{|B_\eta|}{|B_\epsilon|} \int_{B_{\eta(x_1)}} u(z)dz = \frac{|B_\eta|}{|B_\epsilon|} u(x_1) \leq$$
$$... \leq (\frac{|B_\eta|}{|B_\epsilon|})^3 u(y)$$

Take the inf, sup to get the result. □

The Harnack inequality essentially says that a harmonic function cannot oscillate too much in any compactly supported subset of the domain. One might suspect that as a further step, we could say that the smaller the domain, the smaller the oscillation should be. We can quantify this with a simple corollary of the Harnack inequality, a so-called oscillation lemma:

Lemma 1. *Let $u \in H^1(B_1)$ be harmonic in B_1. Then:*

$$osc_{B_{\frac{1}{2}}} u \leq (1 - \theta(n)) osc_{B_1}$$

where $osc_\Omega u := \sup_\Omega u - \inf_\Omega u$.

Proof. Let $w(x) : u(x) - \inf_{B_1} u$. Note the following properties of w:

1. $w \geq 0$ in B_1
2. $\Delta w = 0$ in B_1
3. $\text{osc}_{B_{\frac{1}{2}}} w = \text{osc}_{B_{\frac{1}{2}}} u$
4. Via 1, $\sup_{B_{\frac{1}{2}}} w \leq C \inf_{B_{\frac{1}{2}}} w$

So we get:

$$\text{osc}_{B_{\frac{1}{2}}} u = \text{osc}_{B_{\frac{1}{2}}} w = \sup_{B_{\frac{1}{2}}} w - \inf_{B_{\frac{1}{2}}} w \leq (1 - \frac{1}{C}) \sup_{B_{\frac{1}{2}}} w \leq \theta \sup_{B_1} w = \theta \text{osc}_{B_1} u$$

$\square$

As a corollary, we have rescaled versions of both of these results (just look at the function $\tilde{u}(x) = u(rx)$, and apply the previous results to those):

Corollary 1. *Let $u \in H^1(B_r), \Delta u = 0$ in B_r, $u \geq 0$ in B_r. Then:*

$$\sup_{B_{\frac{r}{2}}} u \leq C(n) \inf_{B_{\frac{r}{2}}} u$$

Corollary 2. *Let $u \in H^1_B(r)$ be harmonic in B_r. Then:*

$$osc_{B_{\frac{r}{2}}} u \leq (1 - \theta(n)) osc_{B_r}$$

It is crucial in this development that C (and therefore θ) does not depend on r.

As a corollary of the oscillation lemma, amazingly, we actually get Holder regularity of harmonic functions! The proof can be summarized with a simple picture. (I hope I remembered to draw it during the lecture).

Corollary 3. *Let $u \in H^1(B_1) \cap L^\infty(B_1)$ be harmonic in B_1. Then:*

$$||u||_{C^{0,\alpha}(B_{\frac{1}{2}})} \leq C ||u||_{L^\infty(B_1)}$$

Proof. WLOG, assume that $||u||_{L^\infty(B_{\frac{1}{)}}} \leq \frac{1}{2}$. It suffices to show:

$$|u(x) - u(y)| \leq C|x - y|^\alpha \ \forall x \neq y \in B_{\frac{1}{2}}$$

for some $\alpha \in (0, 1)$. For simplicity, we show it at $y = 0$. Let $x \in B_{\frac{1}{2}}, k \in \mathbb{N}$ such that $x \in B_{2^{-k}} \setminus B_{2^{-k-1}}$. Then:

$$|u(x) - u(0)| \leq \text{osc}_{B_{2^{-k}}} u \leq (1 - \theta)^k \text{osc}_{B_1} u \leq (1 - \theta)^k = 2^{-\alpha k}$$

for $\alpha = -\log_2(1 - \theta)$. Finally, $2^{-k} \leq 2|x|$, so:

$$|u(x) - u(0)| \leq (2|x|)^\alpha \leq C|x|^\alpha$$

$\square$

Basically, we are just finding α here that fits the picture I drew.

The takeaway from this process is:

$$\underbrace{\text{Harnack}}_{\text{relies on harmonicity}} \implies \underbrace{\text{Oscillation lemma}}_{\text{doesn't see harmonicity}} \implies \underbrace{\text{Holder regularity (as long as } L^\infty)}_{\text{doesn't see harmonicity}}$$

3 The Theorem of de Giorgi-Nash-Moser: Step 2

Going back to our equation: we have a solution $v \in H^1 \cap L^\infty$ of: $div(A(x)\nabla v) = 0$, where A is uniformly elliptic. We <u>don't</u> have a Harnack-type inequality for this type of equation (although using de Giorgi methods, apparently you can get one), but we are able to get an oscillation decay! Once we have that, we can follow the same proof as before to show that $v \in C^{0,\alpha}$.