SMC 2023: HILBERT'S 19^{TH} PROBLEM – Day VIII

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1 The Theorem of de Giorgi-Nash-Moser: Step 2 (cont.)

Today, we will finish the proof of the theorem of de Giorgi-Nash-Moser. It suffices to show the following theorem on oscillation decay:

Theorem 1. *Let $\mathcal{L}(v) = -div(A(x)\nabla v)$, for A uniformly bounded and elliptic. Then, let $v \in H^1(B_2)$ solve:*

$$\mathcal{L}(v) = 0 \text{ in } B_2$$

Then, $osc_{B_{\frac{1}{2}}} < (1 - \theta(n, \lambda, \Lambda))osc_{B_2}$

Note: This immediately gives Holder regularity of v, as mentioned yesterday.

To start, let's give a lemma, the de Giorgi isomperimetric inequality (FR/RO note that this is a "quantitative version of the fact that $H^1(\mathbb{R})$ functions cannot have jumps"):

Lemma 1. *Let $w \in H^1(B_1)$ be such that:*

$$\int B_1 |\nabla w|^2 dx \leq C_0$$

Now, define:

1. $A := \{w \leq 0\} \cap B_1$
2. $D := \{w \geq \frac{1}{2}\} \cap B_1$
3. $E := \{0 < w < \frac{1}{2}\} \cap B_1$

Then:

$$C_0|E| \geq c(n)|A|^2|D|^2$$

Proof. Define $\tilde{w}$ in B_1 as:

$$\tilde{w}(x) = \begin{cases} w & x \in E \\ 0 & x \in A \\ \frac{1}{2} & x \in D \end{cases}$$

(essentially, we are cutting off w between 0 and $\frac{1}{2}$. Note that $\nabla \tilde{w} \equiv 0$ in $B_1 \setminus E$ (recall that the gradient of Sobolev functions vanishes on level sets), and $\int_{B_1} |\nabla \tilde{w}|^2 dx \leq C_0$. Define $\overline{w} := \fint_{B_1} \tilde{w}(x)dx$. Then:

$$|A| \cdot |D| = \int_A |D| dy \leq \int_A \int_D 2w(x) dxdy \leq 2 \int_A \int_D |\tilde{w}(x) - \tilde{w}(y)| dxdy \leq$$

$$2 \int_{B_1} \int_{B_1} |\tilde{w}(x) - \overline{w} + \overline{w} - \tilde{w}(y)| dxdy \leq 2 \int_{B_1} \int_{B_1} (|\tilde{w}(x) - \overline{w}| + |\overline{w} - \tilde{w}(y)|) dxdy =$$

$$4 \int_{B_1} |\tilde{w}(x) - \overline{w}| dx \underbrace{\leq}_{\text{Poincare}} c \int_E |\nabla \tilde{w}(x)| dx$$

Finally, we use Holder's inequality:

$$|A| \cdot |D| \leq c \int_E |\nabla \tilde{w}(x)| dx \leq c ||\nabla \tilde{w}||_{L^2(B_1)} |E|^{\frac{1}{2}} \leq c C_0 |E|^{\frac{1}{2}}$$

□

Lemma 2. *Let $\mathcal{L}$ be as above, $v \in H^1(B_2)$ such that:*

1. *$v \leq 1$ in B_2*
2. *$\mathcal{L}v \leq 0$ in B_2*

Assume that $|\{v \leq 0\} \cap B_1| \geq \mu > 0$. Then:

$$\sup_{B_{\frac{1}{2}}} v \leq 1 - \gamma(n, \lambda, \Lambda, \mu)$$

for some $\gamma > 0$ small.

Proof. Consider the sequence:

$$\{w_k\} := 2^k[v - (1 - 2^{-k})]_+$$

Note that $w_k \leq 1$ in B_2, as $v \leq 1$ in B_2. Moreover, $\mathcal{L}w_k \leq 0$ in B_2. So, using the Cacioppoli energy inequality:

$$\int_{B_1} |\nabla w_k|^2 dx \leq C \int_{B_2} w_k^2 \leq C_0 := C|B_2|$$

By assumption, $|\{v \leq 0\} \cap B_1| \geq \mu \implies |\{w_k \leq 0\} \cap B_1| \geq \mu$. Now, let $\delta > 0$, assume for the sake of contradiction that $\int_{B_1} w_k^2 \geq \delta^2$ for all k. Then, the isoperimetric inequality 1 yields:

$$|\{w_k \geq \frac{1}{2}\} \cap B_1| \geq |\{w_{k+1} > 0\} \cap B_1| \geq \int_{B_1} w_{k+1}^2 dx \geq \delta^2 \underbrace{\implies}_{\text{Iso.}\neq} |\{0 < w_k < \frac{1}{2}\} \cap B_1| \geq \frac{c}{C_0} \delta^4 \mu^2 = \beta > 0$$

where β is independent of k, and depends only on n, δ, μ.

Now, notice that the sets $\{0 < w_k < \frac{1}{2}\}$ are disjoint $\forall\ k$, so the previous inequality cannot hold $\forall\ k$ (if it did, we would get $|B_1| = \infty$, a contradiction). So, $\exists\ k_0$ such that $\int_{B_1} w_{k_0}^2 dx < \delta^2$. So, by the de Giorgi $L^2 \to L^\infty$ lemma:

$$||w_{k_0}||_{L^\infty(B_{\frac{1}{2}}} \leq C\delta \leq \frac{1}{2}$$

for δ sufficiently small. So, $w_{k_0} \leq \frac{1}{2}$ in $B_{\frac{1}{2}} \implies v \leq \frac{1}{2} 2^{-k_0} + (1 - 2^{-k_0}) \leq 1 - 2^{-k_0 - 1} = 1 - \gamma$. □

Finally, the oscillation decay is a simple consequence of this lemma.

Proof. Let $w(x) := \frac{2}{\mathrm{osc}_{B_2} v}[v(x) - \frac{\sup_{B_2} v + \inf_{B_2} v}{2}]$. Then, $-1 \leq w \leq 1$ in B_2. WLOG, let $|\{w \leq 0\} \cap B_1| \geq \frac{1}{2}|B_1|$ (otherwise, take $-w$). Then, via the lemma:

$$w \leq 1 - \gamma \text{ in } B_{\frac{1}{2}}$$

So, $\mathrm{osc}_{B_{\frac{1}{2}}} w \leq 2 - \gamma$. Finally, this implies:

$$\mathrm{osc}_{B_{\frac{1}{2}}} v \leq (1 - \frac{\gamma}{2}) \mathrm{osc}_{B_2} v$$

□