SMC 2023: HILBERT'S 19^{TH} PROBLEM – Day IX

Jeffrey Cheng

June 5, 2023

1 Difference Quotients

Today is the last day. Recall we have the following: let u be a weak solution to $div(DL(\nabla u)) = 0$. Then, we had:

$$u \in H^1 \underbrace{\implies}_{\text{difference quotients}} u \in H^2 \underbrace{\implies}_{\text{de Giorgi-Nash-Moser}} \nabla u \in C^{0,\alpha} \underbrace{\implies}_{\text{Schauder}} u \in C^\infty$$

It remains to show the first step. The techniques here are very similar to section 6.3 in Evans PDE (energy methods to show regularity).

Theorem 1. *Let $u \in H^1(\Omega)$ be a weak solution to:*

$$div(DL(\nabla u)) = 0$$

Then, $\forall\ \tilde{\Omega} \Subset \Omega$, $u \in H^2(\tilde{\Omega})$, and for all i, $w = \partial_i u$ is a weak solution to:

$$div(D^2L(\nabla u)\nabla w) = 0$$

Proof. We will show that for $u \in H^1(B_3)$, for any i, that $w \in H^2(B_1)$ and w solves $div(D^2L(\nabla u)\nabla w) = 0$ weakly (move from localized version to $\tilde{\Omega}$ via a partition of unity argument).

Fix $1 \leq i \leq n$, and $\forall -1 \leq h \leq 1$, consider the *difference quotient map* $T_h : C(\mathbb{R}^n) \to C(\mathbb{R}^n)$ defined by:

$$T_h\phi(x) = \frac{\phi(x + he_i) - \phi(x)}{h}$$

Now, fix a non-negative, smooth, compactly supported cutoff function η such that:

1. $\eta \equiv 1$ in B_1
2. $\text{supp}(\eta) \subset \overline{B_2}$

Now, we can define two operators:

1. $\eta T_h : L^2(B_3) \to L^2(B_3)$
2. $T_h(\eta \cdot) : H^1(B_3) \to H^1_0(B_3)$

These operators have the following properties:

Lemma 1. *For all $\tilde{\phi}, \tilde{\psi} \in L^2(B_3)$, $\phi \in H^1(B_3), \Phi \in [L^2(B_3)]^N$, we have:*

1. $\int T_h(\eta\tilde{\phi})\tilde{\psi}dx = -\int \tilde{\phi}\eta T_{-h}\tilde{\psi}dx$ *(the dual operator to $T_h(\eta\cdot)$ is ηT_{-h})*
2. $\nabla[\eta^2 T_h\phi] = \eta^2 T_h[\nabla\phi] + 2\eta[T_h\psi]\nabla\eta$

3. $\eta T_h[DF(\Phi)](x) = \eta T_h\Phi(x)\int_0^1 D^2F(\Phi(x) + t(\Phi(x+he_i) - \Phi(x)))dt$

4. $||\eta T_h\phi||_{L^2(B_3)} \leq C_\eta||\nabla\phi||_{L^2(B_3)}$

5. *if* $\sup_{|h|\leq 1}||\eta T_h\tilde{\phi}||_{L^2(B_3)} = C < \infty$, *then* $\partial_i\tilde{\phi} \in L^2(B_1)$, *and:*

$$||\partial_i\tilde{\phi}||_{L^2(B_1)} \leq C$$

and $\eta T_h\tilde{\phi} \to \partial_i\tilde{\phi}$ *as* $h \to 0$ *weakly in* $L^2(B_1)$.

Proof. 1. Make a change of variables in the integral $x \mapsto x - he_i$

2. Product rule

3. Calculation using Fundamental theorem of Line Integrals & change of variables

4. $\eta T_h\phi = \eta\int_0^1 \partial_i(\phi(x+he_i))dt$, so $||\eta T_h\phi||_{L^2} \leq ||\eta||^2_{L^\infty}||\partial_i\phi||_{L^2}$

5. Let δ be the standard mollifier, $\delta_\epsilon(x) = \epsilon^{-n}\delta(\frac{x}{\epsilon})$. Let $\tilde{\phi}_\epsilon := \tilde{\phi} * \delta_\epsilon$. $\tilde{\phi}_\epsilon$ is smooth, so for $x \in B_2$, $\eta T_h\tilde{\phi}_\epsilon(x_) \to \eta\partial_i\tilde{\phi}_\epsilon(x)$ as $h \to 0$. But, $\eta T_h\tilde{\phi}_\epsilon = \eta(T_h\tilde{\phi})_\epsilon$ is uniformly bounded w.r.t both h, ϵ (h via 4, ϵ via some convolution inequality type thing). So, pass into the limit in h to find that $\partial_i\tilde{\phi}_\epsilon$ is uniformly bounded in $L^2(B_1)$. Finally, $\partial_i\tilde{\phi} \in L^2(B_1)$, and the convergence is weak via Banach-Alaoglu (see theorem 3 in 5.8.2 of Evans).

□

Let's get back to the proof. Consider the test function $\phi = T_{-h}(\eta^2 T_h u) \in H_0^1(B_3)$. We have:

$$0 = \int \nabla\phi DF(\nabla u)dx =$$

$$-\int \eta\nabla(T_h u)T_h DF(\nabla u)dx - 2\int \nabla\eta(T_h y)\eta T_h DF(\nabla u)dx =$$

$$-\int \eta\nabla(T_h u)[\int_0^1 D^2F(\nabla u(x) + t(\nabla u(x+he_i) - \nabla u(x)))dt]\eta\nabla(T_h u)dx -$$

$$2\int (\nabla\eta)(T_h u)[\int_0^1 D^2F(\nabla u(x) + t(\nabla u(x+he_i) - \nabla u(x)))dt]\eta\nabla(T_h u)dx$$

Finally, we use the ellipticity to get uniform bounds:

$$\int \eta^2|\nabla(T_h u)|^2 dx \leq (2\Lambda\lambda)^2\int |\eta\nabla|^2|T_h u|^2 dx \leq C||u||^2_{H^1(B_3)}$$

So, by 5, $\partial_i u \in H^1(B_1) \implies u \in H^2(B_1)$. Finally, for any $\phi \in H_0^1(B_1)$, $T_{-h}\phi \in H^1(B_2)$. So, we have:

$$0 = -\int \nabla\phi T_h DF(\nabla u)$$

$\partial_i u \in H^1(B_1)$, so $T_h DF(\nabla u) \to \partial_i(DF(\nabla u)) = D^2F(\nabla u)\nabla w$ weakly in $L^2(B_1)$. So, in the limit:

$$0 = -\int \nabla\phi T_h DF(\nabla u) \to \int \nabla\phi D^2F(\nabla u)\nabla w dx$$

□