The De-Giorgi L^2 to L^∞ Lemma

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1 Introduction

Consider the following elliptic equation:

$$\mathcal{L}(u)(x) = -div(A(x)\nabla u(x)) = 0 \qquad \in \Omega$$

for some open, bounded region Ω in $\mathbb{R}^n$, and A a uniformly bounded, uniformly elliptic matrix of coefficients. Given solutions $u \in H^1(\Omega)$ to this equation, the first methods of establishing higher regularity for u were as follows:

1. Energy methods (see Chapter 6.3 of [1] for an overview of this)
2. Perturbative methods, aka Schauder estimates (see [2] for an overview of this)

The main obstruction to being satisfied with these results is that they require regularity of the coefficients and initial data to establish regularity for u. Roughly, the energy methods require that all the coefficients $a_{ij}(x)$ be in $C^k(\Omega)$ to establish that u is in $C^{k+2}(\Omega)$, while the Schauder estimates require that the coefficients $a_{ij}(x)$ be in $C^{k,\alpha}(\Omega)$ to establish that u is in $C^{k+2,\alpha}(\Omega)$. However, in 1957, De Giorgi introduced a breakthrough method to establish partial regularity for u without requiring anything of the coefficients. More precisely, De Giorgi proved the following:

Theorem 1. *Let $u \in H^1(\Omega)$ solve $\mathcal{L}(u) = 0$, for A uniformly bounded & elliptic. Then, $u \in C^{0,\alpha}(\tilde{\Omega})$ for any $\tilde{\Omega} \subset\subset \Omega$, and we have:*

$$||u||_{C^{0,\alpha}(\tilde{\Omega})} \leq C||u||_{L^2(\Omega)}$$

for α depending only on the ellipticity constant and the dimension n.

The proof proceeds in two steps:

1. Given $u \in L^2(\Omega)$, show that $u \in L^\infty(\tilde{\Omega})$.
2. Given $u \in L^\infty(\tilde{\Omega})$, show that $u \in C^{0,\alpha}(\tilde{\Omega})$.

The first step is where De Giorgi introduced the technique of "De Giorgi iteration" and this note is mostly dedicated to giving a clear exposition of the iteration in that step.

Step 1

We will only show the results for the special case $\Omega = B_1$, $\tilde{\Omega} = B_{\frac{1}{2}}$. The general case can be recovered from a simple covering argument (see [2]). The argument here is based off the one in [2], which is in turn derived from [3]. First, we need an energy inequality, sometimes referred to as the Caccioppoli inequality:

Theorem 2. *Let $v \geq 0 \in H^1(B_1)$ be such that $\mathcal{L}v \leq 0$. Then, for any $\phi \in C_0^\infty(B_1)$, we have:*

$$\int_{B_1} |\nabla(v\phi)|^2 dx \leq C||\nabla\phi||^2_{L^2(B_1)} \int_{B_1 \cap supp\phi} u^2 dx$$

Proof. The weak formulation of $\mathcal{L}v \leq 0$ is:

$$\langle A\nabla v, \nabla(\eta)\rangle \leq 0 \;\forall\; \eta \in C_0^\infty(B_1), \eta \geq 0$$

where $\langle\ ,\ \rangle$ denotes the L^2 inner product. Choosing $\eta = \phi^2 v$:

$$\begin{gathered}
\langle A\nabla v, \nabla(\phi v)\rangle \leq 0 \\
\langle A\nabla v, \phi\nabla(\phi v) + \phi v\nabla(\phi)\rangle \leq 0 \\
\langle A\nabla v, \phi\nabla(\phi v)\rangle + \langle A\nabla v, \phi v\nabla(\phi)\rangle \leq 0 \\
\langle A\phi\nabla v, \nabla(\phi v)\rangle + \langle A\phi\nabla v, v\nabla(\phi)\rangle \leq 0 \\
\langle A\phi\nabla v, \nabla(\phi v)\rangle + \langle A\phi\nabla v, v\nabla(\phi)\rangle \leq 0 \\
\langle A[\nabla(\phi v) - v\nabla(\phi)], \nabla(\phi v)\rangle + \langle A[\nabla(\phi v) - v\nabla(\phi)], v\nabla(\phi)\rangle \leq 0 \\
\langle A\nabla(\phi v), \nabla(\phi v) \leq \langle Av\nabla(\phi), v\nabla(\phi)\rangle + \langle Av\nabla(\phi), v\nabla(\phi)\rangle + |\langle A\nabla(\phi v), v\nabla(\phi)\rangle|
\end{gathered}$$

By uniform ellipticity with constant λ, and also boundedness of A with constant Λ:

$$\lambda\int_{B_1} |\nabla(\phi v)|^2 dx \leq 2\Lambda \int_{B_1} v^2|\nabla(\phi)|^2 + \Lambda\int_{B_1} |\nabla(\phi v)||v\nabla(\phi)|dx$$

Now, using Young's inequality and choosing ϵ such that $\Lambda\epsilon < \frac{\lambda}{2}$:

$$\frac{\lambda}{2}\int_{B_1} |\nabla(\phi v)|^2 dx \leq 2\Lambda\int_{B_1} v^2|\nabla(\phi)|^2 + \frac{\Lambda}{4\epsilon}\int_{B_1} v^2|\nabla(\phi)|^2 dx$$

Rearranging terms and gathering constants gives the desired result. □

We will apply the Cacciopoli inequaltiy to functions of the form $u_k = (u - c_k)_+$, for some positive constant c_k. We leave it to the reader to show that if u satisfies $\mathcal{L}u = 0$, then $\mathcal{L}u_+ \leq 0$.

Now, we can proceed to the main result, where the De Giorgi iteration is utilized:

Theorem 3. *Denote $u_+ = max(u, 0)$. Then, $\exists$ a constant δ such that for any u such that $\mathcal{L}(u_+) \leq 0$, where A is uniformly bounded/elliptic, the following holds:*

$$||u_+||_{L^2(B_1)} \leq \delta \implies ||u_+||_{L^\infty(B_{\frac{1}{2}})} \leq 1$$

Proof. Define a sequence of:

1. balls: $B_k := B(0, \frac{1}{2} + \frac{1}{2^{k+1}})$
2. constants: $c_k := 1 - \frac{1}{2^k}$
3. cutoffs: $\phi_k(x) \in C_0^\infty(B_{k-1}), \phi_k = 1$ in B_k, ϕ_k=0 in B_{k-1}^C.

Define $u_k = (u - c_k)_+$, $U_k = \int_{B_k} |u_k(x)|^2 dx$. We will do a De Giorgi iteration on U_k. What that means is that we will establish a non-linear relation of the following form:

$$U_k \leq C^k U_{k-1}^\beta \text{ for } C > 1, \beta > 1$$

Although the C^k term looks scary, in the limit, we will see that as long as U_0 is sufficiently small (this is where our δ in the theorem statement comes from), $\{U_k\} \to 0$.

For $k \geq 1$, we have:

$$U_k = \int_{B_k} |u_k|^2 dx = \int_{B_k} (\phi_k u_k)^2 dx = \int_{B_k} (\phi_k u_k)^2 \chi_{\{u_k \geq 0\}} dx = \int_{B_k} (\phi_k u_k)^2 \chi_{\{u_{k-1} \geq \frac{1}{2^k}\}} dx$$

Now, by Holder with $p = \frac{n}{n-2}, p' = \frac{n}{2}$:

$$\int_{B_k} (\phi_k u_k)^2 \chi_{\{u_{k-1} \geq \frac{1}{2^k}\}} dx \leq$$

$$(\int_{B_k} (\phi_k u_k)^{\frac{2n}{n-2}} dx)^{\frac{n-2}{n}} (\int_{B_k} \chi^{\frac{n}{2}}_{\{u_{k-1} \geq \frac{1}{2^k}\}})^{\frac{2}{n}} dx \leq (\int_{B_k} (\phi_k u_k)^{\frac{2n}{n-2}} dx)^{\frac{n-2}{n}} (\int_{B_k} \chi^{\frac{n}{2}}_{\{\phi_{k-1} u_{k-1} \geq \frac{1}{2^k}\}} dx)^{\frac{2}{n}} \leq$$

$$(\int_{B_k} (\phi_k u_k)^{\frac{2n}{n-2}} dx)^{\frac{n-2}{n}} (\int_{B_{k-1}} \chi^{\frac{n}{2}}_{\{\phi_{k-1} u_{k-1} \geq \frac{1}{2^k}\}} dx)^{\frac{2}{n}}$$

By Chebyshev's inequality:

$$(\int_{B_k} (\phi_k u_k)^{\frac{2n}{n-2}} dx)^{\frac{n-2}{n}} (\int_{B_{k-1}} \chi^{\frac{n}{2}}_{\{\phi_{k-1} u_{k-1} \geq \frac{1}{2^k}\}} dx)^{\frac{2}{n}} \leq$$

$$(\int_{B_k} (\phi_k u_k)^{\frac{2n}{n-2}} dx)^{\frac{n-2}{n}} (||\phi_{k-1} u_{k-1}||^2_{L^2(B_{k-1})} 2^{2k})^{\frac{2}{n}}$$

Now, the second term in the above expression is ok, but we have to deal with the first. By Sobolev+Caccioppoli inequalities:

$$(\int_{B_k} (\phi_k u_k)^{\frac{2n}{n-2}})^{\frac{n-2}{n}} \leq C \int_{B_1} (\nabla(\phi_k u_k))^2 dx \leq$$

$$C||\nabla \phi||^2_{L^\infty(B_1)} \int_{supp\{\phi_k\}} u_k^2 dx \leq C 2^{2k} \int_{B_{k-1}} u_k^2 dx \leq C 2^{2k} \int_{B_{k-1}} \phi_{k-1}^2 u_{k-1}^2 dx = C^k U_{k-1}$$

as $||\nabla(\phi_k)||_{L^\infty(B_1)}$ may be chosen to be less than 2^k. Putting these together, we get:

$$U_k \leq (||\phi_{k-1} u_{k-1}||^2_{L^2(B_{k-1})} 2^{2k})^{\frac{2}{n}} C^k U_{k-1} = C^k U_{k-1}^{1+\frac{2}{n}}$$

Now, we want to show that $lim_{k\to\infty} U_k = 0$. Indeed, notice that:

$$U_k \leq C^k V_{k-1}^{1+\frac{2}{n}} \leq ... \leq C^k C^{(k-1)(1+\frac{2}{n})} ... C^{(1)(1+\frac{2}{n})^{k-1}} U_0^{(1+\frac{2}{n})^{k-1}} = (C^{\sum_{i=1}^k \frac{i}{(1+\frac{2}{n})^i}} U_0)^{(1+\frac{2}{n})^{k-1}}$$

Now, denote $\tilde{C} = C^{\sum_{i=1}^\infty \frac{i}{(1+\frac{2}{n})^i}}$ (one can check that the infinite sum actually converges, as $\frac{i}{(1+\frac{2}{n})^i} \leq \frac{1}{\sqrt{(1+\frac{2}{n})}^i}$ for sufficiently large i, so this is well-defined). So, if we pick U_0 such that $\tilde{C} U_0 < 1$, i.e. we set $\delta < \frac{1}{\tilde{C}}$, then $\{U_k\} \to 0$ as $k \to \infty$. So, we have:

$$lim_{k\to\infty} U_k = 0 \implies lim_{k\to\infty} \int_{B_{(\frac{1}{2}+\frac{1}{2^{k+1}})}} (u - 1 + \frac{1}{2^k})_+^2 dx = 0 \implies \int_{B_{\frac{1}{2}}} (u-1)_+^2 dx = 0 \implies$$

$$||u_+||_{L^\infty(B_{\frac{1}{2}})} \leq 1$$

by the Dominated Convergence theorem. □

Note that by identical methods, we can establish the same result for u_-. Finally, this theorem is one small step away from an L^∞ regularity result:

Theorem 4. *Let $u \in H^1(B_1)$ such that $\mathcal{L}u_+ \leq 0 \in B_1$. Then:*

$$||u_+||_{L^\infty(B_{\frac{1}{2}})} \leq C||u_+||_{L^2(B_1)}$$

Proof. This is a simple consequence of the previous theorem. Write $\tilde{u} = \frac{\sqrt{\delta}}{||u_+||_{L^2(B_1)}}u$. Then, applying the previous theorem to $\tilde(u)$, we see $||\tilde{u}_+||_{L^\infty(B_{\frac{1}{2}})} \leq 1 \implies ||u_+||_{L^\infty(B_{\frac{1}{2}})} \leq \frac{||u_+||_{L^2(B_1)}}{\sqrt{\delta}}$. □

Finally, by writing $u = u_+ - u_-$, we establish the De-Giorgi lemma.

Step 1 (Non-homogeneous)

Now, we show that the same result can be established for the elliptic equation:

$$\mathcal{L}(u)(x) = -div(A(x)\nabla u(x)) = f \qquad \in \Omega$$

assuming that $f \in L^\infty(\Omega)$. The proof of the main result is exactly the same, except for the step where we apply the Caccioppoli inequality. Although we can no longer hope for such an inequality in such a general form, we get around this with a little bit of extra work.

Looking back at the step where we applied the Caccioppoli inequality, it suffices to show that $\int_{B_1}(\nabla(\phi_k u_k))^2 dx \leq C^k U_{k-1}$. Indeed, note that in this case, instead of $\mathcal{L}u_k \leq 0$, we have $\mathcal{L}u_k \leq f$. So, working through the proof of the Caccioppoli inequality (this time with a RHS f, and the specific functions $v = u_k, \phi = \phi_k$), we arrive at the following:

$$\int_{B_1}(\nabla(\phi_k u_k))^2 \leq C||\nabla\phi_k||^2_{L^\infty(B_1)}\int_{supp\{\phi_k\}} u_k^2 dx + \int_{B_1} f\phi_k^2 u_k dx$$

Now, in the homogeneous case, the first term on the RHS has already been shown to be bounded by $C^k U_{k-1}$, so it STS that $\int_{B_1} f\phi_k^2 u_k dx$ may be as well. Indeed, write $\int_{B_1} f\phi_k^2 u_k dx$ as:

$$\int_{B_1} f\phi_k^2 u_k dx = \int_{B_1} f\phi_k^2 u_k \chi_{supp(\phi_k^2 u_k} dx$$

Now, applying Holder with $p = q = 2$:

$$\int_{B_1} f\phi_k^2 u_k dx \leq ||f||_{L^\infty(B_1)}\int_{B_1} f\phi_k^2 u_k \chi_{supp(\phi_k^2 u_k} dx \leq ||f||_{L^\infty(B_1)}(||\phi_k^2 u_k||_{L^2(B_1)}||\chi_{supp(\phi_k^2 u_k)}||_{L^2(B_1)})$$

$||\phi_k^2 u_k||_{L^2(B_1)}|| \leq U_{k-1}^{\frac{1}{2}}$. Similarly, $||\chi_{supp(\phi_k^2 u_k)}||_{L^2(B_1)} \leq 2^k U_{k-1}^{\frac{1}{2}}$ by bumping down k and using Chebyshev's inequality as before. In total, we see that $\int_{B_1} f\phi_k^2 u_k dx \leq ||f||_{L^\infty(B_1)} 2^k U_{k-1} = C^k U_{k-1}$, so we can push the argument through as before.

References

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[3] Alexis F. Vasseur. "The De Giorgi method for elliptic and parabolic equations and some applications". In: *Lectures on the analysis of nonlinear partial differential equations. Part 4.* Vol. 4. Morningside Lect. Math. Int. Press, Somerville, MA, 2016, pp. 195–222.