# Extending A Priori Estimates: Schauder Estimates for the Laplacian

Jeffrey Cheng

October 12, 2022

When tackling regularity issues in PDE, a common idea is to first establish "a priori" estimates: estimates on regularity that can be established if it is given that you are working with a sufficiently regular function. However, once such a priori estimates are established, the final task still remains: to translate the a priori estimates into statements about weaker functions. This can (mostly) always be done in a standard way, so it is often skated over in the literature. This note is meant to be an explicit demonstration of how it is done in one of the simplest cases: the case of the Schauder estimates for the Laplacian.

## The Laplacian: An Overview

In all that follows, $\Omega$ will be an open, bounded, sufficiently regular domain in $\mathbb{R}^n$. The basic problem we are trying to address is finding solutions, and establishing regularity of solutions, to the problem:

$$\Delta u(x) = f \qquad x \in \Omega$$

for some prescribed function $f : \Omega \to \mathbb{R}$. In the case of the Laplacian, there are multiple ways to do this:

1. Convolving the Green's function for the Laplacian with the initial data $f$.
2. Using functional analytic methods (e.g. Lax-Milgram) to find weak solutions $u \in H^1(\Omega)$ (for more on this, see [2].

This note is concerned with an aspect of the second method: once we have a solution in $H^1(\Omega)$, can we say that this solution is in fact more regular?

## A Priori Schauder Estimates

We start by finding "a priori" estimates: that is, given a smooth solution $u \in C^\infty(\Omega)$, we want to determine bounds on some norm $||u||_{C^{k,\alpha}(\Omega)}$. Once we have done that, the hope is that we can translate these bounds to less regular solutions.

The main "a priori" estimate that we will try to generalize is the following:

**Theorem 1.** *Let $\alpha \in (0,1)$, $u(x)$ be a smooth solution to:*

$$\Delta u(x) = f \qquad x \in B_R$$

*where $f \in C^{0,\alpha}(B_R)$. Then, we have an estimate:*

$$||u||_{C^{2,\alpha}(B_{\frac{R}{2}})} \leq C(R)(||u||_{L^\infty(B_R)} + ||f||_{C^{0,\alpha}(B_R)})$$

There is an important covering argument that lets us move from an estimate in a ball of radius $\frac{R}{2}$ to an estimate in a ball of any radius $r \in (0, R)$:

**Theorem 2.** *Let $\alpha \in (0,1)$, $u(x)$ be a smooth solution to:*

$$\Delta u(x) = f \qquad x \in B_R$$

*where $f \in C^{0,\alpha}(B_R)$. Then, for any $r \in (0, R)$, we have an estimate:*

$$||u||_{C^{2,\alpha}(B_r)} \leq C(R,r)(||u||_{L^\infty(B_R)} + ||f||_{C^{0,\alpha}(B_R)})$$

*Proof.* Firstly, if $r \leq \frac{R}{2}$, then there is nothing to be said, so we will assume that $r$ is very close to $R$. The idea is to cover $B_r$ with finitely many smaller balls.

Let $z = (R-r)/2$. Then, by compactness, we may cover $B_r$ with finitely many balls $B_z(x_i)$ for $x_i \in B_r, 1 \leq i \leq m$. Note that $B_z(x_i) \subset B_R$, so applying Theorem 1 (translated) to the balls $B_z(x_i)$), we get:

$$||u||_{C^{2,\alpha}B_z(x_i))} \leq C(R,r)(||u||_{L^\infty(B_{R-r}(x_i))} + ||f||_{C^{0,\alpha}(B_{R-r}(x_i))}) \leq$$
$$C(R,r)(||u||_{L^\infty(B_{R-r}(x_i))} + ||f||_{C^{0,\alpha}(B_{R-r}(x_i))})$$

Finally, as the balls $B_z(x_i)$ cover $B_r$, we get:

$$||u||_{C^{2,\alpha}(B_r)} \leq \sum_{i=1}^{m} ||u||_{C^{2,\alpha}B_z(x_i))} \leq$$
$$mC(R,r)(||u||_{L^\infty(B_{R-r}(x_i))} + ||f||_{C^{0,\alpha}(B_{R-r}(x_i))})$$

□

Note that the constant $C(R,r)$ blows up as $r$ approaches $R$.

## Supporting Lemmas

The first lemma we will need is the following special case of the Arzela-Ascoli theorem:

**Theorem 3.** *(Arzela-Ascoli) Let $\Omega \subset \mathbb{R}^n$, and let $\{u_k\}$ be a sequence of functions that are uniformly bounded:*

$$||u_k||_{C^{0,\alpha}(\overline{\Omega})} \leq C$$

*Then, $\exists$ a subsequence $\{u_{k_j}\}$ that converges uniformly to a function $u$.*

The proof of this follows easily from the classical Arzela-Ascoli theorem, noting that equicontinuity is given by the Holder condition.

**Theorem 4.** *Let $\{u_j\}$ be a sequence of funtions that converge uniformly to $u$ in $\overline{\Omega} \subset \mathbb{R}^n$, and assume that we have a bound:*

$$||u_j||_{C^{k,\alpha}(\overline{\Omega})} \leq C$$

*Then, $u \in C^{k,\alpha}(\overline{\Omega})$, and:*

$$||u||_{C^{k,\alpha}(\overline{\Omega})} \leq C$$

*Proof.* Firstly, consider the case $k = 0$. Note that $u$ is continuous as the uniform limit of continuous functions. Then, for any $j$, we have

$$||u_j||_{L^\infty(\overline{\Omega})} + sup_{x \neq y, x, y \in \overline{\Omega}} \frac{|u_j(x) - u_j(y)|}{|x-y|^\alpha} \leq C$$

As the limit is uniform, taking $\{u_j\} \to u$, we obtain the desired result.

Now, consider $k \geq 1$. Then, note that as $\Omega$ is sufficiently regular, we have inclusions:

$$C^{m,\alpha}(\overline{\Omega}) \subset C^{k,\alpha}(\overline{\Omega}) \ \forall \ m \leq k$$
$$||u_j||_{C^{m,\alpha}(\overline{\Omega})} \leq C$$

So, by recursively applying Theorem 3 to partial derivatives, after passing through many subsequences, we may find a subsequence $\{u_{j_\ell}\}$ such that $\{D^m u_{j_\ell}\}$ converges uniformly to $D^m u$ in $\overline{\Omega}$ for $m \leq k$. So, as before, we may take limits in the inequality:

$$||u_{j_\ell}||_{C^k(\overline{\Omega})} + sup_{x \neq y, x, y \in \overline{\Omega}} \frac{|D^k u_{j_\ell}(x) - D^k u_{j_\ell}(y)|}{|x-y|^\alpha} \leq C$$

to obtain the desired result. □

# The Main Theorem

Using the a priori Schauder estimates, via a standard mollifier argument, we can make the following extension. Our approach is derived from the one found in [3].

**Theorem 5.** *Let $\alpha \in (0,1)$, $u(x) \in H^1(B_1) \cap L^\infty(B_1)$ be a weak solution to:*

$$\Delta u(x) = f \qquad x \in B_1$$

*where $f \in C^{0,\alpha}(B_1)$. Then, for any $r \in (0,1), u \in C^{2,\alpha}(B_r)$ and we have an estimate:*

$$||u||_{C^{2,\alpha}(B_r)} \leq C(r)(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)})$$

Note that the constant $C(r)$ blows up as $r$ approaches 1.

*Proof.* Let $r \in (0,1)$. Consider the standard mollifier:

$$\phi(x) = \begin{cases} C(n) e^{(\frac{1}{||x||^2 - 1})} & -1 \leq ||x|| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where the constant $C(n)$ is chosen such that $\int_{\mathbb{R}^n} \phi(x) dx = 1$. Now, for $\epsilon < \frac{1-r}{2} = \epsilon_0$, define:

$$\phi_\epsilon(x) := \frac{1}{\epsilon^n} \phi(\frac{x}{\epsilon})$$
$$u_\epsilon := u * \phi_\epsilon$$
$$f_\epsilon = f * \phi_\epsilon$$

Note that $u_\epsilon \in C^\infty$. Further, these functions satisfy:

$$\Delta u_\epsilon = f * \phi_\epsilon = f_\epsilon$$

(for a review of the interaction between (weak) derivatives & convolution, see [1]).

Now, $u_\epsilon$ is $C^\infty$, so by applying Theorem 1 with $R = 1 - \epsilon_0$, we obtain:

$$||u_\epsilon||_{C^{2,\alpha}(B_{\frac{1-\epsilon_0}{2}})} \leq C(||u_\epsilon||_{L^\infty(B_{1-\epsilon_0})} + ||f_\epsilon||_{C^{0,\alpha}(B_{1-\epsilon_0})}) \tag{1}$$

After applying the covering argument outlined in Theorem 2 to move from $B_{\frac{1-\epsilon_0}{2}}$ to $B_r$, we get:

$$||u_\epsilon||_{C^{2,\alpha}(B_r)} \leq C(||u_\epsilon||_{L^\infty(B_{1-\epsilon_0})} + ||f_\epsilon||_{C^{0,\alpha}(B_{1-\epsilon_0})}) \tag{2}$$

where $C = C(\frac{1-\epsilon_0}{2}, r)$. Now, note that for $x \in B_{1-\epsilon_0}$, we have:

$$u_\epsilon(x) = \int u(y)\phi_\epsilon(x-y)dy = \int u(x-y)\phi_\epsilon(y)dy = \int_{B_\epsilon} u(x-y)\phi_\epsilon(y)dy \leq$$

$$||u||_{L^\infty(B_1)} \int_{B_\epsilon} \phi_\epsilon(y)dy = ||u||_{L^\infty(B_1)}$$

So, by taking the sup:

$$||u_\epsilon||_{L^\infty(B_{1-\epsilon_0})} \leq ||u||_{L^\infty(B_1)} \tag{3}$$

Similarly, for $x, y \in B_{1-\epsilon_0}$, we see:

$$|f_\epsilon(x) - f_\epsilon(y)| = |\int_{B_\epsilon} (f(x-z) - f(y-z)(\phi_\epsilon(z)dz| \leq$$

$$\int_{B_\epsilon} |f(x-z) - f(y-z)|\phi_\epsilon(z)dz \leq$$

$$||f||_{C^{0,\alpha}(B_1)}|x-y|^\alpha \int_{B_\epsilon} \phi_\epsilon(z)dz = ||f||_{C^{0,\alpha}(B_1)}|x-y|^\alpha$$

and for $x \in B_{1-\epsilon_0}$:

$$f_\epsilon(x) = \int f(y)\phi_\epsilon(x-y)dy = \int f(x-y)\phi_\epsilon(y)dy = \int_{B_\epsilon} f(x-y)\phi_\epsilon(y)dy \leq$$

$$||f||_{L^\infty(B_1)} \int_{B_\epsilon} \phi_\epsilon(y)dy = ||f||_{L^\infty(B_1)} \leq ||f||_{C^{0,\alpha}(B_1)}$$

Combining these estimates, we see that we get a bound:

$$||f_\epsilon||_{C^{0,\alpha}(B_{1-\epsilon_0})} \leq 2||f||_{C^{0,\alpha}(B_1)} \tag{4}$$

Combining 3 & 4 with 2, we get a uniform bound:

$$||u_\epsilon||_{C^{2,\alpha}(B_r)} \leq C(||u||_{L^\infty(B_1)} + ||f||_{C^{0,\alpha}(B_1)}) \tag{5}$$

Finally, note that $\{u_\epsilon\}$ converges to $u$ pointwise as $\epsilon \to 0$. By using Theorem 3 and passing to a subsequence, we have a subsequence $\{u_{k_j}\}$ that converges uniformly to $u$. Finally, by applying Theorem 4, we see that $u \in C^{2,\alpha}(B_r)$, and:

$$||u||_{C^{2,\alpha}(B_r)} \leq C$$

as desired. □

# References


[1] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011, pp. xiv+599. ISBN: 978-0-387-70913-0.

[2] Lawrence C. Evans. *Partial differential equations.* Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1998, pp. xviii+662. ISBN: 0-8218-0772-2.

[3] Xavier Ros-Oton and Xavier Fernandez-Real. *Regularity Theory for Elliptic PDE.* 2020.