Existence of Weak Solutions for Parabolic PDEs

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1 Introduction

Let $\mathcal{L}(u)(x) = -div(A(x)\nabla u(x)) + b\cdot\nabla u + cu$ be an elliptic operator. Then, consider the following parabolic problem:

$$\begin{aligned} u_t + \mathcal{L}(u) = f \qquad &\in \Omega \times [0, t] \\ u = 0 \qquad &\in \partial\Omega \times [0, T] \\ u = g \qquad &\in \Omega \times \{t = 0\} \end{aligned}$$

for $g \in L^2(\Omega)$ given, A uniformly bounded & elliptic, $b, c \in L^\infty(\Omega)$. Fixing $0 \le t \le T$ and integrating by parts, we define:

$$B[u, v; t] = \int_\Omega (A(x)\nabla u\nabla v + b\cdot\nabla uv + cuv)dx$$

Then, we define a weak solution as follows:

Definition. *We call a function $u \in L^2(0, T; H_0^1(\Omega))$, with $u' \in L^2(0, T, ; H^{-1}(\Omega))$ a weak solution to the parabolic problem if:*

1. $\langle u', v\rangle + B[u, v; t] = (f, v)\ \forall\ v \in H_0^1(\Omega)$*, and a.e.* $0 \le t \le T$
2. $u(0) = g$

where $\langle\ ,\ \rangle$ denotes the duality pairing of $H_0^1(\Omega)$ and H^{-1}, and $(\ ,\ )$ denotes the L^2 inner product.

There are two relatively orthogonal ways to show existence of weak solutions: the first is via discretization of the time variable, while the second is via Galerkin approximation: making a limiting argument in the space variables. In this note, we showcase how to do both of these methods.

2 Time Discretization

The strategy of discretization of the time variable is to solve a sequence of elliptic problems, and piece them together to approximate a solution to the parabolic problem. We make this rigorous as follows: taking our original formulation $u_t + \mathcal{L}(u) = f$ and moving $\mathcal{L}(u)$ to the other side, we get:

$$u_t = Lu + f, \qquad \text{where } Lu = div(A(x)\nabla u) - b\cdot\nabla u - cu$$

So, it will suffice to show that we can find a weak solution to the problem:

$$\begin{aligned} u_t = L(u) + f \qquad &\in \Omega \times [0, t] \\ u = 0 \qquad &\in \partial\Omega \times [0, T] \\ u = g \qquad &\in \Omega \times \{t = 0\} \end{aligned}$$

for the elliptic operator $Lu = div(A(x)\nabla u(x)) - b \cdot \nabla u - cuv$. Now, consider the sequence of elliptic problems:

$$L(u_k) = \frac{u_k - u_{k-1}}{\tau} + f \qquad x \in \Omega \qquad \textbf{(*)}$$
$$u_k = 0 \qquad x \in \partial\Omega$$

where $u_0 = g \in L^2(\Omega)$. Firstly, we will use Lax-Milgram to show that there is a unique sequence of weak solutions to these elliptic problems:

The variational form of **(*)** is exactly:

$$B_\tau[u_k, v] = \langle \frac{u_{k-1}}{\tau} + f, v \rangle \qquad \forall\ v \in H_0^1(\Omega)$$

where $B_\tau[u,v] = \int_\Omega A\nabla u \nabla v dx + \int_\Omega b \cdot \nabla uv dx + \int_\Omega cuv dx + \frac{1}{\tau}\int_\Omega uv dx$, where $\frac{u_{k-1}}{\tau} + f \in L^2(\Omega)$ is given. Now, by checking the hypotheses of the Lax-Milgram theorem (a la Section 6.2 in [1]), we see that for τ sufficiently small, there is a unique solution $u_k \in H_0^1(\Omega)$.

Now, for $\tau = \frac{T}{N}$, define the piecewise linear function:

$$u^\tau(t,x) = \sum_{k=1}^{N-1} u_k(x)\chi_{[t_k, t_{k+1})}$$

where $t_k = k\tau$. The claim is that u^τ converges weakly in $L^2(0,T;H_0^1(\Omega)$ to a weak solution of the problem:

$$u_t = L(u) \qquad \in \Omega \times [0,t]$$
$$u = 0 \qquad \in \partial\Omega \times [0,T]$$
$$u = g \qquad \in \Omega \times \{t = 0\}$$

Indeed, to show this, we first need to be able to pass to limits. Considering $L(u_k) = \frac{u_k - u_{k-1}}{\tau}$ and integrating both sides against u_k:

$$\langle L(u_k), u_k)\rangle = -\langle A\nabla u_k, \nabla u_k\rangle - \langle b \cdot \nabla u_k, u_k\rangle - \langle cu_k, u_k\rangle$$
$$\leq -C||u_k||^2_{H_0^1(\Omega)} + \mu||u_k||^2_{L^2(\Omega)}$$

for $C, \mu \geq 0$ by Young's inequality & ellipticity. On the other hand:

$$\langle \frac{u_k - u_{k-1}}{\tau} + f, u_k\rangle = \frac{1}{\tau}\int_\Omega u_k^2 - u_k u_{k-1} dx + \int_\Omega f u_k dx \geq \frac{1}{2\tau}\int_\Omega u_k^2 - u_{k-1}^2 + \int_\Omega f u_k dx =$$
$$\frac{1}{2\tau}(||u_k||^2_{L^2(\Omega)} - ||u_{k-1}||^2_{L^2(\Omega)}) + \int_\Omega f u_k dx$$

Equating these and using Cauchy-Schwarz, we get:

$$\frac{1}{2\tau}(||u_k||^2_{L^2(\Omega)} - ||u_{k-1}||^2_{L^2(\Omega)}) \leq -C||u_k||^2_{H_0^1(\Omega)} + \mu||u_k||^2_{L^2(\Omega)} + ||f||^2_{L^2(\Omega)}||u_k||^2_{L^2} =$$
$$-C||u_k||^2_{H_0^1(\Omega)} + C'||u_k||^2_{L^2(\Omega)}$$

Ignoring the H_0^1 term for a second, we see by the discrete form of Gronwall's inequality that, for $1 - 2\tau C' > 0$:

$$||u_k||^2_{L^2(\Omega)} \leq (1-2')^{-k}||g||^2_{L^2(\Omega)} + \frac{1}{\mu}((1-2\tau\mu)^{-k} - 1)||f||^2_{L^2(\Omega)} \leq ||g||^2_{L^2(\Omega)} + C||f||^2_{L^2(\Omega)}$$

Thus, for sufficiently small τ, we see the above inequality actually tells us:

$$||u_k||^2_{H^1_0(\Omega)} \leq C||g||^2_{L^2(\Omega)} + \frac{C}{2\tau}(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(\Omega)})$$

Finally, by integrating u^τ from 0 to T: $||u^\tau||^2_{L^2(0,T,H^1_0(\Omega))} = \sum_{k=0}^{N-1} \tau||u_k||^2_{H^1_0(\Omega)} \leq \sum_{k=0}^{N-1} \tau(C||g||^2_{L^2(\Omega)} + \frac{C}{2\tau}(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(\Omega)})) \leq C(T)||g||^2_{L^2(\Omega)} + \frac{C}{2}(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(\Omega)})$, showing that u^τ is uniformly bounded in $L^2(0, T, H^1_0(\Omega))$.

So, by Banach-Alaoglu, we see that there is a subsequence (we'll just keep calling it $\{u^\tau\}$) that converges weakly to $u \in L^2(0, T, H^1_0(\Omega)$. It remains to show that u is a weak solution to the original problem. Let $\phi \in C^1_c(0, T; H^1_0(\Omega))$. Then, letting $\langle\ ,\ \rangle$ denote the inner product w.r.t. $L^2(0, T; H^1_0(\Omega))$, and $(\ ,\ )$ denote the inner product w.r.t. $L^(\Omega)$:

$$\langle u_t, \phi\rangle = \int_0^T \langle u_t, \phi\rangle_{H^1_0(\Omega)} dt = -\int_0^T \langle u, \phi_t\rangle_{H^1_0(\Omega)} dt =$$
$$-\langle u, \phi_t\rangle = lim_{\tau\to 0} - \langle u^\tau, D^{-\tau}_t \phi\rangle = lim_{\tau\to 0}\langle D^{-\tau}_t(u^\tau), \phi\rangle =$$
$$lim_{\tau\to 0}\langle L(u^\tau) + f, \phi\rangle = lim_{\tau\to 0} \int_0^T -B[u^\tau, \phi; t]dt + (f, v) = \int_0^T -B[u, \phi; t]dt + \int_0^T (f, v)dt$$

By density, this result also holds for any $\phi \in L^2(0, T, H^1_0(\Omega))$. So, in particular, $\langle u_t, \phi\rangle_{H^1_0(\Omega)} = -B[u, \phi; t] + (f, v)$ for all $\phi \in H^1_0(\Omega)$ and a.e. $0 \leq t \leq T$. Finally, the initial and boundary conditions are satisfied by design: indeed, $u = 0$ on $\partial U \times [0, T]$ as $u \in L^2(0, T; H^1_0(\Omega))$. Finally, $u_0 = g$, as $u^\tau(0, x) = g(x)$ for all x.

3 Galerkin Approximation

Another strategy to show the existence of weak solutions is to instead approximate in the space variables, without touching the time variable. This approach is the one used in [1].

Let $\{w_k\}_{k=1}^\infty$ be an orthonormal basis of $L^2(\Omega)$ that is also an orthogonal basis of $H^1_0(\Omega)$. Let $m \in \mathbb{N}$. The strategy of the Galerkin approximation is to find a function $u_m : [0, T] \to H^1_0(\Omega)$ of the form:

$$u_m(t) = \sum_{k=1}^m d^k_m(t)w_k$$

This function should solve the problem:

$$(u'_m, w_k) + B[u_m, w_k; t] = (f, w_k) \qquad \text{for a.e. } 0 \leq t \leq T, \qquad k = 1, ..., m$$
$$d^k_m(0) = (g, w_k) \qquad k = 1, ..., m$$

As Evans remarks, this u_m solves the "projection" of the original problem onto the finite dimensional subspace $span\{w_1, ..., w_m\}$.

We begin by showing that for each m, there actually exists a solution to the problem posed above.

Theorem 1. *For each integer $m \in \mathbb{N}$, there exists a unique function u_m of the desired form solving the above problem.*

Proof. Assuming that $u_m(t) = \sum_{k=1}^m d^k_m(t)w_k$, we have:

$$(u'_m(t), w_k) = {d^k_m}'(t)$$
$$B[u_m, w_k; t] = \sum_{\ell=1}^m B[w_\ell, w_k; t]d^\ell_m(t)$$

Then, the problem is equivalent to the linear system of m ODEs:

$$d_m^k{}'(t) + \sum_{\ell=1}^{m} B[w_\ell, w_k; t] d_m^\ell(t) = (f(t), w_k) \qquad k = 1, ..., m$$
$$d_m^k(0) = (g, w_k) \qquad k = 1, ..., m$$

By standard existence & uniqueness theory for ODEs, there is a unique absolutely continuous set of m functions d_m^k solving the problem. □

Now, just as before, we need energy estimates to pass to limits:

Theorem 2. *$\{u_m\}$ is uniformly bounded in $L^2(0, T; H_0^1(\Omega))$.*

Proof. The proof utilizes the specific form that we have derived for u_m. u_m solve the problem:

$$(u_m', w_k) + B[u_m, w_k; t] = (f, w_k) \qquad \text{for a.e. } 0 \leq t \leq T, \qquad k = 1, ..., m$$
$$d_m^k(0) = (g, w_k) \qquad k = 1, ..., m$$

Multiplying the first equation by d_m^k and summing from 1 to m, we get:

$$(u_m', u_m) + B[u_m, u_m; t] = (f, u_m) \qquad \text{for a.e. } 0 \leq t \leq T$$

Now, we use the following:

1. $(u_m' u_m) = \frac{d}{dt}(\frac{1}{2}||u||^2_{L_2(\Omega)})$
2. $\beta||u_m||^2_{H_0^1(\Omega)} \leq B[u_m, u_m; t] + \gamma||u_m||^2_{L^2(\Omega)}$
3. $|(f, u_{m|} \leq \frac{1}{2}||f||^2_{L^2(\Omega)} + ||u_m||^2_{L^2(\Omega)}$

Putting these together and re-arranging terms:

$$\frac{d}{dt}(||u||^2_{L_2(\Omega)}) + 2\beta||u_m||^2_{H_0^1(\Omega)} \leq C_1||u_m||^2_{L^2(\Omega)} + C_2||f||^2_{L^2(\Omega)}$$

By ignoring the $H_0^1(\Omega)$ term and using Gronwall, we get a uniform bound:

$$||u_m||^2_{L^2(0,T;H_0^1(\Omega)} \leq C(||g||^2_{L^2(\Omega)} + ||f||^2_{L^2(0,T;L^2(\Omega))})$$

□

We also need to pass to limits with $\{u_m'\}$:

Theorem 3. *$\{u_m'\}$ is uniformly bounded in $L^2(0, T; H^{-1}(\Omega))$.*

Proof. Firstly, we show that the operator norm $||u_m'||_{H^{-1}(\Omega)}$ is uniformly bounded. Let $v \in H_0^1(\Omega)$ with $||v||_{H_0^1(\Omega)} \leq 1$. Then, we can write:

$$v = v^1 + v^2$$

where $v^1 \in S = span\{w_1, ..., w_k\}$ and $v^2 \in S^\perp$. Note $||v^1||_{H_0^1(\Omega)} \leq 1$ as the norm of the projection operator is ≤ 1. Then, by multiplying and summing the equations as before, we get:

$$(u_m', v^1) + B[u_m, v^1; t] = (f, v^1)$$

Note that as u_m is a linear combination of w_k for $1 \leq k \leq m$, we also have:

$$\langle u_m', v\rangle = (u_m', v) = (u_m', v^1) = (f, v^1) - B[u_m, v^1; t]$$

By Cauchy-Schwarz and the boundedness of B, this implies:

$$|\langle u_m', v\rangle| \leq C(||f||_{L^2(\Omega)} + ||u_m||_{H_0^1(\Omega)}$$

Taking the supremum over all such v:

$$||u_m'||_{H^{-1}(\Omega)} \leq C(||f||_{L^2(\Omega)} + ||u_m||_{H_0^1(\Omega)}$$

Finally, we simply integrate this bound to get a bound in $L^2(0, T; H^{-1}(\Omega)$:

$$\int_0^T ||u_m'||^2_{H^{-1}(\Omega)} dt \leq C \int_0^T (||f||_{L^2(\Omega)} + ||u_m||_{H_0^1(\Omega)}) dt \leq C||f||^2_{L^2(0,T;L^2(\Omega))} + C||u_m||^2_{L^2(0,T;H_0^1(\Omega)} \leq$$
$$C(||f||^2_{L^2(0,T;L^2(\Omega))} + ||g||^2_{L^2(\Omega)})$$

by using our uniform bounds for $||u_m||_{L^2(0,T;H_0^1(\Omega)}$. □

Finally, we can pass to limits and show that the limiting function is a solution to the original problem. By Banach-Alaoglu, there is a subsequence (again, we'll just keep calling it $\{u_m\}$ that converges weakly to u in $L^2(0, T : H_0^1(\Omega))$. Applying Banach-Alaoglu again, we get a subsubsequence such that further, $\{u_m'\}$ converges weakly in $L^2(0, T : H^{-1}(\Omega))$ to u'.

Now, fix $N \in \mathbb{N}$ and choose a function $v \in C^1(0, T; H_0^1(\Omega))$ of the form:

$$v(t) = \sum_{k=1}^{N} d^k(t) w_k$$

where d^k are smooth functions. Choosing $m \geq N$, multiplying $(u_m', w_k) + B[u_m, w_k; t] = (f, w_k)$ by d^k, and summing from 1 to m:

$$\langle u_m', v\rangle + \int_0^T B[u_m, v; t] dt = \langle f, v\rangle$$

where $\langle\ ,\ \rangle$ denotes the $L^2(0, T; H_0^1(\Omega))$ inner product. Passing to weak limits:

$$\langle u', v\rangle + \int_0^T B[u, v; t] dt = \langle f, v\rangle \iff$$
$$\int_0^T \langle u', v\rangle_{H_0^1(\Omega)} + \int_0^T B[u, v; t] dt = \int_0^T (f, v) dt$$

As such v are dense, this equality holds for all $v \in L^2(0, T; H_0^1(\Omega))$. In particular:

$$\langle u', v\rangle_{H_0^1(\Omega)} + B[u, v; t] = (f, v)$$

for all $v \in H_0^1(\Omega)$ and a.e. $0 \leq t \leq T$.

It remains to show that $u(0) = g$. By choosing $v \in C^1(0, T; H_0^1(\Omega))$ with $v(T) = 0$ and integrating by parts, we have:

$$\int_0^T \langle v', u\rangle_{H_0^1(\Omega)} + \int_0^T B[u, v; t] dt = \int_0^T (f, v) dt + (u(0), v(0))$$

However, doing this with u_m and them passing to weak limits, we see:

$$\int_0^T \langle v', u\rangle_{H_0^1(\Omega)} + \int_0^T B[u, v; t] dt = \int_0^T (f, v) dt + (g, v(0))$$

as $\{u_m(0)\} \to g$ in $L^2(\Omega)$. As $v(0)$ is arbitrary, this shows $g = u(0)$.

4 Uniqueness of Solutions

We end with Evans' proof that the solution to such a problem is unique. This does not see any of the specific details to either method above.

Theorem 4. *Let u, v be weak solutions to the problem:*

$$\begin{aligned} u_t + \mathcal{L}(u) = f &\qquad \in \Omega \times [0, t] \\ u = 0 &\qquad \in \partial\Omega \times [0, T] \\ u = g &\qquad \in \Omega \times \{t = 0\} \end{aligned}$$

Then, $u = v$.

Proof. It suffices to show that the only solution to the problem with $f = g = 0$ is $u = 0$. So, letting u be a weak solution to the problem, and testing the equality with u, we get:

$$\langle u', u\rangle + B[u, u; t] = 0 \implies \frac{d}{dt}(\frac{1}{2}||u||^2_{L_2(\Omega)}) + B[u, u; t] = 0$$

By the energy estimates for B :

$$\frac{d}{dt}(\frac{1}{2}||u||^2_{L_2(\Omega)}) + B[u, u; t] = 0 \implies \frac{d}{dt}(\frac{1}{2}||u||^2_{L_2(\Omega)}) \leq \gamma||u||^2_{L^2(\Omega)}$$

By Gronwall, this implies $u = 0$. □

References

[1] Lawrence C. Evans. *Partial differential equations.* Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 1998, pp. xviii+662. ISBN: 0-8218-0772-2.