Ideal structure of the C^*-algebra of Thompson group T

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Abstract

In a recent paper Uffe Haagerup and Kristian Knudsen Olesen show that for Richard Thompson's group T, if there exists a finite set H which can be decomposed as disjoint union of sets H_1 and H_2 with $\sum_{g \in H_1} \pi(g) = \sum_{h \in H_2} \pi(h)$ and such that the closed ideal generated by $\sum_{g \in H_1} \lambda(g) - \sum_{h \in H_2} \lambda(h)$ coincides with $C^*_\lambda(T)$, then the Richard Thompson group F is not amenable. In particular, if $C^*_\lambda(T)$ is simple then F is not amenable. Here we prove the converse, namely, if F is not amenable then we can find two sets H_1 and H_2 with the above properties. The only currently available tool for proving simplicity of group C^*-algebra is Power's condition. We show that it fails for $C^*_\lambda(T)$ and present an apparent weakening of that condition which could potentially be used for various new groups H to show the simplicity of $C^*_\lambda(H)$. While we use our weakening in the proof of the first result, we also show that the new condition is still too strong to be used to show the simplicity of $C^*_\lambda(T)$. Along the way, we give a new application of the Ping-Pong Lemma to find free groups as subgroups in groups of homeomorphisms of the circle generated by elements with rational rotation number.

1 Introduction

There has been a long-standing interest in the question of the amenability of Richard Thompson's group F, introduced in Thompson's notes of 1965 (see the survey [5] for a general background on the three Thompson groups $F < T < V$), with many failed attempts to prove either the amenability or non-amenability of F. The groups $F < T < V$ arise in many areas of mathematics for reasons which are not entirely understood. One plausible explanation is that they express in some fundamental way connections through Category Theory with associativity and versions of commutativity (see [2, 13, 11] for some discussion of these connections), which of course are fundamental aspects of any theory involving products. Regardless of the cause, it is still the case that these groups arise naturally in many areas of mathematics including dynamics, logic, topology, and more obviously geometric group theory. One fetching example of such an appearance is in e.g., the relationship between the group F and the theory of associahedra, and in particular, the theory related to the the proof of the Four Colour Theorem [3]. In any case, it is well known now that the Richard Thompson groups are fundamental. In this paper, we will be investigating some structures related to T which have implications towards the amenability of F, by exploring some related questions from the theory of C^* algebras.

Since the foundational paper [18] of Powers, there has been a long-standing interest in whether there could exist a group G with unique trace, for which its reduced C^* algebra $C^*_\lambda(G)$ is not simple. The paper [18] contains various conditions on a group which imply uniqueness of trace and/or simplicity of the algebra. (See Section 3 for a discussion of an often-used condition of Powers.) The question of the equivalence of these two properties is explicitly stated in the papers [6], [7] of Pierre de la Harpe, and his

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excellent overview of the problem [8]. For more on the general problem we refer the reader to the paper [19] of Robin Tucker-Drob.

Uffe Haagerup and Kristian Knudsen Olesen in [15] show that the simplicity of $C^*_\lambda(T)$ implies the non-amenability of F via a construction given below (see Theorem 1 below), while it is well known that T is a group with unique trace. Thus, in this paper we begin to investigate the ideal structure of $C^*_\lambda(T)$.

Haagerup and Olesen's idea showing that that the simplicity of the algebra $C^*_\lambda(T)$ implies the non-amenability of F runs as follows. Consider T "acting" on the interval $[0, 1]$. Assume that the stabiliser of 0, which is the standard copy of the Thompson group F in T, is amenable. Since the action of T on $\mathbb{Z}[\frac{1}{2}]$ is transitive, we have that the representation induced by this action, $\pi : T \to B(l_2(\mathbb{Z}[\frac{1}{2}]))$, is weakly contained in the left regular representation. From this one sees that there is a unique $*$-homomorphism from $C^*_\lambda(T)$ into the C^*-algebra generated by $\pi(T)$.

Consider now a finite subset H of T so that $H = H_1 \sqcup H_2$ and with

$$\sum_{g \in H_1} \pi(g) - \sum_{h \in H_2} \pi(h) = 0.$$

The simplicity of $C^*_\lambda(T)$ now implies that the ideal generated by $\sum_{g \in H_1} \lambda(g) - \sum_{h \in H_2} \lambda(g)$ is proper. However, this is not possible since π is non-trivial, so F must be non-amenable.

Our main result is to show a (partial) converse of the above program (we except the simplicity of $C^*_\lambda(T)$).

Theorem 1. *Thompson's group F is non-amenable if and only if there exists a finite set H which can be decomposed as disjoint union of sets H_1 and H_2 with $\sum_{g \in H_1} \pi(g) = \sum_{h \in H_2} \pi(h)$ and such that the closed ideal generated by $\sum_{g \in H_1} \lambda(g) - \sum_{h \in H_2} \lambda(h)$ coincides with $C^*_\lambda(T)$.*

The rest of this article proceeds as follows.

In Section 2 we present the proof of the main result, modulo our Lemma 13.

In the Section 3 we discuss a theorem of Powers and another of Kesten which together produce Condition 1, a main test which is often used to detect the simplicity of $C^*_\lambda(G)$ for a group G. We show that Condition 1 fails to apply to Thompson's group T, and we offer a new test (Condition 2) which appears to be a natural weakening of Condition 1, and which may be of use for various groups G to show that $C^*_\lambda(G)$ is simple. Like Condition 1, the new Condition 2 gives the simplicity of $C^*_\lambda(G)$ for a group G if for all finite subsets H of $G \setminus \{e\}$, we can carry out a certain construction (creating large free subgroups in a certain way from H). We also show here that Condition 2 is still not weak enough to show that the algebra $C^*_\lambda(T)$ is simple, while giving some evidence that Condition 2 is properly weaker than the commonly used Condition 1.

The authors are unaware of any group with unique trace, for which Condition 1 fails. We believe T is a first example of such a group.

In Section 4 we provide a short discussion of the historical Ping-Pong Lemma and we prove a version of the Ping-Pong Lemma (Lemma 12) useful for detecting free subgroups when considering a group generated by two homeomorphisms of S^1 with rational rotation numbers. Lemma 12 is an essential ingredient in the proof of Lemma 13.

In Section 5, we give proofs that we can carry out the construction of Condition 2 for many cases of finite $H \subset T \setminus \{e\}$, including the cases we need to prove Lemma 13. We also describe some cases of H where we cannot carry out the construction of Condition 2, but where related constructions do produce large free subgroups.

In our final Section 6 we state some remaining questions which we find interesting.

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2 Non-amenability of F and a condition on ideals of its C^*-algebra.

Theorem 2. *The Thompson group F is not amenable if and only if there exists a finite set H which can be decomposed as disjoint union of sets H_1 and H_2 with $\sum_{g \in H_1} \pi(g) = \sum_{h \in H_2} \pi(h)$ and such that the closed ideal generated by $\sum_{g \in H_1} \lambda(g) - \sum_{h \in H_2} \lambda(h)$ coincides with $C^*_\lambda(T)$.*

Proof. One part of the theorem follows from the draft [15] of Haagerup and Olesen.

It is left to show that if F is not amenable then there is a finite set H which satisfies the conditions of the theorem. Let x_1 and x_2 be the standard generators of the copy of Thompson's group F in T, supported in $[\frac{1}{2}, 1] \subset S^1$, with supports in $[\frac{1}{2}, 1]$ and $[\frac{3}{4}, 1]$ respectively. Let $g_1 = x_1^r$ and $g_2 = x_2^r$ be the conjugated copies of these generators, where r is rotation by $1/2$, so that g_1 and g_2 act on the interval $[0, \frac{1}{2}] \subset S^1$ and generate a copy of F there, with trivial action on $[\frac{1}{2}, 1] = [\frac{1}{2}, 0]$. Define $H = \{e, g_1, g_2, x_1, x_2, g_1 x_1, g_2 x_2\}$. Define $H_1 = \{g_1, g_2, x_1, x_2\}$ and H_2 to be the rest of the set H. Obviously, $\sum_{g \in H_1} \pi(g) - \sum_{h \in H_2} \pi(h) = 0$.

Let us show that the ideal, J, generated by $\sum_{g \in H_1} \lambda(g) - \sum_{h \in H_2} \lambda(h)$ is the whole reduced C^*-algebra of T. Note that $\|1 + \lambda(g_1) + \lambda(g_2)\| \leq C < 3$ by assumption. Moreover, the point $p = \frac{7}{8}$ is not fixed by any element from the set $E = \{x_1, x_2, g_1 x_1, g_2 x_2\}$. Thus we can apply Lemma 13 for the set E: let $\varepsilon > 0$ and let g and $c_1, \ldots, c_n \in C_T(g)$ be such that

$$\| \sum_{s \in E} \sum_{i=1}^{n} \lambda((sg)^{c_i}) \| \leq \varepsilon n.$$

Note that the element $b = \lambda(g - \frac{1}{3}[e + g_1 + g_2]g + \frac{1}{3} \sum_{s \in E} sg)$ is in J, thus $\frac{1}{n} \sum_{i=1}^{n} \lambda(c_i) b \lambda(c_i^{-1}) \in J$. The distance between the element $\frac{1}{n} \sum_{i=1}^{n} \lambda(c_i) b \lambda(c_i^{-1})$ and $\lambda(g)$ is strictly smaller then 1 for large n. Indeed,

$$\|\lambda(g) - b\| \leq \frac{1}{3}\|1 + \lambda(g_1) + \lambda(g_2)\| + \frac{1}{3n}\| \sum_{s \in E} \sum_{i=1}^{n} \lambda((sg)^{c_i}) \| \leq C + \varepsilon.$$

thus we have found an invertible element in J, therefore $J = C_\lambda^*(T)$. $\qquad\square$

3 Powers' test

In [18] Powers' gives the following test for the simplicity of the algebra $C_\lambda^*(G)$ over a group G.

Theorem 3. *If for all non-empty $H \subset G$ with $|H| < \infty$, $e \notin H$ and for all positive integers n there is a set $\{c_1, c_2, \ldots, c_n\} \subset G$ so that*

$$\lim_{n \to \infty} \frac{1}{n}\|\Sigma_{i=1}^{n} \lambda(c_i h c_i^{-1})\| = 0, \ \forall h \in H,$$

then $C_\lambda^*(G)$ is simple.

Let G be a group generated by a finite set S with $S = S^{-1}$, then $\frac{1}{|S|}\|\Sigma_{h \in S} \lambda(h)\|$ is equal to the spectral radius of the simple random walk on the Cayley graph of G with respect to S, denoted by $\rho(G, S)$. The spectral radius of the simple random walk have been computed for many groups. Kesten, [17], showed that if $S = \{g_1, \ldots, g_n\}$ is *a free set*, i.e., $g_1, \ldots, g_n$ are standard generators of the free group of rank n, then the spectral radius is

$$\rho(G, S) = \frac{\sqrt{2n - 1}}{n}$$

Thus the following condition implies the hypothesis of the Theorem 3.

Condition 1. *For all finite subsets $H \subset G$ with $e \notin H$ and for all positive integers n there is a set $\{c_1, c_2, \ldots, c_n\} \subset G$ so that*

$$\langle h^{c_1}, h^{c_2}, \ldots, h^{c_n} \rangle$$

is a free subgroup of G of rank n for all $h \in H$.

If g is a bijection from a set X to itself, denote by $Supp(g) := \{x \in X \mid g \cdot x \neq x\}$ and $Fix(g) := X \backslash Supp(g)$, the support and the set of points fixed by g, respectively.

The following remark holds true for groups of permutations of a set X.

Remark 4. *Let X be a set, and G the group of bijections from X to X. Suppose h_1, $h_2 \in G \backslash \{1\}$ so that $Supp(h_1) \cap Supp(h_2) = \emptyset$. If c_1, $c_2 \in G$ so that $Supp(h_1^{c_1}) \cup Supp(h_1^{c_2}) = X$ then $Supp(h_2^{c_1}) \cap Supp(h_2^{c_2}) = \emptyset$.*

3

Proof. Suppose
$$X = Supp(h_1^{c_1}) \cup Supp(h_1^{c_2})(= c_1 \cdot Supp(h_1) \cup c_2 \cdot Supp(h_1)).$$

If there is $x \in X$ so that $x \in Supp(h_2^{c_1}) \cap Supp(h_2^{c_2})$, then $x = c_1 \cdot y$ and $x = c_2 \cdot z$, where y and z are in $Fix(h_1)$. In particular, $x \in c_1 \cdot Fix(h_1) \cap c_2 \cdot Fix(h_1)$. This implies that $c_1 \cdot Supp(h_1) \cup c_2 \cdot Supp(h_1) \neq X$. □

Remark 4 immediately implies that we cannot use Condition 1 when approaching the question of the simplicity of the algebra $C_\lambda^*(T)$.

Corollary 5. *Suppose that $H \subset T$ admits elements h_1 and h_2 so that $Supp(h_1) \cap Supp(h_2) = \emptyset$. Then for $n \geq 2$ there is no set of elements $\{c_1, c_2 \ldots, c_n\}$ so that $\langle h^{c_1}, h^{c_2}, \ldots, h^{c_n} \rangle$ is a free group on n generators for all $h \in H$.*

Proof. Suppose $H := \{h_1, h_2, \ldots, h_k\}$ is a finite set with cardinality at least two, and h_1 and h_2 are in H so that $Supp(h_1) \cap Supp(h_2) = \emptyset$. Further suppose that $n \geq 2$ is fixed and c_1, $c_2, \ldots$, c_n are chosen so that for all $h \in H$, we have $\langle h^{c_1}, h^{c_2}, \ldots, h^{c_n} \rangle$ is free on n generators. As proven and Brin and Squier's paper [4], the group of piecewise linear homeomorphisms of the unit interval has no non-abelian free subgroups, so we see immediately that $Supp(h_1^{c_1}) \cup Supp(h_1^{c_2}) = S^1$. Now by Remark 4 we know that $Supp(h_2^{c_1}) \cap Supp(h_2^{c_2}) = \emptyset$ Therefore $\langle h_2^{c_1}, h_2^{c_2} \rangle \cong \mathbb{Z} \times \mathbb{Z}$. □

We now offer an apparently weaker version of Condition 1 which will be used throughout the remainder of this article. First, we need a supporting theorem.

Below, let $C_G(g)$ be the centralizer of an element g in G.

Theorem 6. *Let $H \subset G$ be a finite set and there is an element $w \in H$ such that for all positive integers n there is a set $\{c_1, c_2, \ldots, c_n\} \subset G$ and $r, s \in G$ such that $c_i \in C_G(swr)$ for all i and*

$$\lim_{n \to \infty} \frac{1}{n} ||\Sigma_{i=1}^n \lambda(c_i sgrc_i^{-1})|| = 0, \text{ for all } g \in H\backslash\{w\},$$

then for all coefficients β_g indexed by H with $\beta_w \neq 0$, the ideal generated by $\sum_{g \in H} \beta_g \lambda(g)$ is equal to $C_\lambda^*(G)$.

Proof. Let I be an ideal in $C_\lambda^*(G)$ generated by $b := \sum_{g \in H} \beta_g \lambda(g)$. Assume that I is proper. The closure of I is proper, thus we can assume I is closed. Note that $\Sigma_{i=1}^n \lambda(c_i s) b \lambda(rc_i^{-1}) \in I$. Since $c_i \in C_G(swr)$ we have

$$||\lambda(swr) - \frac{1}{\beta_w n}\Sigma_{i=1}^n \lambda(c_i s)b\lambda(rc_i^{-1})|| = \frac{1}{\beta_w n}||\Sigma_{g \in H\backslash\{w\}}\Sigma_{i=1}^n\beta_g\lambda(c_i sgrc_i^{-1})||$$

$$\leq \frac{1}{\beta_w}\Sigma_{g \in H\backslash\{w\}}|\beta_g|\frac{1}{n}||\Sigma_{i=1}^n\lambda(c_i sgrc_i^{-1})||$$

$$= \max(|\beta_g|/\beta_w : g \in H) \cdot \max(\frac{1}{n}||\Sigma_{i=1}^n\lambda(c_i sgrc_i^{-1})|| : g \in H).$$

By our assumptions, the last quantity can be arbitrarily small for large n. Thus there is an element in I which is on distance less then 1 to a unitary operator, this implies that it is invertible and $I = C_\lambda^*(G)$. □

Applying the theorem above to the set $H \cup \{e\}$ shows that the following condition implies simplicity of $C_\lambda^*(G)$:

Condition 2. *For all finite non-empty subsets $H \subset G$, $e \notin H$ and for all positive integers n there are r, $s \in G$ and a set $\{c_1, c_2, \ldots, c_n\} \subset C_G(sr)$ such that the set $\{c_k(sgr)c_k^{-1} : k = 1, \ldots, n\}$ is free for all $g \in H$.*

Condition 1 implies Condition 2 and it seems that the other implication is false. However, Condition 2 is still inadequate for showing that $C_\lambda^*(T)$ is simple.

Lemma 7. *There are g_1, $g_2 \in T\backslash\{e\}$ so that for any r, $s \in T$ there are no elements c_1, c_2, c_3, and $c_4 \in C_T(sr)$ with both $G_1 = \langle (sg_1 r)^{c_1}, (sg_1 r)^{c_2}, (sg_1 r)^{c_3}, (sg_1 r)^{c_4} \rangle$ and $G_2 = \langle (sg_2 r)^{c_1}, (sg_2 r)^{c_2}, (sg_2 r)^{c_3}, (sg_2 r)^{c_4} \rangle$ free on four generators.*

4

Proof. Let g_1, $g_2 \in T$ so that $Supp(g_1) = (0, 1/2)$ and $Supp(g_2) = (1/2, 1)$. Let r and $s \in T$ and suppose c_1, c_2, c_3 and $c_4 \in C_T(sr)$. Set $k_{ij} = (sg_ir)^{c_j}$, for i, $j \in \{1, 2, 3, 4\}$, and suppose that c_1, c_2, c_3 and c_4 were so chosen so that $G_i = \langle k_{i1}, k_{i2}, k_{i3}, k_{i4} \rangle$ is free on four generators for $1 \leq i \leq 2$.

Consider the intervals $X_{i1} = (c_1r^{-1}) \cdot Fix(g_i)$, $X_{i2} = (c_2r^{-1}) \cdot Fix(g_i)$, $X_{i3} = (c_3r^{-1}) \cdot Fix(g_i)$, and $X_{i4} = (c_4r^{-1}) \cdot Fix(g_i)$. If $x_{ij} \in X_{ij}$, then $k_{ij} \cdot x_{ij} = c_jsg_irc_j^{-1} \cdot x_{ij} = (c_jsg_i) \cdot y_{ij} = (c_js) \cdot y_{ij} = (c_jsrc_j^{-1}) \cdot x_{ij} = (sr) \cdot x_{ij}$, as for all i, j we have $y_{ij} \in Fix(g_i)$. That is, k_{ij} acts as sr over X_{ij}.

Further, consider the elements $f_{i,ab} = k_{ia}^{-1}k_{ib}$, where $i \in \{1, 2\}$ and $a \neq b \in \{1, 2, 3, 4\}$. It is immediate that $\langle f_{i,ab}, f_{i,cd} \rangle$ is free on two generators if either $b \neq c$ or $d \neq a$. Therefore, by Brin and Squier's result (from [4]) that $PL_o(I)$ has no non-abelian free subgroups, we know that $Fix(f_{i,ab}) \cap Fix(f_{i,cd}) = \emptyset$ for $i \in \{1, 2\}$ and either $b \neq c$ or $d \neq a$. Now, for instance, if there is an index i and some point $p \in X_{i1} \cap X_{i2} \cap X_{i3}$, then both $f_{i,12}$ and $f_{i,13}$ must fix p, which is a contradiction. Therefore we see that X_{i1}, X_{i2}, and X_{i3} cannot share a common point for any index i. By the same argument, for any valid indices i, a, b, and c (where $i \in \{1, 2\}$ and $a \neq b \neq c \neq a$) we see that $X_{ia} \cap X_{ib} \cap X_{ic} = \emptyset$.

One now sees immediately that for any valid indices i, a,b, and c (where $i \in \{1, 2\}$ and $a \neq b \neq c \neq a$) we must also have that $X_{ia} \cup X_{ib} \cup X_{ic} = S^1$. This follows as otherwise there is some point p in the intersection $X_{ja} \cap X_{jb} \cap X_{jc}$ for the index $j \neq i$ (since $X_{1*} = \overline{S^1 \backslash X_{2*}}$ for any index $*$).

Suppose that for some indices i, $a \neq b$ we have that $X_{ia} \subset X_{ib}$, and let c and d be the two remaining distinct indices of $\{1, 2, 3, 4\} \backslash \{a, b\}$. Let p be an endpoint of X_{ib}. We have that p must be in both X_{ic} and X_{id}, otherwise their will be some point $q \in S^1 \backslash X_{ib}$ which is near to p so that q is not in either of $X_{ia} \cup X_{ib} \cup X_{ic} = X_{ib} \cup X_{ic}$ or $X_{ia} \cup X_{ib} \cup X_{id} = X_{ib} \cup X_{id}$. But this contradicts the fact that $X_{ib} \cap X_{ic} \cap X_{id} = \emptyset$.

It now immediately follows that for any index i and two distinct indices a and b, we have that $X_{ia} \cap X_{ib}$ is a non-empty closed interval (possibly a single point) while $X_{ia} \cup X_{ib}$ is also a closed interval which misses some points in S^1.

But now we are done as follows. For any index i the intervals X_{i1} X_{i2} and X_{i3} cover the circle, and have the properties that each pair of sets intersects in an interval, and no pair covers the whole circle. Now consider X_{i4}. It must likewise intersect both X_{i1} and X_{i2} non-trivially, and the union of X_{i1}, X_{i2} and X_{i4} also covers the whole circle. Therefore the end of X_{i1} which is not in X_{i2} is in both X_{i3} and X_{i4}. Hence $X_{i1} \cap X_{i3} \cap X_{i4} \neq \emptyset$, which implies that the group G_i cannot be free on four generators, as $f_{i,13}$ and $f_{i,14}$ share a common fixed point and will not generate a free subgroup of G_i. $\qquad \square$

Remark 8. *We observe that it is still plausible that even with g_1 and g_2 as in the proof above (supports over $(0, 1/2)$ and $(1/2, 1)$, respectively), one could plausibly find r, s, and c_1, c_2, and $c_3 \in C_T(sr)$ so that setting $k_{ij} = c_jsg_irc_j^{-1}$ as above we would have $H_r = \langle k_{r1}, k_{r2}, k_{r3} \rangle$ free on three generators for both $r = 1$ and $r = 2$, where the related claim for even two generator free groups could not be conceived of under Condition 1.*

4 A Ping-Pong Lemma for orientation preserving homeomorphisms of S^1

In this section, we prove a version of the Ping-Pong Lemma which we are using in our main argument. In the notations below we write all actions as left actions, in keeping with the tradition in the C^* literature, although much Thompson groups literature uses right action. In particular, if $x \in S^1$ and $s,t \in T$, we write tx for the image of x under t, and the conjugation $s^t := sts^{-1}$, which means, apply s^{-1} first, then t, and then s. We consider finite sets with repetitions.

In support of that lemma we ask the reader to recall an ordinary statement of Fricke and Klein's Ping-Pong Lemma (first proven in [12], but we give a different statement), and two further facts, one quite classical.

Lemma 9. *(Ping-Pong Lemma) Let G be a group of permutations on a set X, and let a, $b \in G$, where $b^2 \neq 1$. If X_a and X_b are two subsets of X so that neither is contained in the other, and for all integers n we have $b^n \cdot X_a \subset X_b$ whenever $b^n \neq 1$, and $a^n \cdot X_b \subset X_a$ whenever $a^n \neq 1$, then $\langle a, b \rangle$ factors naturally as the free product of $\langle a \rangle$ and $\langle b \rangle$. In particular, $\langle a, b \rangle \cong \langle a \rangle * \langle b \rangle$.*

Suppose that $f : S^1 \rightarrow S^1$ is an orientation preserving homeomorphism of the circle $S^1 = \mathbb{R}/\mathbb{Z}$, then f may be lifted to a homeomorphism of $\mathbb{R}$ by $F(x + m) = F(x) + m$ for every x and m. The rotation

number of f is defined to be $Rot(f) = \lim_{n \to \infty}(F^n(x) - x)/n$. The following theorem is generally relevant to the arguments in the final section of this paper, and appears first in [14], although there now exist many different proofs, the shortest of which appears to be in [1].

Theorem 10. *Every element of Thompson's group T has rational rotation number.*

The last tool we need in order to establish our own version of the Ping Pong lemma is the following classical result of Poincarè.

Lemma 11. *(Poincarès Lemma, circa 1905) If f is an orientation preserving homeomorphism of S^1 and f has rotation number p/q in lowest terms, then there is an orbit in S^1 of size exactly q under the action of $\langle f \rangle$.*

We are now in a good position to quote and prove our main technical tool.

Lemma 12. *Suppose a and b are orientation preserving homeomorphisms of the circle S^1 with rational rotation numbers $Rot(a) = p/q$ and $Rot(b) = r/s$ in lowest non-negative terms where*

1. *b is not torsion, and*

2. *if $x \in Fix(b^s)$ and $j \in \mathbb{Z}$ with $a^j \neq 1_T$, then we have $a^j x \notin Fix(b^s)$,*

then, there is a positive integer k so that a and b^k are a free basis for the group $\langle a, b^k \rangle$.

Proof. In the proof below, let us take a, $b \in T$ and p, q, r, $s \in \mathbb{N}$ as in the statement of the lemma. Set $b_0 := b$. We will occasionally update to a new version of b, which will be given by a new index. The new b will always be an integral power of the previous indexed b.

Set $b_1 := b_0^s$. The element b_1 will have rotation number $0/1$ in lowest non-negative terms. For b_1, we have $Fix(b_1)$ is not empty, and also not the whole circle (else b was originally a torsion element in T).

Let $\mathscr{I} \subset S^1$ be such that for each component C of $Supp(b_1)$, we have $|C \cap \mathscr{I}| = 1$, and associate each such C with its unique point in $\mathscr{I}$, so that $\mathscr{I}$ becomes an index set for the components of $Supp(b_1)$. We observe that $\mathscr{I}$ comes with an inherent circular order as a subset of S^1. Let L_b represent the set of limit points of $\mathscr{I}$ which are not in $\mathscr{I}$, and observe that $L_b \subset Fix(b_1)$.

For each positive integer d, set $\Delta_d := [-d, d] \cap (\mathbb{Z} \backslash \{0\})$, the set of non-zero integers a distance d or less from zero. Now for all positive integers d we can set ϵ_d to be one half of the distance from $Fix(b_1)$ to the set $\cup_{i \in \Delta_d} a^i \cdot Fix(b_1)$. Noting that these ϵ_d are all well defined and non-zero (unless a is torsion) as the sets involved are compact and as $a^m \cdot Fix(b_1) \cap a^n \cdot Fix(b_1) \neq \emptyset$ implies that either $m = n$ or that a is torsion and $n - m$ is divisible by the order of a.

Our analysis now splits, depending on whether or not a is torsion. In the case that a is torsion, our proof is somewhat easier, so we will execute that proof immediately.

Case: a is torsion with order q.

In this case, the value ϵ_{q-1} explicitly measures one half of the distance between $Fix(b_1)$ and the union of the images of $Fix(b_1)$ under the action of non-trivial powers of a. Set $\mathcal{U}$ to be the open ϵ_{q-1} neighbourhood of $Fix(b_1)$, and observe that for each integer $i \in \{1, 2, \ldots, q-1\}$ we have $a^i \cdot Fix(b_1) \cap \mathcal{U} = \emptyset$. For each non-zero $i \in \{1, 2, \ldots q - 1\}$ set $\mathcal{U}_i$ to be the ϵ_{q-1} neighbourhood of $a^i \cdot Fix(b)$. Again, for all such indices i, $\mathcal{U} \cap \mathcal{U}_i = \emptyset$. Set

$$X_b := \mathcal{U} \bigcap_{1 \leq i < q} (a^{q-i} \cdot \mathcal{U}_i).$$

Now by construction we have that the image set $a^i \cdot X_b \subset \mathcal{U}_i$, but $\mathcal{U}_i \cap X_b = \emptyset$ since $\mathcal{U}_i \cap \mathcal{U} = \emptyset$ and $X_b \subset \mathcal{U}$. Thus, X_b is an open set containing $Fix(b_1)$ that is disjoint from its image under the action of any nontrivial power of a.

As X_b and the components of support of b_1 altogether cover the circle, there is a finite set of open interval components of $Supp(b_1)$ which together with X_b covers the circle. In turn, this implies there is a minimal positive integer v so that for all $x \in S^1 \backslash X_b$, we have $b_1^v x \in X_b$ and $b_1^{-v} x \in X_b$.

Now we can set

$$X_a := \cup_{1 \leq i < q}(a^i \cdot X_b).$$

With this choice of X_a and X_b we have arranged that we satisfy the hypotheses of Lemma 9 for the elements a and a^k where $k = v \cdot s$.

<u>*Case: a is not torsion.*</u>

Throughout this case, given a set $X \subset S^1$, and $\epsilon > 0$, we shall use the notation $N_\epsilon(X)$ to denote the open ϵ-neighbourhood of X, that is, all points in S^1 a distance less than ϵ from some point in X.

In this case with a not torsion, we must specify the set $F_a := Fix(a^q)$, which is a closed non-empty subset of the circle which is disjoint from $Fix(b_1)$. Choose a specific $\epsilon > 0$ so that $N_\epsilon(Fix(b_1)) \cap N_\epsilon(F_a) = \emptyset$, noting that such an epsilon value exists as $Fix(b_1)$ and F_a are disjoint compact subsets of S^1.

Let m be a positive integer so that both $a^{mq} \cdot Fix(b_1) \subset N_\epsilon(F_a)$ and $a^{-mq} \cdot Fix(b_1) \subset N_\epsilon(F_a)$. This m exists as a^q acts as a monotone strictly increasing, or as a monotone strictly decreasing function over each component of its support, and as the limit point of any point in a component of support of a^q under increasing powers of a^q must be a fixed point of a^q (and similarly under a negative powers of a^q), and as $Fix(b_1)$ is a compact set and hence is contained in a union of finitely many components of support of a^q.

We now observe that for n an integer with $|n| > m$, we have that $a^{nq} \cdot Fix(b_1) \subset F_a$ as well. We would like to argue a stronger result now that there is a positive constant N so that for all $j > N$ we have $a^j \cdot Fix(b_1) \subset N_\epsilon(F_a)$ and $a^{-j} \cdot Fix(b_1) \subset N_\epsilon(F_a)$.

To make this argument, the main point to observe is that there is an induced action of $\langle a \rangle$ on the set of components of support of a^q which partitions these components into (possibly infinitely many) orbits of size q. Further, as a commutes with a^q and a is orientation preserving, it is easy to see that each such orbit consists of components of support where the action of a^q is increasing on all components of the orbit, or decreasing on all components of the orbit.

It is also the case that there are only finitely many components of support of a^q which are not already wholly contained in $N_\epsilon(F_a)$. Let $C_1, C_2, \ldots, C_w$ represent these components, and observe that $Fix(b_1)$ is contained in the union K of these compact intervals. For each component C_j, let I_j be the closed interval $C_j \backslash N_\epsilon(F_a)$. Now each of these components C_j are in an orbit of length q amongst the components of support of a^q, and in each such orbit the action of a^q on each component is in the same direction. Hence there is a finite number N so that for all $j > N$ and intervals I_m, we have that $a^j \cdot I_m \subset N_\epsilon(F_a)$, and also $a^{-j} \cdot I_m \subset N_\epsilon(F_a)$.

Now define J as below:

$$J := \cup_{i \in \Delta_N}((a^i \cdot Fix(b_1)) \cap K).$$

Where we recall that $\Delta_n = [-n, n] \cap (\mathbb{Z} \backslash \{0\})$ for any particular $n \in \mathbb{N}$.

It is immediate that J is a compact set which is disjoint from $Fix(b_1)$. As such, there is a $\delta > 0$ so that $\delta < \epsilon$ and the δ-neighbourhood $N_\delta(Fix(b_1))$ of $Fix(b_1)$ is disjoint from the set V_δ defined as

$$V_\delta := \cup_{i \in \Delta_N}(a^i \cdot N_\delta(Fix(b_1)))$$

and noting that as $\delta < \epsilon$ we also have that $N_\delta(Fix(b_1))$ is disjoint from $N_\epsilon(F_a)$.

Now set $X_b := N_\delta(Fix(b_1))$ and $X_a := N_\epsilon(F_a) \cup V_\delta$.

By construction, there is an integer $z > 0$ so that b_1^z takes the complement of X_b (and so, X_a) into X_b, while all non-trivial powers of a take X_b into X_a. Hence the integer $k = s \cdot z$ has the property that a and b^k freely generate a free group of rank 2. $\qquad\square$

5   Applying Condition 2, and variants, in T

Here we list lemmas, where the Condition 2 can be used.

Lemma 13. *Let H be a finite set of nontrivial elements in T so that there is some point $p \in \cap_{h \in H} Supp(h)$. Then, for any positive integer n there is an element $g \in T$ and $\{c_1, c_2, \ldots, c_n\}$ so that $c_i \in C_T(g)$ for all i, and so that for all $h \in H$ we have the set*

$$G_h := \{(gh)^{c_i} \mid i \in \{1, 2, \ldots, n\}\}$$

is a free basis for a free group of rank n.

Proof. Let H and p as in the statement of the lemma, and let $n \in \mathbb{N}$ be given. For each $h \in H$, let $Rot(h) := r_h/s_h$ written in lowest terms (NB, any finite periodic orbit under the action of $\langle h \rangle$ is of length s_h). By the definition of p, we see there is a non-empty interval (a, b) with $p \in (a, b)$ so that for all $h \in H$ we have $(a, b) \cdot h^j \cap (a, b) = \emptyset$ for all $1 \leq j < s_h$.

Now let $g \in T$ be an element with rotation number $Rot(g) = 0$ which fixes exactly the point p. We can choose g so that for all $h \in H$ the product gh has precisely two fixed points which are generated

7

from the contact of the graph of g with the graph of the element h^{-1}, by choosing the graph of g to be near to a step function (the nearly vertical component of the graph of g should be steeper than any slope of any element of H, and the nearly horizontal component of the graph of g should have slope closer to zero than any slope of any element of H, and, in a small epsilon box around (p, p) which misses the graphs of the elements of H, the graph of g is unrestrained). As g fixes the point p, it is the case that that for all elements $h \in H$, we have $k_h \cdot p = gh \cdot p = g \cdot (h \cdot p) \neq p$ and further that under the action of $\langle k_h \rangle$, p is in an infinite orbit which limits to the two ends of the component of support of k_h which contains p. In these conditions we can immediately apply Lemma 12 to claim that for each $h \in H$, there is a power g^{ρ_h} of g so that k_h and the element g^{ρ_h} freely generate a free group of rank 2. Now take $\theta := LCM\{\rho_h \mid h \in H\}$, so that g^θ is free for k_h for each h.

Now, it is immediate that setting $c_i := g^{i \cdot \theta}$ for indices i in the range $1 \leq i \leq n$, we can create sets $X_h := \{(gh)^{c_i}\}_{i=1}^n$ so that $\langle X_h \rangle$ is free on n-generators for all $h \in H$.

$\square$

The following is some weakening of Condition 2 which we can achieve in Thompson's group T. Unfortunately, by raising to the power p, we find a free subgroup of $K := \{((hg)^{c_i}) \mid i \in \{1, 2, \ldots, n\}\}$ which is generally not finite index.

Lemma 14. *Let H be a finite set of nontrivial elements in T with cardinality p. Then there is an element $g \in T$ such that for any positive integer n there are elements $\{c_1, c_2, \ldots, c_n\}$ so that $c_i \in C_T(g)$ for all i, and so that for all $h \in H$ the set*

$$G_h := \{((hg)^{c_i})^p \mid i \in \{1, 2, \ldots, n\}\}$$

freely generates a free group of rank n.

Proof. Index the $h \in H$ so that $H = \{h_1, h_2, \ldots, h_p\}$ for some minimal positive integer p. As each element $t \in T$ has a rational rotation number $Rot(t) \in \mathbb{Q}/\mathbb{Z}$, let $Rot(h_i) = r_i/s_i$ expressed in lowest positive terms. For each index i, inductively choose a point $x_i \in Supp(h_i)$ so that the set

$$X_i := \left\{ h_j^{-1} \cdot x_i, x_i, h_j \cdot x_i \mid 1 \leq j \leq i \right\}$$

is disjoint from the union $\cup_{j<i} X_j$, and where each x_i is chosen as a dyadic rational so that no point in X_i is an end of a component of support of $h_j^{s_j}$ for any index j. Note that in all cases we are choosing an x_i so that a finite set of images miss a finite set of points in the circle, and this is easy to do. Set $P := \{x_i \mid 1 \leq i \leq p\}$. Reindex P so that $0 \leq x_1 < x_2 < \cdots < x_p < 1$, and reindex the sets X_i correspondingly.

Now choose, for each i, an interval (a_i, b_i) centred around x_i, and set

$$I_i := \left\{ h_j^{-1} \cdot (a_i, b_i), (a_i, b_i), h_j \cdot (a_i, b_i) \mid 1 \leq j \leq i \right\}$$

where each (a_i, b_i) is chosen small enough so that each element of I_i is disjoint from the union $\cup_{j<i, B \in I_j} B$, and where (a_i, b_i) is disjoint from $h_j^{-1} \cdot (a_i, b_i)$ and $h_j \cdot (a_i, b_i)$ whenever $x_i \in Supp(h_j)$, and where $h_j^{-1} \cdot (a_i, b_i)$ and $h_j \cdot (a_i, b_i)$ intersect each other non-trivially only if h_j is torsion of order two. We can insist these intervals are possibly smaller still, so that the complement of the union $\tilde{U} := \cup_{1 \leq j \leq p, B \in I_j} B$ is a union of closed intervals each of which has non-empty interior. We observe that by our choices of the sets X_i, we can so choose our intervals (a_i, b_i).

Now let $\tilde{g}$ be a non-torsion element with rotation number $1/p$ which admits exactly one orbit of size p under the action of $\langle \tilde{g} \rangle$, where the orbit is the set P, and where $\tilde{g}$ cyclically permutes the x_i in order, taking x_1 to x_2 and etc.

The element $\gamma := \tilde{g}^p$ has p components of support $S_1 := (x_1, x_2)$, $S_2 := (x_2, x_3)$, ..., $S_p := (x_p, x_1)$, and γ is either increasing on each component of support, or decreasing on each component of support. We re-choose $\tilde{g}$ if necessary so that it satisfies all previous conditions and so that γ is increasing on each component of support, and note that such elements $\tilde{g}$ exist.

Now set $U := \cup_{1 \leq j \leq p} (a_i, b_i)$, which union also has the property that the complement $C := S^1 \setminus U$ is a union of p disjoint closed intervals $D_i := [b_i, a_{i+1}]$ (where we set $a_{p+1} := a_1$) each of which is not a point. As such is the case, we can find a positive integer d so that $\gamma^d \cdot C \subset U$ as after d iterations of γ, the set D_i will be moved so as to have image just to the left of x_{i+1} (where again, we set $x_{p+1} := x_1$).

8

Now set $g := \gamma^d \tilde{g}$. The element g so constructed has rotation number $1/p$ as before, and acts on the set P as $\tilde{g}$ does, but it has the further property that for all integers $s \neq 0$, $g^s \cdot C \subset U$.

Now set, for each index i, $k_i := gh_i$. We will now show that for all k_i, the point x_j is not in any finite periodic orbit of k_i, for any index j.

Let i and j be indices so that k_i and x_j are an element and point as defined above, respectively. Let q be an index and consider the interval $(a_q, x_q]$ under the action of g, and then consider the interval $(a_j, x_j]$ under the action of k_i.

Under the action of g, we know already that $x_q \mapsto x_{q+1}$, and that the component $[b_{q-1}, a_q]$ of C maps into the interval (a_{q+1}, x_{q+1}). In particular, the whole interval $[b_{q-1}, x_q]$ is mapped into $(a_{q+1}, x_{q+1}]$ by g and so the interval $[a_q, x_q]$ is mapped into (a_{q+1}, x_{q+1}) as well, and precisely the point x_q will map to x_{q+1}.

Now consider the action of k_i on $(a_j, x_j]$. If x_j is in the support of h_i, then $h_i \cdot (a_j, x_j] \subset C$, whereupon that image is moved into some interval (a_*, x_*) by the action of g.

In particular, $k_i^z \cdot x_j = x_{j+z}$ (in cyclic order) as long as for all $0 \leq y \leq z$ we have $x_{j+y} \notin Supp(h_i)$. If instead for some minimal integer $s \in \{0, 1, \ldots, p-1\}$ we have $x_{j+s} \in Supp(h_i)$ then on k_i^{s+1} will move x_j out of P as we will have $k_i^{s+1} \cdot x_j \in (a_*, x_*)$ for some index $*$. Furthermore, if this happens, all further iterates of x_j will fail to re-enter P.

However, x_i is itself in the support of h_i, so this eventuality happens (in p steps or fewer), and so no x_j is on a finite periodic orbit under the action of $\langle k_i \rangle$, for any indices j and i.

In particular, for all indices j, we see that $(k_j^{tp} \cdot P) \cap P = \emptyset$ for all $t \in \mathbb{Z} \setminus \{0\}$.

We can now quote Lemma 12, using $a := k_j^p$ for each index j and $b = g$ to claim that for each index j there is an integer z_j so that the group $\langle k_i^p, g^{z_j} \rangle$ is free on two generators (in this setup, $P = Fix(b^s) = Fix(g^p)$). Now set $z := LCM(\{z_1, z_2, \ldots, z_p\})$ the least common multiple of the values z_j, and then we have that for all indices j, the elements k_j^p and g^z generate a free group. In particular, setting $c_i := g^{iz}$ for $i \in \{1, 2, \ldots, n\}$ we see that $\Gamma_j := \left\{ (k_j^p)^{c_i} = (k_j^{c_i})^p \mid 1 \leq i \leq n, i \in \mathbb{Z} \right\}$ freely generates a free group of rank n.

But now, recalling that $k_i = gh_i$, and that therefore $[g, c_i] = 1$, we have proved our claim. $\qquad \square$

6   Some questions

The last result in particular shows that our Condition 2 is almost (in some sense) enough to show that $C_\lambda^*(T)$ is simple. But not quite! This, together with our partial converse of the Haagerup-Olesen result, encourages us to ask the following question.

Question 1. *Is the non-amenability of Thompson's group F equivalent to the simplicity of the algebra $C_\lambda^*(T)$.*

As the setup of Condition 2 is more flexible than that of Condition 1, we find Condition 2 easier to use. Still, we have not actually proven that the conditions are not equivalent. Thus it would be quite useful to give a positive answer to the following question.

Question 2. *Is there a group G which fails to satisfy Condition 1 but which does satisfy Condition 2?*

In [16], [9] and [10] several approaches to amenability via group actions were developed. It is an interesting question to relate these approaches to properties of a group's C^*-algebra.

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