FINITELY PRESENTED GROUPS RELATED TO KAPLANSKY'S DIRECT FINITENESS CONJECTURE

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ABSTRACT. We consider a family of finitely presented groups, called Universal Left Invertible Element (or ULIE) groups, that are universal for existence of one–sided invertible elements in a group ring $K[G]$, where K is a field or a division ring. We show that for testing Kaplansky's Direct Finiteness Conjecture, it suffices to test it on ULIE groups, and we show that there is an infinite family of non-amenable ULIE groups. We consider the Invertibles Conjecture and we show that it is equivalent to a question about ULIE groups. By calculating all the ULIE groups over the field $K = \mathbb{F}_2$ of two elements, for ranks $(3, n)$, $n \leq 11$ and $(5, 5)$, we show that the Direct Finiteness Conjecture and the Invertibles Conjecture (which implies the Zero Divisors Conjecture) hold for these ranks over $\mathbb{F}_2$.

1. INTRODUCTION

In the middle of the last century, Kaplansky showed (see [11], p. 122) that for every field K of characteristic 0 and every discrete group Γ, the group ring $K[\Gamma]$ (which is, actually, the K–algebra with basis G and multiplication determined by the group product on basis elements and the distributive law) is *directly finite*, namely, that for every $a, b \in K[\Gamma]$ the equation $ab = 1$ implies $ba = 1$. This is clearly equivalent to saying that all one–sided invertible elements in $K[G]$ are invertible. However, the situation for fields of positive characteristic is unresolved; the following conjecture of Kaplansky is still open:

Conjecture 1.1. *For every discrete group Γ and every field K, the group ring $K[\Gamma]$ is directly finite.*

We will call this Kaplansky's Direct Finiteness Conjecture, or simply the Direct Finiteness Conjecture (DFC).

Ara, O'Meara and Perera proved [1] that the DFC holds (and also when K is a division ring) for residually amenable groups. Elek and Szabó [5] generalized this result to a large class of groups, namely the sofic groups, (also with K a division ring, and they proved also stable finiteness). Since currently there are no known examples of non-sofic groups, the Kaplansky DFC is even more intriguing. Moreover

it is well known that, in the case of finite fields, Gottschalk's conjecture [9] implies Kaplansky's DFC (see [5] for a proof).

The notion of a sofic group was introduced by Gromov in [10] as a group with Cayley graph that satisfies a certain approximation property. He showed that Gottschalk's conjecture is satisfied for sofic groups. Many interesting properties are known about sofic groups. The class of sofic groups is known to be closed under taking direct products, subgroups, inverse limits, direct limits, free products, and extensions by amenable groups (by [6]) and under taking free products with amalgamation over amenable groups (see [2], [7] and [13]).

In this paper, we describe finitely presented groups that are universal for existence of one–sided invertible elements in a group algebra. To test Kaplansky's DFC, it will be enough to test it on these universal groups. In fact, this idea, at least in the case of the field of two elements, has been around in discussions among several mathematicians for some time. See for example the MathOverflow posting [16] of Andreas Thom, or Roman Mikhailov's preprint [12]. Who was the first to describe these groups is unclear to the authors, and we believe that these groups may have been rediscovered by several persons at different times. After we posted an earlier version of this paper (which lacked Sections 5 and 6 and is still available on the arXiv), a paper of Pascal Schweitzer [14] about similar calculations for the Zero Divisors Conjecture appeared. These efforts were independent of each other.

To illustrate, let us work over the field $K = \mathbb{F}_2$ of two elements. If $a, b \in \mathbb{F}_2[G]$ and $ab = 1$, then we may write

$$a = a_0 + a_1 + \cdots + a_{m-1} \text{ and } b = b_0 + b_1 + \cdots + b_{n-1} \tag{1}$$

for group elements $a_0, \ldots, a_{m-1}$ that are distinct and group elements $b_0, \ldots, b_{n-1}$ that are distinct. The identity $ab = 1$ implies that $a_i b_j = 1$ for some i and j; after renumbering, we may assume $i = j = 0$, and then, replacing a by $a_0^{-1}a$ and b by bb_0^{-1}, we may assume $a_0 = b_0 = 1$. Now distributing the product ab we get

$$\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} a_i b_j = 1$$

and, thus, there is a partition π of $\{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$ with one singleton set $\{(0,0)\}$ and all other sets containing two elements, such that if $(i, j) \overset{\pi}{\sim} (k, \ell)$ (i.e., if (i, j) and (k, ℓ) belong to the same set of π), then $a_i b_j = a_k b_\ell$. Consider the finitely presented group

$$\Gamma_\pi = \langle a_0, a_1, \ldots, a_{m-1}, b_0, b_1, \ldots, b_{n-1} \mid a_0 = b_0 = 1, (a_i b_j = a_k b_\ell)_{\{(i,j),(k,\ell)\}\in\pi}\rangle, \tag{2}$$

where the relations are indexed over all pairs $\{(i, j), (k, \ell)\}$ of the partition π. Then there is a group homomorphism $\Gamma_\pi \to G$ sending the given generators of Γ_π to their namesakes. Furthermore, the corresponding elements a and b in $\mathbb{F}_2[\Gamma_\pi]$, defined by equation (1), satisfy also $ab = 1$. If $ba = 1$ holds in $\mathbb{F}_2[\Gamma_\pi]$, then it holds in $\mathbb{F}_2[G]$ as well. Therefore, to test Kaplansky's Direct Finiteness Conjecture over $\mathbb{F}_2$, it will suffice to test it on the groups Γ_π.

We call these groups (and their analogues for more general K) ULIE groups, short for Universal Left Invertible Element groups. In this paper, we will show that studying the ULIE groups will be enough to answer Kaplansky's Direct Finiteness Conjecture, and we will prove a few facts about them, including that there is an infinite family of non-amenable ULIE groups. With the aid of computers, we have found all ULIE groups (for the field $\mathbb{F}_2$) up to sizes 3×11 and 5×5 and used soficity results to obtain partial confirmation of Kaplansky's DFC over $\mathbb{F}_2$.

The group ring $K[G]$ is said to be stably finite if all matrix algebras $M_n(K[G])$ ($n \in \mathbb{N}$) are directly finite. At first we wondered whether an effort similar to the one described in this paper for direct finiteness should be made for stable finiteness, but Corollary 2.3 of [4] shows that there would be no advantage in doing so.

Throughout this paper, if K is said to be a division ring, then it may also be a field, and will be assumed to be nonzero. We let 1 denote the identity element of a group G, or the multiplicative identity of a division ring K or of a group ring $K[G]$, depending on the context.

We would like to mention two other well known conjectures about group rings. Let us call the following the *Invertibles Conjecture* (IC). See Conjecture 2 of [17] for a statement when K is the complex numbers.

Conjecture 1.2. *If K is a division ring and G is a group and if $K[G]$ contains a one–sided invertible element that is not of the form kg for $g \in G$ and $k \in K$, then G has torsion.*

As is well known, it implies the famous *Zero Divisors Conjecture* (ZDC):

Conjecture 1.3. *If K is a division ring and G a group and if $K[G]$ contains zero divisors, then G has torsion.*

See, for example, a posting [15] by Andreas Thom on MathOverflow a proof.

In Section 2, we introduce ULIE groups and show that for solving Kaplansky's Direct Finiteness Conjecture (or various subcases thereof), it is enough to consider ULIE groups and we state that our calculations (summarized in Section 6) imply that the DFC holds for ranks $(3, n)$ with $n \leq 11$ and $(5, 5)$. In Section 3, we exhibit an infinite family of non-amenable ULIE groups.

In Section 4, we show that the Invertibles Conjecture can be reformulated in terms of certain quotients of ULIE groups and we state that our calculations imply that the Invertibles Conjecture holds for ranks $(3, n)$ with $n \leq 11$ and $(5, 5)$.

In Section 5 we describe the algorithm we employed to list all the ULIE groups over the field $\mathbb{F}_2$ of two elements for given ranks and in Section 6 we report on the results of these calculations.

2. Universal Left Invertible Element groups

Let K be a division ring (or field). Consider a group G and elements a and b in the group ring $K[G]$, satisfying $ab = 1$. We suppose that not both a and b are supported

on single elements of G, and then neither of them may be, and we are interested in the question of whether $ba = 1$ must then hold. We may write

$$a = r_0 a_0 + \cdots + r_{m-1} a_{m-1}, \qquad b = s_0 b_0 + \cdots + s_{n-1} b_{n-1} \qquad (3)$$

for integers $m, n \geq 2$, for nonzero elements $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1}$ of K and for distinct elements $a_0, \ldots, a_{m-1}$ of G and distinct elements $b_0, \ldots, b_{n-1}$ of G. We then say that the *rank* of a is m and of b is n, and that the *support* of a is $\{a_0, a_1, \ldots, a_{m-1}\}$ and of b is $\{b_0, \ldots, b_{n-1}\}$. We must have $a_i b_j = 1$ for at least one pair (i, j), and by renumbering, we may assume $a_0 b_0 = 1$. Replacing a by $a_0^{-1} a$ and b by $b b_0^{-1}$, we may assume $a_0 = b_0 = 1$. Replacing a by $r_0^{-1} a$ and b by $b r_0$, we may also assume $r_0 = 1$.

Let π be the partition of the set

$$\{0, \ldots, m-1\} \times \{0, \ldots, n-1\} \qquad (4)$$

defined by

$$(i, j) \overset{\pi}{\sim} (i', j') \text{ if and only if } a_i b_j = a_{i'} b_{j'}, \qquad (5)$$

where $(i, j) \overset{\pi}{\sim} (i', j')$ means that (i, j) and (i', j') belong to the same set of the partition π. Then we have, for all $E \in \pi$,

$$\sum_{(i,j) \in E} r_i s_j = \begin{cases} 1, & (0, 0) \in E, \\ 0, & (0, 0) \notin E. \end{cases} \qquad (6)$$

Definition 2.1. We call π the *cancellation partition* for the pair (a, b) with respect to the orderings $(a_0, \ldots, a_{m-1})$ and $(b_0, \ldots, b_{n-1})$ of their supports.

Definition 2.2. Given a partition π of the set (4), we consider the group Γ_π with presentation

$$\Gamma_\pi = \langle a_0, a_1, \ldots, a_{m-1}, b_0, b_1, \ldots, b_{n-1} \mid a_0 = b_0 = 1, (a_i b_j = a_{i'} b_{j'})_{(i,j) \overset{\pi}{\sim} (i',j')} \rangle, \qquad (7)$$

where the relations are indexed over the set of all pairs $((i, j), (i', j'))$ of elements of (4) that belong a same set of the partition π.

Definition 2.3. Let $m, n \in \{2, 3, \ldots\}$ and let π be a partition of the set (4). We say that π is *degenerate* if, in the group Γ_π above, the group elements $a_0, a_1, \ldots, a_{m-1}$ are not distinct or the group elements $b_0, b_1, \ldots, b_{n-1}$ are not distinct.

Remark 2.4. A partition π is clearly degenerate if we have $(i, j) \overset{\pi}{\sim} (i, j')$ for any $j \neq j'$ or $(i, j) \overset{\pi}{\sim} (i', j)$ for any $i \neq i'$, i.e., if π groups together any two elements in the same row or column.

Definition 2.5. Let $m, n \in \{2, 3, \ldots\}$ and let π be a partition of the set (4). Let K be a division ring and let $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1}$ be nonzero elements of K. We say that π is *realizable with* $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1}$ if the equalities (6) hold for all $E \in \pi$. We say that π is *realizable over* K if it is realizable with some nonzero elements of K, and we say that π is *realizable* if it is realizable over some division ring K.

Remark 2.6. A realizable partition can have at most one singleton, which would then be $\{(0, 0)\}$.

We order the partitions of a set in the usual way, writing $\pi \leq \sigma$ if every element of π is a subset of an element of σ. Then Γ_σ is a quotient of Γ_π by the map sending canonical generators to their namesakes, so σ nondegenerate implies π nondegenerate.

Definition 2.7. Let K be a division ring and let $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1} \in K \backslash \{0\}$. A partition π of the set (4) is *minimally realizable with* $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1}$ if it is minimal among all the partitions that are realizable with $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1}$. We say π is *minimally realizable over* K if it is minimally realizable with some choice of $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1} \in K \backslash \{0\}$, and it π is simply *minimally realizable* if it is minimally realizable over some division ring K.

Remark 2.8. The partitions of the set (4) that are minimally realizable over the field $\mathbb{F}_2$ of two elements are precisely the partitions having only pairs except for the singleton set $\{(0,0)\}$. Thus, the existence of such a partition implies that m and n are both odd.

Remark 2.9. The notion of being minimally realizable over K is ostensibly different from being minimal among the partitions that are realizable over K, just as being minimally realizable is different from being minimal among the realizable partitions. This is because the quality of being minimally realizable is bound up with a particular choice of field elements $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1}$. We see that this is important, for example in the proof of Theorem 2.14. However, see Remark 2.11 for more on this.

Definition 2.10. Let K be a division ring and let $m, n \geq 2$ be integers. Let $\mathrm{ULIE}_K(m, n)$ be the set of all groups Γ_π as in Definition 2.2 as π runs over all partitions π of the set (4) that are both nondegenerate and minimally realizable over K. We will say that an *ULIE$_K$ group* is one that belongs to the set $\bigcup_{m,n \geq 2} \mathrm{ULIE}_K(m, n)$. Similarly, we let $\mathrm{ULIE}(m, n)$ denote the set of all groups Γ_π as π runs over all partitions π of (4) that are both nondegenerate and minimally realizable, and say that an *ULIE group* is one that belongs to the set $\bigcup_{m,n \geq 2} \mathrm{ULIE}(m, n)$. (ULIE is an acronym for *Universal Left Invertible Element*.) Finally, we let $\mathrm{ULIE}_K^{(-)}(m, n) \subseteq \mathrm{ULIE}_K(m, n)$ be equal to$\mathrm{ULIE}_K(m, n)$ if $m \neq n$ and, if $m = n$, we let it consist of the complement of the set of $\mathrm{ULIE}_K(m, n)$–groups Γ_π for partitions π that are minimally realizable for some $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1} \in K$ and so that the partition forces equality $r_0 a_0 + \cdots + r_{m-1} a_{m-1} = s_0 b_0 + \cdots + s_{n-1} b_{n-1}$ in the group ring $K[\Gamma_\pi]$.

Remark 2.11. We have introduced the ULIE groups in order to restrict the class of groups Γ that would need to be tested for $K[\Gamma]$ being directly finite, in order to prove Kaplansky's conjecture. We see this in Theorems 2.14 and 2.19 and Corollaries 2.15 and 2.20 below. However, there is no harm in increasing the class of groups that are tested. Keeping this in mind, one may find that it is better not to worry about minimally realizable partitions π, but, for example, for a given rank pair (m, n) to test simply all groups Γ_π over all nondegenerate partitions π of $\{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$, rather than first to decide which of them are minimally realizable or even realizable. Of course, if we restrict to the field $K = \mathbb{F}_2$ of two elements, then, as seen in Remark 2.8, the minimally realizable partitions have a particularly simple form. But for general K this is not clear to us.

In connection with the Invertibles Conjecture, in Theorem 4.5 the implication (i) $\implies$ (ii) does depend on taking realizable partitions, though they need not be minimally realizable.

Definition 2.12. For a division ring K and for integers $m, n \geq 2$, we will say that *Kaplansky's Direct Finiteness Conjecture holds over K for rank pair (m, n)* if for all groups G and all $a, b \in K[G]$ with rank of a equal to m and rank of b equal to n, $ab = 1$ implies $ba = 1$. We will say that *Kaplansky's Direct Finiteness Conjecture holds over K for rank m* if it holds for all rank pairs (m, n) with $n \geq 2$, namely, if for all groups G, right invertibility of $a \in K[G]$ with $\mathrm{rank}(a) = m$ implies invertibility of a. We will say that *Kaplansky's Direct Finiteness Conjecture holds over K* if it holds over K for all rank pairs, namely, if $K[G]$ is directly finite for all groups G.

Remark 2.13. Given a group G, we can define the group G^{op} to be the set G equipped with the opposite binary operation: the product of g and h in G^{op} is defined to be the element hg of G. Using G^{op}, we easily see that Kaplansky's Direct Finiteness Conjecture holds over K for rank pair (m, n) if and only if it holds over K for rank pair (n, m). Furthermore, this implies that Kaplansky's Direct Finiteness Conjecture holds for rank m if and only if for any group G and for any $a \in K[G]$ of rank m, one–sided invertibility a of implies invertibility of a.

The idea of the following theorem was explained (and an adequate proof in the case $K = \mathbb{F}_2$ was given) in the introduction.

Theorem 2.14. *Let K be a division ring and let $m, n \geq 2$ be integers. Suppose that for every group $\Gamma \in \mathrm{ULIE}_K^{(-)}(m, n)$, the group ring $K[\Gamma]$ is directly finite. Then Kaplansky's Direct Finiteness Conjecture holds over K for rank pair (m, n).*

Proof. Let G be any group and let c and d be elements of $K[G]$ having ranks m and n, respectively, and assume $cd = 1$. We must show $dc = 1$. We may write $c = r_0 c_0 + \cdots + r_{m-1} c_{m-1}$ for distinct elements $c_0, \ldots, c_{m-1}$ of G, and $d = s_0 d_0 + \cdots + s_{n-1} d_{n-1}$ for distinct elements $d_0, \ldots, d_{n-1}$ of G and for nonzero elements $r_0, \ldots r_{m-1}, s_0, \ldots, s_{n-1}$ of K.

After renumbering, we may without loss of generality assume $c_0 d_0 = 1$; replacing c by $c_0^{-1} c$ and d by dd_0^{-1}, we may assume $c_0 = d_0 = 1$. Let σ be the partition of the set (4) that is the cancellation partition for the pair (c, d) with respect to the given orderings of their supports. Then there is a group homomorphism $\psi : \Gamma_\sigma \to G$ sending each a_i to c_i and each b_j to d_j. This implies that σ is nondegenerate. Clearly, it is realizable with $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1}$. Let $\pi \leq \sigma$ be a partition of the set (4) that is minimally realizable with $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1}$. Thus, taking $a = r_0 a_0 + \cdots + r_{m-1} a_{m-1}$ and $b = s_0 b_0 + \cdots + s_{n-1} b_{n-1}$ in $K[\Gamma_\pi]$, where the a_i and b_j are the named generators in the group Γ_π from Definition 2.2, by virtue of the defining relations in (7), we have $ab = 1$. Combining ψ with the natural quotient group homomorphism $\Gamma_\pi \to \Gamma_\sigma$, we get a group homomorphism $\phi : \Gamma_\pi \to G$ that extends linearly to a unital ring homomorphism $K[\Gamma_\pi] \to K[G]$ sending a to c and b to d. By hypothesis either $K[\Gamma_\pi]$ is directly finite or we have $a = b$ in $K[\Gamma_\pi]$. In either case, we conclude $ba = 1$ in $K[\Gamma_\pi]$. Therefore, we have $dc = 1$ in $K[G]$. $\qquad\square$

Corollary 2.15. *Let K be a division ring. Then Kaplansky's conjecture holds over K if and only if for all $ULIE_K$ groups Γ, the group ring $K[\Gamma]$ is directly finite.*

A strategy for testing Kaplansky's Direct Finiteness Conjecture over a division ring K is, thus, to check for direct finiteness of the groups belonging to $ULIE_K(m, n)$ for various ranks m and n. In fact, as we will now show, if we proceed by starting small and incrementing m and n by only one each time, then we can restrict to testing a slightly smaller set of groups.

Definition 2.16. We consider the set (4). We view this set as laid out like an $m \times n$ matrix, with rows numbered 0 to $m - 1$ and columns numbered 0 to $n - 1$. Given a parition π of this set, we let $\overset{\pi}{\sim}$ be the corresponding equivalence relation, whose equivalence classes are the sets of the partition. Let $\overset{r}{\sim}$ be the equivalence relation on $\{0, \ldots, m-1\}$ that is generated by $\overset{\pi}{\sim}$ under the projection onto the first coordinate, namely, generated by the relations

$$\{i \overset{r}{\sim} i' \mid \exists j, j' \text{ with } (i, j) \overset{\pi}{\sim} (i', j')\}.$$

We say π is *row connected* if $\overset{r}{\sim}$ has only one equivalence class.

Similarly, let $\overset{c}{\sim}$ be the equivalence relation on $\{0, \ldots, n-1\}$ that is generated by $\overset{\pi}{\sim}$ under the projection onto the second coordinate, namely, generated by the relations

$$\{j \overset{c}{\sim} j' \mid \exists i, i' \text{ with } (i, j) \overset{\pi}{\sim} (i', j')\}.$$

We say π is *column connected* if $\overset{c}{\sim}$ has only one equivalence class.

Lemma 2.17. *Let K be a division ring and let $M, N \geq 2$ be integers. Let G be a group and suppose $c, d \in K[G]$ have ranks M and N, respectively, both have the identity element of G in their supports and satisfy $cd = 1$. Write $c = r_0 c_0 + \cdots r_{M-1} c_{M-1}$ and $d = s_0 d_0 + \cdots s_{N-1} d_{N-1}$ for group elements c_i and d_j and assume $c_0 = d_0 = 1$. Let σ be the cancellation partition of (c, d) with respect to these orderings of the supports.*

(a) Suppose Kaplansky's conjecture holds over K for all rank pairs (m, N), with $2 \leq m < M$. Then σ is row connected. Moreover, if $\pi \leq \sigma$ is a partition that is realizable with $r_0, \ldots, r_{M-1}, s_0, \ldots, s_{N-1}$, then π is row connected.

(b) Suppose Kaplansky's conjecture holds over K for all rank pairs (M, n), with $2 \leq n < N$. Then σ is column connected. Moreover, if $\pi \leq \sigma$ is a partition that is realizable with $r_0, \ldots, r_{M-1}, s_0, \ldots, s_{N-1}$, then π is column connected.

Proof. For part (a), let Γ_π be the group with presentation (7). Then there is a group homomorphism $\phi : \Gamma_\pi \to G$ sending each a_i to c_i and each b_j to d_j. Since $c_0, \ldots, c_{M-1}$ are distinct elements of G and $d_0, \ldots, d_{N-1}$ are distinct elements of G, it follows that π is nondegenerate. Letting $a = r_0 a_0 + \cdots + r_{M-1} a_{M-1}$ and $b = s_0 b_0 + \cdots + s_{N-1} b_{N-1}$, we also have $ab = 1$ in $K[\Gamma_\pi]$.

Suppose, to obtain a contradiction, that π is not row connected. Then, after renumbering if necessary, we may assume there is $\ell \in \{1, \ldots, M - 1\}$ such that $\{0, \ldots, \ell - 1\}$ is a union of equivalence classes of $\overset{r}{\sim}$, i.e., such that $i \overset{r}{\not\sim} i'$ whenever $0 \leq i < \ell \leq i' < M$. (Actually, using nondegeneracy and Remark 2.4, we must have

$2 \leq \ell \leq M - 2$.) We have $a = a' + a''$, where

$$a' = r_0 a_0 + \cdots + r_{\ell-1} a_{\ell-1}$$
$$a'' = r_\ell a_\ell + \cdots + r_{M-1} a_{M-1}.$$

Using the defining relations of Γ_π and the fact that π is realizable with $r_0, \ldots, r_{M-1}$, $s_0, \ldots, s_{N-1}$, we get $a'b = 1$ and $a''b = 0$.

Since $\mathrm{rank}(a') = \ell < M$ and $\mathrm{rank}(b) = N$, by hypothesis we have $ba' = 1$. But then, in $K[\Gamma_\pi]$, we have

$$0 = 0a' = (a''b)a' = a''(ba') = a''.$$

Since π is nondegenerate and all the r_j are nonzero, this gives a contradiction.

The proof of part (b) is similar: assuming π is not column connected, we get analogously $b = b' + b''$ with $ab' = 1$ and $ab'' = 0$, and this yields $b'a = 1$ and $b'' = 0$, giving a contradiction.   □

Definition 2.18. Let K be a division ring and let $m, n \geq 2$ be integers. Let $\mathrm{ULIE}_K^{(1)}(m, n)$ be the set of all groups Γ_π as in Definition 2.2 as π runs over all partitions π of the set (4) that are nondegenerate, minimally realizable over K, row connected and column connected. We will say that an *$ULIE_K^{(1)}$ group* is one that belongs to the set $\bigcup_{m,n \geq 2} \mathrm{ULIE}_K^{(1)}(m, n)$. Similarly, we let $\mathrm{ULIE}^{(1)}(m, n)$ denote the set of all groups Γ_π as π runs over all partitions π of (4) that are nondegenerate, minimally realizable, row connected and column connected, and say that an *$ULIE^{(1)}$ group* is one that belongs to the set $\bigcup_{m,n \geq 2} \mathrm{ULIE}^{(1)}(m, n)$. Finally, we let $\mathrm{ULIE}_K^{(1-)}(m, n) \subseteq \mathrm{ULIE}_K^{(1)}(m, n)$ be equal to $\mathrm{ULIE}_K^{(1)}(m, n)$ if $m \neq n$ and, if $m = n$, we let it consist of the complement of the set of $\mathrm{ULIE}_K^{(1)}(m, n)$–groups Γ_π for partitions π that are minimally realizable for some $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1} \in K$ and so that the partition forces equality $r_0 a_0 + \cdots + r_{m-1} a_{m-1} = s_0 b_0 + \cdots + s_{n-1} b_{n-1}$ in the group ring $K[\Gamma_\pi]$.

Now Lemma 2.17 gives the following variant of Theorem 2.14:

Theorem 2.19. *Let K be a division ring and let $M, N \geq 2$ be integers. Suppose that for every group*

$$\Gamma \in \bigcup_{\substack{2 \leq m \leq M \\ 2 \leq n \leq N}} \mathrm{ULIE}_K^{(1-)}(m, n), \tag{8}$$

the group ring $K[\Gamma]$ is directly finite. Then Kaplansky's Direct Finiteness Conjecture holds over K for rank pair (M, N).

Proof. Arguing first by induction on $M + N$, we may assume that Kaplansky's Direct Finiteness Conjecture holds over K for all rank pairs (m, n) appearing in (8) provided $(m, n) \neq (M, N)$. We now proceed as in the proof of Theorem 2.14, with (M, N) replacing (m, n), except we note that the equality $ab = 1$ in $K[\Gamma_\pi]$ implies, grace of Lemma 2.17, that π is row connected and column connected.   □

Corollary 2.20. *Let K be a division ring. Then Kaplansky's conjecture holds over K if and only if for all $ULIE_K^{(1-)}$ groups Γ, the group ring $K[\Gamma]$ is directly finite.*

From the calculations reported in Section 6, we now have the following:

Proposition 2.21. *Let m and n be odd integers with either (a) $\min(m,n) = 3$ and $\max(m,n) \leq 11$ or (b) $m = n = 5$. Then Kaplansky's Direct Finiteness Conjecture holds over the field $\mathbb{F}_2$ of two elements for rank pair (m,n).*

Proof. All of the $\mathrm{ULIE}_{\mathbb{F}_2}^{(-)}(m,n)$ groups Γ_π have been computed and they are summarized in Section 6. In case (a), they are all amenable while in case (b), all are amenable except for two, which by the main result of [2], [7] and [13], are seen to be sofic. Hence, by [5], each group ring $\mathbb{F}_2[\Gamma_\pi]$ is directly finite. Now Theorem 2.14 applies. □

## 3. An infinite family of non-amenable ULIE groups

We describe infinitely many nondegenerate partitions that yield non-amenable $\mathrm{ULIE}_{\mathbb{F}_2}^{(1)}$ groups. These groups are, however, known to be sofic.

For an integer $n \geq 2$, we describe a pair partition π of the set

$$\big(\{0, 1, \ldots, 2n\} \times \{0, 1, \ldots, 2n\}\big) \backslash \{(0,0)\}.$$

The pair partition is described on a $(2n+1) \times (2n+1)$ grid (rows and columns numbered from 0 to $2n$) by (a) drawing lines between positions and (b) writing numbers in the positions. If there is a straight line between positions (i,j) and (k,ℓ) or if the same number is written in positions (i,j) and (k,ℓ), then this indicates $\{(i,j),(k,\ell)\} \in \pi$. See Figure 1. For example, the cross in the upper left corner indicates the pairings $(1,1) \sim (2,2)$ and $(1,2) \sim (2,1)$. Also, the numbers 1 and 2

FIGURE 1. A $(2n+1) \times (2n+1)$ partial pair partition.
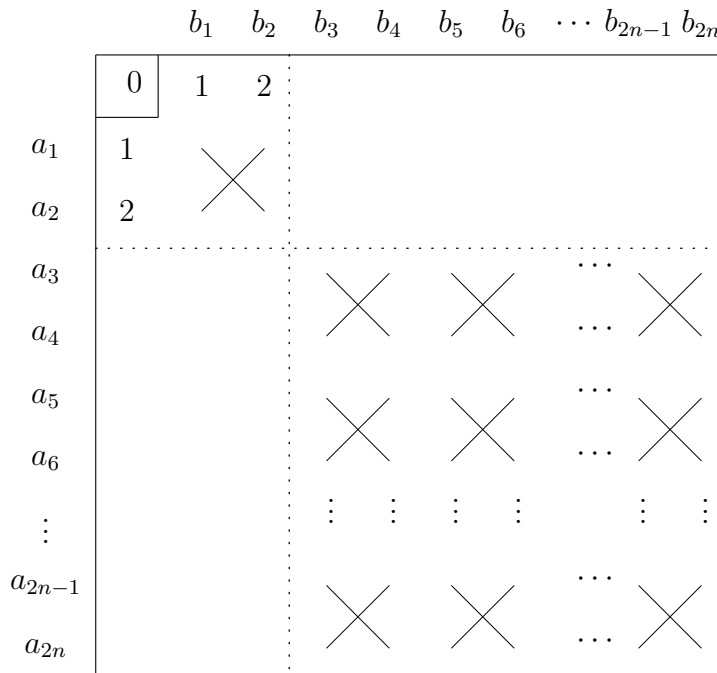
in the picture indicate, respectively, the pairings $(0, 1) \sim (1, 0)$ and $(0, 2) \sim (2, 0)$. The long $3 \times (2n - 2)$ block on the upper right and the tall $(2n - 2) \times 3$ block on the lower left, both partially outlined with dotted lines, are still to be filled in. We have written the group elements above and to the left of the grid, to remind us that a pairing between positions (i, j) and (k, ℓ) leads to the relation $a_i b_j = a_k b_\ell$ in the group Γ_π.

The finitely presented group with generators $a_1, \ldots, a_{2n}, b_1, \ldots, b_{2n}$ and relations dictated by the pairings indicated in Figure 1 is isomorphic to the group with presentation

$$\langle s, t, a_1, a_3, a_5 \ldots, a_{2n-1}, b_3, b_5, \ldots, b_{2n-1} \mid s^2 = t^2 = [a_1, s] = 1 \rangle, \qquad (9)$$

where $[x, y]$ means the multiplicative commutator $xyx^{-1}y^{-1}$, with the isomorphism implemented by

$$
\begin{aligned}
a_j &\mapsto a_j & &(j \text{ odd}) \\
b_j &\mapsto b_j & &(j \text{ odd}, \ j \geq 3) \\
a_2 &\mapsto a_1 s & &\hspace{4em}(10) \\
b_2 &\mapsto a_1 s & &\hspace{4em}(11) \\
a_k &\mapsto a_{k-1} t & &(k \text{ even}, \ k \geq 4) \hspace{2em}(12) \\
b_k &\mapsto t\, b_{k-1} & &(k \text{ even}, \ k \geq 4) \hspace{2em}(13)
\end{aligned}
$$

We relabel the group elements to incorporate the identifications (10)–(13). Thus, the top row in Figure 1 becomes

$$a_1 \quad a_1 s \quad b_3 \quad t b_3 \quad b_5 \quad t b_5 \quad \cdots \quad b_{2n-1} \quad t b_{2n-1}$$

while the left–most column in Figure 1 becomes (the transpose of)

$$a_1 \quad a_1 s \quad a_3 \quad a_3 t \quad a_5 \quad a_5 t \quad \cdots \quad a_{2n-1} \quad a_{2n-1} t$$

Now we fill in the remaining pairings to create a complete pair partition. The upper right–hand $3 \times (2n - 2)$ block gets filled in as indicated in Figure 2, while the lower left–hand $(2n - 2) \times 3$ block gets filled in as indicated in Figure 3. This completes the pair partition π of $(\{0, 1, \ldots, 2n\} \times \{0, 1, \ldots, 2n\}) \backslash \{(0, 0)\}$. The group

FIGURE 2. The numbers we put into the upper right $3 \times (2n - 2)$ block.

	b_3	$t b_3$	b_5	$t b_5$	$\cdots$	b_{2n-1}	$t b_{2n-1}$
	3	4	5	6	$\cdots$	$2n - 1$	$2n$
a_1	$2n + 1$	$2n + 2$	$2n + 3$	$2n + 4$	$\cdots$	$4n - 3$	$4n - 2$
$a_1 s$	$4n - 1$	$4n$	$4n + 1$	$4n + 2$	$\cdots$	$6n - 5$	$6n - 4$

Γ_π equals the quotient of the group (9) by the additional relations corresponding the the numbers 3 to $6n - 4$, according to Figures 2 and 3.

Let R_j denote the relation implied by the pairing indicated by the number j in Figures 2 and 3. We have

$$R_3: \qquad b_3 = a_3 a_1 s$$
$$R_4: \qquad t b_3 = a_3 t a_1 s$$

$$\vdots \qquad\qquad \vdots$$

$$R_{2n-1}: \qquad b_{2n-1} = a_{2n-1} a_1 s$$
$$R_{2n}: \qquad t b_{2n-1} = a_{2n-1} t a_1 s.$$

These are equivalent to the relations

$$b_j = a_j a_1 s \tag{14}$$
$$[t, a_j] = 1 \tag{15}$$

for all j odd, $3 \leq j \leq 2n - 1$. For this same range of j values, relations R_{2n+1} to R_{4n-2} give us

$$a_1 b_j = a_j \tag{16}$$
$$a_1 t b_j = a_j t, \tag{17}$$

which, using (14) and (15) and $[a_1, s] = s^2 = 1$, are seen to be equivalent to

$$a_j^{-1} a_1 a_j = a_1^{-1} s \tag{18}$$
$$[t, a_1] = 1. \tag{19}$$

Again for the same range of j values, R_{4n-1} to R_{6n-4} give us

$$a_1 s b_j = a_j a_1 \tag{20}$$
$$a_1 s t b_j = a_j t a_1. \tag{21}$$

FIGURE 3. The numbers we put into the lower left $(2n - 2) \times 3$ block.

	a_1	$a_1 s$	
a_3	$2n + 1$	$4n - 1$	3
$a_3 t$	$2n + 2$	$4n$	4
a_5	$2n + 3$	$4n + 1$	5
$a_5 t$	$2n + 4$	$4n + 2$	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{2n-1}	$4n - 3$	$6n - 5$	$2n - 1$
$a_{2n-1}t$	$4n - 2$	$6n - 4$	$2n$

Using (14) and $[a_1, s] = s^2 = 1$, the first of these is equivalent to

$$a_j^{-1}a_1sa_j = s, \tag{22}$$

while using also (15) and (19), we see that (21) yields

$$[t, s] = 1.$$

Taking (18) and (22) together gives

$$a_j^{-1}sa_j = a_1,$$

which implies $a_1^2 = 1$.

Therefore, in the group Γ_π, the relations

$$s^2 = t^2 = [a_1, s] = [a_1, t] = [s, t] = 1, \tag{23}$$

$$([a_j, t] = 1)_{3 \le j \le 2n-1, \, j \text{ odd}}, \tag{24}$$

$$(a_j^{-1}sa_j = a_1)_{3 \le j \le 2n-1, \, j \text{ odd}}, \tag{25}$$

$$(a_j^{-1}a_1a_j = a_1s)_{3 \le j \le 2n-1, \, j \text{ odd}} \tag{26}$$

hold, and we easily see that they imply the relations (14), (15), (16), (17), (20) and (21). Thus, Γ_π has presentation with generators $s, t, a_1, a_3, \ldots, a_{2n-1}$ and relations (23)–(26).

We see that the relations (23)–(26) are equivalently described by:

(i) t is in the center
(ii) the subgroup H generated by s and a_1 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$
(iii) conjugation by a_j for every $j \in \{3, 5, \ldots, 2n-1\}$ implements the same automorphism α, of H, which is the automorphism of order 3 that cycles the nontrivial elements of H.

The group Γ_π is, therefore, isomorphic to

$$\overset{t}{\mathbb{Z}_2} \times \left(\left(\overset{s}{\mathbb{Z}_2} \times \overset{a_1}{\mathbb{Z}_2} \right) \rtimes_{\alpha * \cdots * \alpha} \left(\underbrace{\overset{a_3}{\mathbb{Z}} * \overset{a_5}{\mathbb{Z}} * \cdots * \overset{a_{2n-1}}{\mathbb{Z}}}_{n-1 \text{ times}} \right) \right), \tag{27}$$

where the symbols appearing above the cyclic groups indicate the corresponding generators of the groups. When $n \ge 3$, this group is non-amenable. The semidirect product group appearing above is isomorphic to the free product of $n - 1$ copies of the amenable group

$$(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\alpha \mathbb{Z}$$

with amalgamation over $\mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, the group Γ_π is sofic, (by the main result of [2], [7] and [13]).

4. The Invertibles Conjecture

See Conjecture 1.2 for a statement of the Invertibles Conjecture. Here are some related finer considerations.

Definition 4.1. Let K be a division ring. We will say that the *Invertibles Conjecture holds over K* if $K[G]$ contains no one–sided invertible elements of rank > 1 for all torsion–free groups G. For integers $m, n \ge 2$, we will say that the *Invertibles Conjecture holds for rank pair (m, n)* if for all division rings K and all torsion–free

groups G, $K[G]$ contains no two elements a and b having ranks m and n, respectively, such that $ab = 1$. We will say that the *Invertibles Conjecture holds for rank m* if it holds for rank pairs (m, n), for all integers $n \geq 2$, namely, if the existence of a right–invertible element of rank m in a group algebra $K[G]$ implies G has torsion. (By the method described in Remark 2.13, we may replace "right–invertible" by "one–sided invertible" in the previous sentence.) Intersections of these properties (e.g., over K for rank pair (m, n)) have the obvious meaning.

It is well known and not difficult to show that if $a \in K[G]$ is one-sided invertible and has rank 2, then it is invertible and is of the form $sh(1 - rg)$ for $s, r \in K\backslash\{0\}$ and for $g, h \in G$ where g has finite order $n > 1$ and $r^n \neq 1$. As a consequence we have:

Theorem 4.2. *The Invertibles Conjecture holds for rank 2.*

We first describe the smallest normal subgroup whose corresponding quotient is torsion–free. This is surely well known, but it doesn't take long.

Definition 4.3. Given a group Γ, let $N_{\mathrm{tor}}^{(1)}(\Gamma)$ be the smallest normal subgroup of Γ that contains all torsion elements of Γ. We now recursively define normal subgroups $N_{\mathrm{tor},n}$ of Γ, $n \geq 1$, by letting $N_{\mathrm{tor},1} = N_{\mathrm{tor}}^{(1)}(\Gamma)$ and, given $N_{\mathrm{tor},n}$, letting $\phi_n : \Gamma \rightarrow \Gamma/N_{\mathrm{tor},n}$ be the quotient map and $N_{\mathrm{tor},n+1} = \phi_n^{-1}(N_{\mathrm{tor}}^{(1)}(\Gamma/N_{\mathrm{tor},n}))$. Let $N_{\mathrm{tor}}(\Gamma) = \bigcup_{n=1}^{\infty} N_{\mathrm{tor},n}$. Clearly, $N_{\mathrm{tor}}(\Gamma)$ is a normal subgroup of Γ.

Proposition 4.4. *If $\Gamma/N_{\mathrm{tor}}(\Gamma)$ is nontrivial, then it is torsion–free. Moreover, if N is a normal subgroup of Γ so that Γ/N is torsion–free, then $N_{\mathrm{tor}}(\Gamma) \subseteq N$.*

Proof. If $g \in \Gamma$ and $g^k \in N_{\mathrm{tor}}(\Gamma)$ for some $k \in \mathbb{N}$, then $g^k \in N_{\mathrm{tor},n}$ for some $n \in \mathbb{N}$, and, consequently, $g \in N_{\mathrm{tor},n+1}$, so $g \in N_{\mathrm{tor}}(\Gamma)$. This implies the first statement. If Γ/N is torsion–free, then clearly $N_{\mathrm{tor}}^{(1)}(\Gamma) \subseteq N$. Now for any $g \in \Gamma$ so that $g^k \in N_{\mathrm{tor}}^{(1)}(\Gamma)$, if $g \notin N$, then the class of g would have finite order in Γ/N contrary to hypothesis; thus, $N_{\mathrm{tor},2} \subseteq N$. Continuing in this way, we see by induction that $N_{\mathrm{tor},n} \subseteq N$ for all n. So $N_{\mathrm{tor}}(\Gamma) \subseteq N$. $\square$

Theorem 4.5. *Let K be a division ring and let $m, n \geq 2$ be integers. Then the following are equivalent:*

(i) The Invertibles Conjecture holds over K for rank pair (m, n).
(ii) for every $\mathrm{ULIE}_K(m, n)$–group Γ with its canonical generators $1 = a_0, a_1, \ldots, a_{m-1}$ and $1 = b_0, b_1, \ldots, b_{n-1}$, letting $\phi : \Gamma \rightarrow \Gamma/N_{\mathrm{tor}}(\Gamma)$ be the quotient map, we have $\phi(a_i) = \phi(a_{i'})$ for some $0 \leq i < i' \leq m - 1$ or $\phi(b_j) = \phi(b_{j'})$ for some $0 \leq j < j' \leq n - 1$.

Proof. For (ii) $\Longrightarrow$ (i), suppose the Invertibles Conjecture over K for rank pair (m, n) fails. Then there is a torsion–free group G such that $K[G]$ contains elements $\tilde{a}$ of rank m and $\tilde{b}$ of rank n such that $\tilde{a}\tilde{b} = 1$. After allowable modifications, we may without loss of generality write $\tilde{a} = 1 + r_1\tilde{a}_1 + \cdots r_{m-1}\tilde{a}_{m-1}$ and $\tilde{b} = s_0 1 + s_1\tilde{b}_1 + \cdots + s_{n-1}\tilde{b}_{n-1}$ for some $\tilde{a}_1, \ldots, \tilde{a}_{m-1}$ distinct, nontrivial elements of G and $\tilde{b}_1, \ldots, \tilde{b}_{n-1}$ distinct, nontrivial elements of G and for $r_1, \ldots, r_{m-1}, s_0, \ldots, s_{n-1} \in K\backslash\{0\}$. Letting σ be

the cancellation partition for $\tilde{a}\tilde{b} = 1$ and taking $\pi \leq \sigma$ that is minimally realizable for $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1}$, we have a group homomorphism from the $\text{ULIE}_K(m, n)$ group Γ_π into G that sends canonical generators a_i to $\tilde{a}_i$ and b_i to $\tilde{b}_i$. Of, course, we have $ab = 1$ in $K[\Gamma_\pi]$, where $a = r_0 a_0 + \cdots + r_{m-1} a_{m-1}$ and $b = s_0 b_0 + \cdots + s_{n-1} b_{n-1}$. Since G is torsion–free, by Proposition 4.4, the kernel of the above homomorphism contains $N_{\text{tor}}(\Gamma_\pi)$. Since $1, \tilde{a}_1, \ldots, \tilde{a}_{m-1}$ are distinct and $1, \tilde{b}_1, \ldots, \tilde{b}_{n-1}$ are distinct it follows that the images of $1, a_1, \ldots, a_{m-1}$ in the quotient $\Gamma/N_{\text{tor}}(\Gamma_\pi)$ are distinct, as are the images of of $1, b_1, \ldots, b_{n-1}$, and (ii) fails.

For (i) $\implies$ (ii), suppose that for some $\text{ULIE}_K(m, n)$–group $\Gamma = \Gamma_\pi$ and for ϕ the quotient map to $\Gamma/N_{\text{tor}}(\Gamma)$, the elements $1, \phi(a_1), \ldots, \phi(a_{m-1})$ are distinct and $1, \phi(b_1), \ldots, \phi(b_{n-1})$ are distinct. Now the partition π is realizable with some $r_0, \ldots, r_{m-1}, s_0, \ldots, s_{n-1} \in K \backslash \{0\}$, so letting $a = r_0 1 + r_1 a_1 + \cdots + r_{m-1} a_{m-1}$ and $b = s_0 1 + s_1 b_1 + \cdots + s_{n-1} b_{n-1}$ in $K[\Gamma]$, we have $ab = 1$. Extending the quotient map ϕ linearly to a ring homomorphism $K[\Gamma] \to K[\Gamma/N_{\text{tor}}(\Gamma)]$, we get that $\phi(a)$ has rank m and $\phi(b)$ has rank n and $\phi(a)\phi(b) = 1$. In particular, $\Gamma/N_{\text{tor}}(\Gamma)$ is nontrivial. By Proposition 4.4, it is torsion–free. So the Invertibles Conjecture fails over K for rank pair (m, n). $\qquad\square$

Remark 4.6. It is well known and easy to show that for a torsion–free abelian group G and K a division ring, $K[G]$ has no invertible elements of rank strictly greater than 1. Thus, the setting of Theorem 4.5, if $\Gamma/N_{\text{tor}}(\Gamma)$ is abelian, then $1, \phi(a_1), \ldots, \phi(a_{m-1})$ cannot be distinct.

Example 4.7. For the non-amenable ULIE groups Γ_π considered at (27) in Section 3, we easily see $\Gamma/N_{\text{tor}}(\Gamma)$ is a copy of the free group on $n-1$ generators, but the quotient map sends a_1 to the identity.

If, instead of considering each rank pair (m, n) individually, we start small and increase one rank at a time, then we can get away with considering a smaller set of partitions and corresponding ULIE groups.

Keeping in mind the special role of $(0, 0)$ in

$$\{0, \ldots, m - 1\} \times \{0, \ldots, n - 1\} \tag{28}$$

as pertains to ULIE groups, we make the following definition.

Definition 4.8. Let $m, n \geq 2$ be integers and let π be a partition of (28). An *invariant subgrid* of π is a pair (R, C) with $0 \in R \subset \{0, 1, \ldots, m - 1\}$ and $0 \in C \subset \{0, 1, \ldots, n - 1\}$, $|R| \geq 2$ and $|C| \geq 2$, so that whenever $(i, j) \in R \times C$ and $(i, j) \overset{\pi}{\sim} (i', j') \in \{0, \ldots, m - 1\} \times \{0, \ldots, n - 1\}$, then $(i', j') \in R \times C$. The subgrid (R, C) is *proper* if either $|R| < m$ or $|C| < n$.

Note that partitions without proper invariant subgrids must be row and column connected.

Lemma 4.9. *Let K be a division ring and let $M, N \geq 2$ be integers. Suppose G is a torsion–free group and suppose $c, d \in K[G]$ having ranks M and N, respectively, both have the identity element of G in their supports and satisfy $cd = 1$. Write $c = r_0 c_0 + \cdots r_{M-1} c_{M-1}$ and $d = s_0 d_0 + \cdots s_{N-1} d_{N-1}$ for group elements c_i and d_j*

and assume $c_0 = d_0 = 1$. Let σ be the cancellation partition of (c,d) with respect to these orderings of the supports and let $\pi \leq \sigma$ be any partition that is realizable with $r_0, \ldots, r_{M-1}, s_0, \ldots, s_{N-1}$. Suppose the Invertibles Conjecture holds over K for all rank pairs (m,n), with $2 \leq m \leq M$ and $2 \leq n \leq N$ and $(m,n) \neq (M,N)$. Then π has no proper invariant subgrids.

Proof. Suppose for contradiction that π has a proper invariant subgrid (R, C). Without loss of generality we may suppose $R = \{0, \ldots, m-1\}$ and $C = \{0, \ldots, n-1\}$. Let π' be the restriction of π to $R \times C$. Then π' is realizable with $r_0, \ldots, r_{m-1}$ and $s_0, \ldots, s_{n-1}$. Let $\pi'' \leq \pi'$ be a partition of $R \times C$ that is minimally realizable with $r_0, \ldots, r_{m-1}$ and $s_0, \ldots, s_{n-1}$. Let $\Gamma_{\pi''}$ be the corresponding $\mathrm{ULIE}_K(m,n)$–group with its canonical generators $a_0, \ldots, a_{m-1}$ and $b_0, \ldots, b_{n-1}$. Then there is a group homomorphism $\psi : \Gamma_{\pi''} \to G$ so that $\psi(a_i) = c_i$ and $\psi(b_j) = d_j$. Since G is torsion–free, by Proposition 4.4, $N_{\mathrm{tor}}(\Gamma_{\pi''}) \subseteq \ker \psi$. By hypothesis, the Invertibles Conjecture holds over K for rank pair (m,n). Thus, by Theorem 4.5, the mapping ψ must identify either two distinct a_i and $a_{i'}$ with each other or two distinct b_j and $b_{j'}$ with each other, which contradicts that $c_0, \ldots, c_{m-1}$ are distinct and $d_0, \ldots, d_{n-1}$ are distinct. $\square$

Definition 4.10. Let K be a division ring and let $m, n \geq 2$ be integers. Let $\mathrm{ULIE}_K^{(2)}(m,n)$ be the set of all groups Γ_π as in Definition 2.2 as π runs over all partitions π of the set (28) that are nondegenerate, minimally realizable over K and have no proper invariant subgrids. We will say that an $ULIE_K^{(2)}$ *group* is one that belongs to the set $\bigcup_{m,n \geq 2} \mathrm{ULIE}_K^{(2)}(m,n)$. Similarly, we let $\mathrm{ULIE}^{(2)}(m,n)$ denote the set of all groups Γ_π as π runs over all partitions π of (28) that are nondegenerate, minimally realizable (over some division ring) and have no proper invariant subgrids, and say that an $ULIE^{(2)}$ *group* is one that belongs to the set $\bigcup_{m,n \geq 2} \mathrm{ULIE}^{(2)}(m,n)$.

Now Lemma 4.9 gives the following variant of Theorem 4.5:

Theorem 4.11. *Let K be any nonzero field or division ring and let $M, N \geq 2$ be integers. Suppose that for every group*

$$\Gamma \in \bigcup_{\substack{2 \leq m \leq M \\ 2 \leq n \leq N}} \mathrm{ULIE}_K^{(2)}(m,n) \tag{29}$$

with its canonical generators $1 = a_0, a_1, \ldots, a_{m-1}$ and $1 = b_0, b_1, \ldots, b_{n-1}$, letting $\phi : \Gamma \to \Gamma/N_{\mathrm{tor}}(\Gamma)$ be the quotient map, we have $\phi(a_i) = \phi(a_{i'})$ for some $0 \leq i < i' \leq m - 1$ or $\phi(b_j) = \phi(b_{j'})$ for some $0 \leq j < j' \leq n - 1$. Then the Invertibles Conjecture holds over K for rank pair (M, N).

Proof. Arguing first by induction on $M + N$, we may assume the Invertibles Conjecture holds over K for all rank pairs (m,n) appearing in (29) provided $(m,n) \neq (M,N)$. Now we proceed as in the proof of (ii) $\implies$ (i) in Theorem 4.5, but using (M,N) instead of (m,n), except we note that the equality $ab = 1$ in $K[\Gamma_\pi]$ implies, grace of Lemma 4.9, that π has no proper invariant subgrids. $\square$

From the calculations reported in Section 6, we now have the following:

Proposition 4.12. *Let m and n be odd integers with either (a) $\min(m,n) = 3$ and $\max(m,n) \leq 11$ or (b) $m = n = 5$. Then the Invertibles Conjecture holds over the field $\mathbb{F}_2$ of two elements for rank pair (m,n).*

Proof. All of the $\mathrm{ULIE}_{\mathbb{F}_2}(m,n)$ groups Γ_π in the cases (a) and all the $\mathrm{ULIE}^{(-)}(5,5)$ groups in case (b) have been computed and they are described in [3] and summarized in Section 6. In all cases, one easily verifies that the quotient groups $\Gamma_\pi/N_{\mathrm{tor}}(\Gamma_\pi)$ are abelian. As described in Remark 4.6, it follows that the quotient map $\phi : \Gamma \to \Gamma_\pi/N_{\mathrm{tor}}(\Gamma_\pi)$ fails to be one–to–one on $\{1, a_1, \ldots, a_{m-1}\}$. For the $\mathrm{ULIE}_{\mathbb{F}_2}(5,5)$–groups that are not in $\mathrm{ULIE}_{\mathbb{F}_2}^{(-)}(5,5)$, this lack of injectivity of ϕ on $\{1, a_1, \ldots, a_{m-1}\}$ is verified directly in [3].Now Theorem 4.5 applies. $\square$

5. A procedure for computations of ULIE groups over $\mathbb{F}_2$

Our aim is, for certain m and n, to compute the ULIE groups of all nondegenerate pairings of the $(2m + 1) \times (2n + 1)$ grid

$$E = (\{0, 1, \ldots, 2m\} \times \{0, 1, \ldots, 2n\}) \backslash \{(0, 0)\}. \tag{30}$$

Our strategy is to use a C++ code to enumerate the pairings and for each pairing π to call GAP [8] to compute the finitely presented group Γ_π as in equation (7) and to determine whether the pairing π degenerates. One difficulty with this strategy is that the Knuth–Bendix procedure employed by GAP to try to decide when a given word is equivalent to the identity in a finitely presented group may not terminate in a reasonable amount of time and is not even guaranteed to ever terminate. Any groups for which this approach fails to determine degeneracy and/or to decide the $ba = 1$ question, must be handled separately. We handle this by having a timeout routine inside the C++ code that aborts the GAP computation after a couple of seconds and marks this pairing matrix for manual analysis. We also separate the construction of valid pairings from the degeneracy analysis in GAP for performance reasons: the C++ code can enumerate pairings much more efficiently.

In order to limit the number of costly calls into GAP, we have considered a natural equivalence relation on pairings, and run GAP on only one pairing from each equivalence class. All relevant pairings are constructed in a recursive procedure by filling in entries one by one in a pairing matrix. This procedure forms a tree with pairings being the leaves of the tree. Branches that can not produce valid pairing matrices (because they are handled in a different branch due to the equivalence relation, or because they will never produce a valid pairing matrix later) are skipped as soon as it is known.

For permutations σ and τ of $\{0, 1, \ldots, 2m\}$ and $\{0, 1, \ldots, 2n\}$, respectively, both of which fix 0, let $\tilde{\pi}$ be the image of π under the permutation $\sigma \times \tau$, and let $\tilde{a}$ and $\tilde{b}$ be the elements of $\Gamma_{\tilde{\pi}}$ that are analogous to a and b. Then Γ_π is degenerate if and only if $\Gamma_{\tilde{\pi}}$ is degenerate, and $ba = 1$ in $\mathbb{F}_2[\Gamma_\pi]$ if and only if $\tilde{b}\tilde{a} = 1$ in $\mathbb{F}_2[\Gamma_{\tilde{\pi}}]$. Thus, the equivalence relation on the set of pairings that we use is the one induced by this natural action of $S_{2m} \times S_{2n}$.

We will encode pairings as $(2m+1) \times (2n+1)$ matrices, as described below. Keeping with the convention that the elements in the support of a are numbered starting with

a_0, and similarly for b, we will index the entries of a $(2m+1) \times (2n+1)$ matrix A as a_{ij} with $0 \le i \le 2m$ and $0 \le j \le 2n$.

Definition 5.1. A $(2m+1) \times (2n+1)$ *pairing matrix* is a $(2m+1) \times (2n+1)$ matrix A with

- -1 in the $(0,0)$ entry
- all other entries of A coming from the set $\{1, 2, \ldots, 2mn + m + n\}$
- each element of $\{1, 2, \ldots, 2mn + m + n\}$ appearing in exactly two entries of A
- no row or column of A containing a repeated value.

Definition 5.2. A $(2m+1) \times (2n+1)$ *partial pairing matrix* is a $(2m+1) \times (2n+1)$ matrix A with

- -1 in the $(0,0)$ entry
- all other entries of A coming from the set $\{0, 1, 2, \ldots, 2mn + m + n\}$
- no element of $\{1, 2, \ldots, 2mn + m + n\}$ appearing in more than two entries of A
- no row or column of A containing a repeated nonzero value.

We think about traversing an $(2m+1) \times (2n+1)$ matrix, starting at the $(0,1)$ entry, by proceeding towards the right until we reach the $(0, 2n)$ entry, then taking the next row, starting at the $(1, 0)$ entry, moving from left to right until the $(1, 2n)$, and so on, row after row, until we reach the $(2m, 2n)$ entry. In fact, we will eventually construct pairing matrices by filling in the entries in this order. We say that a partial pairing matrix A is *stacked* if, when we traverse the matrix as described above, if we once encounter a zero, then all the following entries are zero. This is expressed more precisely below.

Definition 5.3. A partial pairing matrix $A = (a_{ij})_{0 \le i \le 2m, 0 \le j \le 2n}$ is *stacked* if $a_{ij} = 0$ implies $a_{i\ell} = 0$ for all $\ell > j$ and $a_{k\ell} = 0$ for all $k > i$ and all $0 \le \ell \le 2n$.

Definition 5.4. We say that a partial pairing matrix $A = (a_{ij})_{0 \le i \le 2m, 0 \le j \le 2n}$ is *consecutively numbered* if for the list

$$a_{0,1}, a_{0,2}, \ldots, a_{0,2n}, \, a_{1,0}, a_{1,1}, \ldots, a_{1,2n},$$
$$a_{2,0}, a_{2,1}, \ldots, a_{2,2n}, \, \ldots, \tag{31}$$
$$a_{2m,0}, a_{2m,1}, \ldots, a_{2m,2n},$$

when relabeled as $b_1, b_2, \ldots, b_{4mn+2m+2n}$, the set $\{b_1, \ldots, b_q\}$ of every initial segment (ignoring repeats and rearranging) is equal to a set of the form $\{0, 1, \ldots, r\}$ or $\{1, 2, \ldots, r\}$ for some non-negative integer r.

A partial pairing matrix $A = (a_{ij})_{0 \le i \le 2m, 0 \le j \le 2n}$ yields an equivalence relation on the set E as in (30) given by

$$(i, j) \sim (k, \ell) \iff a_{ij} = a_{k\ell} \ne 0,$$

and the equivalence classes are all singletons or pairs. They are all pairs if and only if A is a pairing matrix, and we will call such an equivalence relation a *restricted pairing* of E. Note that the restricted pairings are precisely the partitions of the set

E into pairs and singletons so that no entry is paired with another in the same row or column.

Given a partial pairing matrix A, there is a unique partial pairing matrix B that yields the same equivalence relation as A and such that B is consecutively numbered. We call B the *consecutive renumbering* of A. To compute the consecutive renumbering B one has to traverse through all the entries of A row-wise and replace the numbers according to a map that gets created during the traversal. If an entry for a certain number already exists in the mapping, it is used, otherwise the smallest unused positive number will be taken.

We will consider the action α of the product $S_{2m} \times S_{2n}$ of symmetric groups on the set of all $(2m + 1) \times (2n + 1)$ partial pairing matrices, by permutations of rows numbered $1, 2, \ldots, 2m$ and columns numbered $1, 2, \ldots, 2n$. Restricting this action to the set of all pairing matrices, it descends to an action β of $S_{2m} \times S_{2n}$ on the set of all restricted pairings of the set E in (30).

We will now describe an algorithm that will generate a set $R_{m,n}$ consisting of one consecutively numbered pairing matrix for each orbit of β (i.e., whose restricted pairing belongs to the given orbit of β). We will begin with the matrix A_0 which has zero in every entry except for a -1 in the $(0, 0)$ entry, and we proceed via a branching process, filling in the entries of the matrix, one after the other, so that we generate a tree, $T_{m,n}$, of stacked, consecutively numbered $(2m + 1) \times (2n + 1)$ pairing matrices rooted at A_0, and so that all the branches flowing from a given matrix A are obtained from A be replacing a single zero entry by a nonzero entry. If the first zero entry of A is in the (i, j) entry, then the branches at this node are determined by the set V of possible nonzero values to place in the (i, j) position. Clearly, V will be a subset of the union $H \cup N$, where H is the set of strictly positive integers that appear in exactly one entry of A (the so-called half–pairs of A) and do not appear in the ith row or jth column of A, and N is either the singleton set $\{\ell + 1\}$, where ℓ is the largest value that appears as an entry of A or, if $\ell = 2mn + m + n$, then N is the empty set.

In order to specify V, consider the total ordering $<$ on the set of all $(2m + 1) \times (2n + 1)$ partial pairing matrices, which is defined as the lexicographic ordering on the sequences (31) associated to partial pairing matrices $A = (a_{ij})_{1 \leq i \leq m,\, 1 \leq j \leq n}$. Then $V = H' \cup N$, where H' is the set of all $h \in H$ such that, if A' is matrix obtained from A by setting the first zero entry (i.e., the (i, j) entry) to be h, then whenever

$$(\sigma, \rho) \in S_{2m} \times S_{2n} \tag{32}$$

is such that $\alpha((\sigma, \rho), A')$ is a stacked partial pairing matrix and B is the consecutive renumbering of it, we do not have $B < A'$.

In other words, we branch off at (i, j) with entry v only if there is no permutation with a smaller consecutive numbering than the current partial pairing matrix. Otherwise this case is already handled in a different branch of the tree and we would generate duplicate results.

Using the above algorithm, we will construct a tree, call it $\widetilde{T}_{m,n}$, that may have leaves that are not pairing matrices, i.e., have zeros in them. For example, in the

case $m = n = 1$, $\begin{pmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 4 & 0 \end{pmatrix}$ is such a leaf. We will call such a leaf *stunted*. We obtain the tree $T_{m,n}$ by pruning $\widetilde{T}_{m,n}$, lopping off all stunted leaves and all branches that end in only stunted leaves. Finally, the set $R_{m,n}$ consists of the matrices found at the leaves of $T_{m,n}$.

Note that the algorithm described here has several good properties: First, no duplicate pairings will be created. This simplifies the analysis and speeds up the computation as described before. Second, there is no global state to be kept around. All the operations are done with the current partial pairing matrix. We can determine if a branch has been done already using the ordering and without keeping track what we already touched. Third, the branching allows us to do parallel computations without any communication between processes. In a pre-process we can run the algorithm where we stop the recursion after the partial pairing matrix has k entries for some $k < (2m+1)(2n+1)$. If we write out these partial pairing matrices, we can start independent jobs for each of those matrices in parallel.

The code that implements this algorithm and the raw output of it are included in the directory `ULIE.computations` that was submitted to (and is retrievable from) the version of this work in the arXiv (identifier 1112.1790), as part of the source code.

6. Summary of the computations

The list of interesting ULIE groups of sizes $3 \times n$ for $n \in \{3, 5, 7, 9, 11\}$ and 5×5 can be found online in [3]. The conclusions drawn from these regarding cases of Kaplansky's Direct Finiteness Conjecture and the Invertibles Conjecture have already been mentioned in Propositions 2.21 and 4.12. Table 1 contains a short summary table of our findings concerning the equivalence classes of nondegenerate pairing matrices. More specifically, we consider the equivalence classes of pairing matrices

TABLE 1. The number of equivalence classes of nondegenerate pairing matrices yielding different sorts of groups.

size	finite	infinite abelian	infinite nonabelian amenable	nonamenable
3 by 3	2	1	0	0
3 by 5	9	0	0	0
3 by 7	16	0	2	0
3 by 9	22	1	1	0
3 by 11	27	0	2	0
5 by 5, $a \neq b$	2741	78	86	2

for the equivalence relation described in Section 5. For each size, we list the number of equivalence classes of nondegenerate pairing matrices whose corresponding ULIE groups are, respectivly, finite, infinite abelian, infinite nonabelian amenable and non–amenable.

For the 5×5 case, we considered separately the pairing matrices that were equivalent to ones of the form

$$
\begin{pmatrix}
\times & 1 & 2 & 3 & 4 \\
1 & * & * & * & * \\
2 & * & * & * & * \\
3 & * & * & * & * \\
4 & * & * & * & *
\end{pmatrix}. \tag{33}
$$

These pairing matrices yield immediately the identification $a = b$ in the group ring, so are uninteresing for the Kaplansky conjecture (which tests whether $ab = 1$ implies $ba = 1$). However, this case $a = b$ needs to be treated to check the Invertibles Conjecture. We found 100 equivalence classes of pairing matrices in the 5×5, $a = b$ case; however, we did not determine precisely which are nondegenerate. These are described in the online document [3] and they all yield groups satisfying the criterion appearing in Thoerem 4.5(ii).

The equivalence classes of 5×5 matrices not containing any of the form (33) are summarized in Table 1 in the line "5 by 5, $a \neq b$." The two non–amenable groups are described below.

Group 6.1. $\langle x, y : x^4 = 1, x^2 y = y x^2 \rangle \cong \mathbb{Z}_4 *_{\mathbb{Z}_2} (\mathbb{Z} \times \mathbb{Z}_2)$, the amalgamated free product of $\mathbb{Z}_4 \cong \langle x : x^4 = 1 \rangle$ and $\mathbb{Z} \times \mathbb{Z}_2 \cong \langle y, z : z^2 = 1, yz = zy \rangle$ over $\mathbb{Z}_2$ by the identification $x^2 = z$, from the pairing matrix

$$
\begin{pmatrix}
\times & 1 & 2 & 3 & 4 \\
1 & 5 & 3 & 2 & 6 \\
6 & 4 & 7 & 8 & 5 \\
9 & 10 & 11 & 12 & 7 \\
10 & 9 & 12 & 11 & 8
\end{pmatrix}
$$

Group 6.2. $\langle x, y, z : x^2 = z^2 = 1, xy = yx, xz = zx \rangle \cong \mathbb{Z}_2 \times (\mathbb{Z} * \mathbb{Z}_2)$, from the pairing matrix

$$
\begin{pmatrix}
\times & 1 & 2 & 3 & 4 \\
1 & 5 & 3 & 2 & 6 \\
4 & 6 & 7 & 8 & 5 \\
9 & 10 & 11 & 12 & 7 \\
10 & 9 & 12 & 11 & 8
\end{pmatrix}
$$

Remark 6.3. The computations of ULIE groups showed that Kaplansky's Direct Finiteness Conjecture and the Invertibles Conjecture are valid for sizes $(3, n)$, $n \leq 11$ and $(5, 5)$. As expected, the ULIE groups themselves exhibited greater variety and complexity as the sizes of the pairing matrices increased and in Section 3 we found an infinite family of non-amenable ULIE groups. Whether computations of ULIE groups of larger sizes will disprove Kaplansky's Direct Finiteness Conjecture (and, thus, provide an example of a non-sofic group) remains to be seen.

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