A companion to the mini-course on full topological groups of Cantor minimal systems

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Contents

Chapter 1

The full topological group of Cantor minimal systems. Definition and basic facts.

We begin with basic definitions. The Cantor space is denoted by $\mathbf{C}$, it is characterized up to a homeomorphism as a compact, metrizable, perfect and totally disconnected topological space. The group of all homeomorphisms of the Cantor space is denoted by $Homeo(\mathbf{C})$. *A Cantor dynamical system* $(T, \mathbf{C})$ is the Cantor space together with its homeomorphism T.

Let A be a finite set, we will call it an alphabet. A basic example of Cantor space is the set of all sequences in A indexed by integers, $A^{\mathbb{Z}}$, and considered with product topology. A sequence $\{\alpha_i\}$ converges to α in this space if and only if for all n there exists i_0 such that for all $i \geq i_0$, we have that α_i coincides with α on the interval $[-n, n]$.

The basic example of a Cantor dynamical system is *the shift* on $A^{\mathbb{Z}}$, i.e., the map $s : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by

$$s(x)(i) = x(i + 1)$$

for all $x \in A^{\mathbb{Z}}$.

The system $(T, \mathbf{C})$ is *minimal* if there is no non-trivial closed T-invariant subset in $\mathbf{C}$. Equivalently, the closure of the orbit of T of any point p in $\mathbf{C}$

coincides with **C**:
$$\overline{\{T^ip : i \in \mathbb{Z}\}} = \mathbf{C}$$

One of the basic examples of the Cantor minimal system is *the odometer*, defined by the map $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$:

$$\sigma(x)(i) = \begin{cases} 0, & \text{if } i < n, \\ 1, & \text{if } i = n, \\ x(i), & \text{if } i > n \end{cases}$$

where n is the smallest integer such that $x(n) = 0$, and $\sigma(1) = 0$. One can verify that the odometer is minimal homeomorphism.

While shift is not minimal, one can construct many Cantor subspaces of $A^{\mathbb{Z}}$ on which the action of the shift is minimal. Closed and shift-invariant subsets of $A^{\mathbb{Z}}$ are called *subshifts*.

A sequence $\alpha \in A^{\mathbb{Z}}$ is *homogeneous*, if for every finite interval $J \subset \mathbb{Z}$, there exists a constant $k(J)$, such that the restriction of α to any interval of the size $k(J)$ contains the restriction of α to J as a subsequence. In other words, for any interval J' of the size $k(J)$, there exist $t \in \mathbb{Z}$ such that $J + t \subset J'$ and $\alpha(s + t) = \alpha(s)$ for every $s \in J$.

Theorem 1.0.1. Let A be a finite set, T be the shift on $A^{\mathbb{Z}}$, $\alpha \in A^{\mathbb{Z}}$ and

$$X = \overline{Orb_T(\alpha)}.$$

Then the system (T, X) is minimal if and only if α is homogeneous.

Proof. Assume that the sequence $\alpha \in A^{\mathbb{Z}}$ is homogeneous. Let $\beta \in \overline{Orb_T(\alpha)}$. It is suffice to show that $\alpha \in \overline{Orb_T(\beta)}$. Fix $n > 0$, then there exist $k(n, \alpha)$ such that the restriction of the sequence α to any interval of the length $k(n, \alpha)$ contains a copy of the restriction of α to the interval $[-n, n]$. Thus, since $\beta \in \overline{Orb_T(\alpha)}$ then the restriction of β to the interval $[-k(n, \alpha), k(n, \alpha)]$ contains a copy of α restricted to $[-n, n]$. This implies that there exists a power i of T such that $T^i(\beta)(j) = \alpha(j)$ for all $j \in [-n, n]$. Since n is arbitrary large, we can find a sequence i_n of powers of T, such that $T^{i_n}(\beta)$ converges to α, therefore (T, X) is minimal.

Assume (T, X) is a minimal system. To reach a contradiction assume that α is not homogeneous. Then there exists an interval $[-n, n]$, such that for any k there exists a subinterval of length k in α, which does not contain the restriction of α to $[-n, n]$. Thus there exists a sequence m_k such that the interval $[-k, k]$ of $T^{m_k}(\alpha)$ does not contain the restriction of α to $[-n, n]$. Since the space is compact we can find a convergent subsequence in $T^{m_k}(\alpha)$. Let β be a limit point. Then $\alpha \notin \overline{Orb_T(\beta)}$, which gives a contradiction. Hence α is homogeneous. $\square$

The full topological groups. The central object of this Chapter is the full topological group of a Cantor minimal system.

The full topological group of $(T, \mathbf{C})$, denoted by $[[T]]$, is the group of all $\phi \in Homeo(\mathbf{C})$ for which there exists a continuous function $n : \mathbf{C} \to \mathbb{Z}$ such that

$$\phi(x) = T^{n(x)}x \text{ for all } x \in \mathbf{C}.$$

Since C is compact, the function $n(\cdot)$ takes only finitely many values. Moreover, for every its value k, the set $n^{-1}(k)$ is clopen. Thus, there exists a finite partition of C into clopen subsets such that $n(\cdot)$ is constant on each piece of the partition.

Kakutani-Rokhlin partitions. Let T be a minimal homeomorphism of the Cantor space $\mathbf{C}$, we can associate a partition of $\mathbf{C}$ as follows.

Let D be a non-empty clopen subset of $\mathbf{C}$. It is easy to check that for every point $p \in \mathbf{C}$ the minimality of T implies that the forward orbit $\{T^k p : k \in \mathbb{N}\}$ is dense in $\mathbf{C}$. Define the first return function $t_D : D \to \mathbb{N}$:

$$t_D(x) = \min(n \in \mathbb{N} : T^n(x) \in D).$$

Since $t_D^{-1}[0, n] = T^{-n}(D)$, it follows that t_D is continuous. Thus we can find natural numbers $k_1, \ldots, k_N$ and a partition

$$D = D_1 \sqcup D_2 \sqcup \ldots \sqcup D_N$$

into clopen subsets, such that t_D restricted to D_i is equal to k_i for all $1 \leq i \leq N$.

This gives a decomposition of $\mathbf{C}$, called *Kakutani-Rokhlin partition*:

$$\begin{aligned}\mathbf{C} =& (D_1 \sqcup T(D_1) \sqcup \ldots \sqcup T^{k_1}(D_1)) \sqcup \\ & \sqcup (D_2 \sqcup T(D_2) \sqcup \ldots \sqcup T^{k_2}(D_2)) \sqcup \ldots \\ & \ldots \sqcup (D_N \sqcup T(D_N) \sqcup \ldots \sqcup T^{k_N}(D_N))\end{aligned}$$

The family $D_i \sqcup T(D_i) \sqcup \ldots \sqcup T^{k_i}(D_i)$ is called *a tower* over D_i. The base of the tower is defined to be D_i and the top of the tower is $T^{k_i}(D_i)$.

Refining of the Kakutani-Rokhlin partitions. Let $\mathcal{P}$ be a finite clopen partition of $\mathbf{C}$ and let

$$\begin{aligned}\mathbf{C} =& (D_1 \sqcup T(D_1) \sqcup \ldots \sqcup T^{k_1}(D_1)) \sqcup \\ & \sqcup (D_2 \sqcup T(D_2) \sqcup \ldots \sqcup T^{k_2}(D_2)) \sqcup \ldots \\ & \ldots \sqcup (D_N \sqcup T(D_N) \sqcup \ldots \sqcup T^{k_N}(D_N))\end{aligned}$$

be the Kakutani-Rokhlin partition over a clopen set D in $\mathbf{C}$. There exist a refinement of the partition of $D_i = \bigsqcup_{j=1}^{j_i} D_{i,j}$ such that the partition

$$\begin{aligned}&(D_{1,1} \sqcup T(D_{1,1}) \sqcup .. \sqcup T^{k_1}(D_{1,1})) \sqcup .. \sqcup (D_{1,j_1} \sqcup T(D_{1,j_1}) \sqcup .. \sqcup T^{k_1}(D_{1,j_1})) \ldots \\ &(D_{N,1} \sqcup T(D_{N,1}) \sqcup .. \sqcup T^{k_N}(D_{N,1})) \sqcup .. \sqcup (D_{N,j_N} \sqcup T(D_{N,j_N}) \sqcup .. \sqcup T^{k_N}(D_{N,j_N}))\end{aligned}$$

of $\mathbf{C}$ is a refinement of $\mathcal{P}$. Indeed, this can be obtained as follows. Assume there exists a clopen set $A \in \mathcal{P}$ such that $A \cap T^i(D_j) \neq \emptyset$ and $A \Delta T^i(D_j) \neq \emptyset$ for some i, j. Then we refine the partition $\mathcal{P}$ by the sets $T^s(T^{-i}(A) \cap D_j)$, $0 \leq s \leq k_j$. Since $\mathcal{P}$ is finite partition this operation is exhaustive.

1.1 Basic facts on dynamics on the Cantor space

We collect basic facts and definitions on dynamics on the Cantor space and several lemmas on minimal homeomorphisms to which we will refer several times in the later sections.

The *support* of a homeomorphism T of the Cantor space $\mathbf{C}$ is defined by

$$supp(T) = \overline{\{x \in \mathbf{C} : T(x) \neq x\}}.$$

Generally the support of a homeomorphism does not need to be open set. However for any minimal homeomorphism T, the support of each element of the full topological group $[[T]]$ is a clopen set. This follows immediately from the definition of $[[T]]$.

A homomorphism $T \in Homeo(\mathbf{C})$ is called *periodic*, if every orbit of T is finite, and *aperiodic* if every orbit is infinite. It has *period* n if every orbit has exactly n elements.

Lemma 1.1.1. Let $T \in Homeo(\mathbf{C})$ be a periodic homeomorphism of period n. Then there exists a clopen set $A \subset \mathbf{C}$ such that

$$\mathbf{C} = \bigsqcup_{i=0}^{n-1} T^i(A)$$

Proof. For any $x \in \mathbf{C}$ let $U_x \subset \mathbf{C}$ be a clopen neighborhood such that $T^i(U_x) \cap U_x = \emptyset$ for all $1 \leq i \leq n$, where n is the period of T. Since $\mathbf{C}$ is compact there are $x_1, \ldots, x_k$ such that $\mathbf{C} = \bigcup_{1 \leq i \leq k} U_{x_i}$. Let $A_1 = U_{x_1}$ and

$$A_{j+1} = A_j \bigcup \left(U_{x_{j+1}} \setminus \bigcup_{i=1}^{n-1} T^i(A_j) \right).$$

It is trivial to check that the statement holds for A_k. □

Lemma 1.1.2. Let $T \in Homeo(\mathbf{C})$ be a minimal homeomorphism. Then for any $g \in [[T]]$ and $n \in \mathbb{N}$ the set

$$\mathcal{O}_n = \{x \in \mathbf{C} : |Orb_g(x)| = n\}$$

is clopen.

Proof. Let $\mathbf{C} = \bigcup_{i \in I} A_i$ be a finite clopen partition of $\mathbf{C}$ such that the restriction of g to each piece of the partition coincides with some power of T.

Denote by $\{B_j\}_{j\in J}$ the refinement of $\{A_i\}_{i\in I}$ and the sets $\{T^{-k}(A_i)_{i\in I}\}$, $1 \le k \le n$. Thus

$$g|_{B_j} = T^{m_j}|_{B_j},$$

for some $\{m_j\}_{j\in J}$.

Let $x \in \mathcal{O}_n$, we will show that there is a neighborhood of x inside $\mathcal{O}_n$. Let $j_0, \ldots, j_n$ be such that $T^k x \in B_{j_k}$ for all $0 \le k \le n$. Then we have $g^n(x) = x$ for all $x \in \mathcal{O}_n$ and therefore

$$T^s x = x, \text{ where } s = \sum_{0\le k\le n} m_{j_k}.$$

But this can happen only if $s = 0$, hence $B_{j_0} \subseteq \mathcal{O}_n$. This implies that $\mathcal{O}_n$ is open.

Moreover, we have the decomposition

$$\mathcal{O}_n = \{x \in \mathbf{C} : g^n(x) = x\} \setminus \bigcup_{m<n} \{x \in \mathbf{C} : g^m(x) = x\},$$

which implies that $\mathcal{O}_n$ is closed. □

Invariant Borel measures on the Cantor set. The set of Borel measures on a compact space X is separable, compact in the weak*-topology coming from the dual of the space of all continuous functions on X, $C(X)$.

Let T be a homeomorphism of X, denote by $\mathcal{M}(T)$ the space of all T-invariant Borel measures on X. The classical Krylov-Bogolyubov theorem, [16] states that $\mathcal{M}(T)$ is non-empty. For example, for a fixed point $x \in X$ it contains a cluster point μ of the following sequence

$$\mu_n = \frac{1}{n}\sum_{i=1}^{n} T^i \circ \delta_x,$$

where δ_x is a Dirac measure concentrated on a point $x \in X$. To verify that μ is T-invariant, let $f \in C(X)$, then

$$\int f d\mu_n = \frac{1}{n}\sum_{i=1}^{n} f(T^i(x))$$

and

$$\int f d(T \circ \mu_n) = \frac{1}{n}\sum_{i=1}^{n} f(T^{i+1}(x)).$$

Then

$$\left|\int f d\mu_n - \int f d(T \circ \mu_n)\right| \leq 2/n\|f\|,$$

which implies the claim.

The following lemma will be useful in the process of constructing new elements in the full topological group.

Lemma 1.1.3. Let $A \subset \mathbf{C}$ be a clopen set and $T \in Homeo(\mathbf{C})$ be a minimal homeomorphism. Then for every $\varepsilon > 0$ there exists a partition of A into clopen sets

$$A = \bigsqcup_{i \in I} A_i$$

such that for every $\mu \in \mathcal{M}(T)$ and $i \in I$ we have $\mu(A_i) < \varepsilon$.

Proof. It is sufficient to show the statement for $A = \mathbf{C}$. Let $\varepsilon = 1/n$ and choose a clopen set $D \subset \mathbf{C}$ such that the sets $T^k(D)$, $1 \leq k \leq n$, are pairwise disjoint. Since μ is T-invariant, we have $\mu(D) \leq 1/n$. Now the Kakutani-Rokhlin partition of $\mathbf{C}$ over D satisfies the statement. $\square$

Chapter 2

Simplicity of the commutator subgroup

The commutator subgroup of the group Γ is the group generated by all elements of the form $[g,h] = ghg^{-1}h^{-1}$ and denoted by Γ'. In this section we will show simplicity of $[[T]]'$ for a Cantor minimal system $(T, \mathbf{C})$. This is a result of Matui, [69]. We will follow a simplified proof of Bezuglyi and Medynets, [13].

We start with the following theorem of Glasner and Wiess, [38], which will be crucial in the proof.

Theorem 2.0.4 (Glasner-Weiss)**.** Let T be a minimal homeomorphism of a Cantor set $\mathbf{C}$ and let $A, B \subseteq \mathbf{C}$ be clopen subsets such that for every $\mu \in \mathcal{M}(T)$, we have $\mu(B) < \mu(A)$. Then there exists $g \in [[T]]$ with $g(B) \subset A$ and $g^2 = id$.

Proof. Let $f = \chi_A - \chi_B$, then f is continuous and by the assumptions $\int f d\mu > 0$ for all $\mu \in \mathcal{M}(T)$. There exist a constant $c > 0$ such that

$$\inf(\int f d\mu : \mu \in \mathcal{M}(T)) > c.$$

Indeed, assume that this is not the case and the infimum reaches 0 on some sequence of measures in $\mathcal{M}(T)$. If μ is a cluster point of this sequence in the weak*-topology, we obtain $\mu(A) = \mu(B)$, which is a contradiction.

Let us show now that there exists n_0, such that for all $x \in \mathbf{C}$ and all

$n \geq n_0$ we have

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i x) \geq c. \tag{2.1}$$

To reach a contradiction assume that there exists a increasing sequence $\{n_k\}$ of natural numbers and a sequence of points $\{x_k\}$ for which

$$-1 \leq \frac{1}{n_k}\sum_{i=0}^{n_k-1} f(T^i x_k) \leq c.$$

As in the proof of Krylov-Bogoliubov theorem, we set

$$\mu_k = \frac{1}{n_k}\sum_{i=0}^{n_k-1} T^i \circ \delta_{x_k}.$$

Let μ be a cluster point of μ_k in the weak*-topology. Then μ is in $\mathcal{M}(T)$, on the other hand we have

$$\int f d\mu \leq c,$$

which is a contradiction.

Let n_0 be such that (2.1) holds for all $n \geq n_0$ and all $x \in \mathbf{C}$. Choose $D \subset \mathbf{C}$ be such that $T^i(D) \cap D = \emptyset$ for all $i \leq n_0$. This implies that the hight of each tower over D is greater then n_0. Let $D_1, \ldots, D_N$ be a refinement of Kakutani-Rokhlin partition of D:

$$\begin{aligned}\mathbf{C} =& (D_1 \sqcup T(D_1) \sqcup \ldots \sqcup T^{k_1}(D_1)) \sqcup \\ & \sqcup (D_2 \sqcup T(D_2) \sqcup \ldots \sqcup T^{k_2}(D_2)) \sqcup \\ & \ldots \sqcup (D_N \sqcup T(D_N) \sqcup \ldots \sqcup T^{k_N}(D_N)),\end{aligned}$$

with property that each piece of the partition is contained in one of the sets $A \backslash B, B \backslash A, A \cap B$ or $(A \cup B)^c$. By assumptions, $k_i \geq n_0$ for all $1 \leq i \leq N$. Now taking any x in D the inequality (2.1) implies that for a fixed j the number of the components $T^i(D_j)$, $1 \leq i \leq k_j$ that belong to A is greater then the number of components that belong to B. We will define g by mapping between the pieces of the partition. Define g on D_j as a symmetry that maps $T^i(D_j)$ that belong to B onto a component that belong to A by applying a power of T. Since Kakutani-Rokhlin partition is clopen, we can define g as a trivial map on the rest of the sets. Obviously, g is in $[[T]]$ and $g^2 = id$. $\square$

Below we give a sequence of lemmas due to Bezuglyi and Medynets, from which we deduce the main result of this section: the simplicity of the full topological group of a Cantor minimal system.

Lemma 2.0.5. Let T be a minimal homeomorphism of the Cantor set. For any $g \in [[T]]$ and $\delta > 0$ there exists a decomposition $g = g_1 g_2 \ldots g_n$ such that $\mu(\mathrm{supp}(g_i)) \leq \delta$ for all $\mu \in \mathcal{M}(T)$.

Proof. Assume firstly that $g \in [[T]]$ is periodic. Since $g \in [[T]]$ then by Lemma 1.1.1 and Lemma 1.1.2 we can find clopen sets A_k, $k \in I$, such that $g|_{A_k}$ has order k and

$$\mathbf{C} = \bigsqcup_{k \in I} \bigsqcup_{i=0}^{k-1} g^i(A_k).$$

By Lemma 1.1.3 we can partite A_k into clopen sets

$$A_k = \bigsqcup_{j=1}^{n_k} B_j^{(k)}$$

such that $\mu(B_j^{(k)}) < \delta/k$ for all $B_j^{(k)}$ and $\mu \in \mathcal{M}(T)$.

Now set

$$C_{k,j} = \bigsqcup_{i=0}^{k-1} g^i(B_j^{(k)}).$$

Define $g_{k,j}$ to be g on $C_{k,j}$ and identity on its complement. Since all sets are clopen $g_{k,j}$ is continuous and $g_{k,j} \in [[T]]$. Obviously, $g = \prod_{k,j} g_{k,j}$ and $\mu(supp(g_{k,j})) < \delta$.

Assume now that g is non-periodic. Let $k \in \mathbb{N}$ be such that $1/k < \delta$ and define

$$\mathcal{O}_{\geq k} = \{x \in \mathbf{C} : Orb_g(x) \text{ has at least } k \text{ elements}\}$$

By Lemma 1.1.3 we have that the complement of $\mathcal{O}_{\geq k}$ is clopen and thus $\mathcal{O}_{\geq k}$ is clopen. Therefore, for any $x \in \mathcal{O}_{\geq k}$ there exists a clopen neighborhood U_x such that $g^i(U_x) \cap U_x = \emptyset$ for all $1 \leq i < k$. By compactness there are $x_1, \ldots, x_n \in \mathcal{O}_{\geq k}$ such that $\mathcal{O}_{\geq k} = \bigcup_{1 \leq i \leq n} U_{x_i}$. Define $B_1 = U_{x_1}$ and

$$B_{i+1} = B_i \bigsqcup \left(U_{x_{i+1}} \setminus \bigcup_{l=-k+1}^{k+1} g^l(B_i) \right).$$

Then $B = B_n$ meets every orbit of g in $\mathcal{O}_{\geq k}$. Moreover, $g^i(B) \cap B = \emptyset$ for all $1 \leq i < k$, which implies $\mu(B) \leq 1/k < \delta$ for all $\mu \in \mathcal{M}(T)$. Since the transformation T is minimal we have that the function

$$F : x \mapsto \min\{l \geq 1 : g^l(x) \in B\}$$

is continuous. Define

$$g_B(x) = \begin{cases} g^k(x), & \text{if } x \in B \text{ and } k = F(x), \\ x, & x \notin B. \end{cases}$$

It is easy to see that $g_B \in [[T]]$, $\mu(supp(g_B)) < \delta$ and $g_B^{-1} \circ g$ is periodic. Thus the statement of the lemma follows from the previous case. □

Lemma 2.0.6. Let $T \in Homeo(\mathbf{C})$ be a minimal homeomorphism. Then for any $f \in [[T]]'$ and $\delta > 0$ there exists $g_1, \ldots, g_n, h_1, \ldots, h_n \in [[T]]$ such that $f = [g_1, h_1] \ldots [g_n, h_n]$ and $\mu(supp(g_i) \cup supp(h_i)) < \delta$ for all $\mu \in \mathcal{M}(T)$.

Proof. Let $f = [g, h]$ for some $g, h \in [[T]]$. By Lemma 2.0.5 we can find $g_1, \ldots, g_n$ and $h_1, \ldots, h_n$ in $[[T]]$ such that $g = g_1 \ldots g_n$, $h = h_1 \ldots h_n$ with $\mu(supp(g_i)) < \delta/2$ and $\mu(supp(h_i)) < \delta/2$ for all $\mu \in \mathcal{M}(T)$. Since f is in the group generated by $[g_i, h_j]$, $1 \leq i, j \leq n$, we obtain the statement. □

The following is a generalization of Glasner-Weiss to the commutator subgroup.

Lemma 2.0.7. Let $T \in Homeo(\mathbf{C})$ be a minimal homeomorphism. If A and B are clopen subsets of $\mathbf{C}$ such that $3\mu(B) < \mu(A)$ for all $\mu \in \mathcal{M}(T)$, then there exists $f \in [[T]]'$ such that $f(B) \subset A$.

Proof. By replacing A by $A \backslash B$ we have $2\mu(B) < \mu(A)$ for all $\mu \in \mathcal{M}(T)$ and $A \cap B = \emptyset$. Now by Theorem 2.0.4 we can find a symmetry $g \in [[T]]$ such that $g(B) \subset A$. Then

$$\mu(g(B)) = \mu(B) < \mu(A) - \mu(B) = \mu(A \backslash g(B))$$

thus again by Theorem 2.0.4 we can find a symmetry $h \in [[T]]$ such that $h(g(B)) \subset g(A) \backslash B$. It is trivial to check, using the properties of g and h, that $g = (hg)h^{-1}(hg)^{-1}$, we obtain $hg = [h, hg]$ and $hg(B) \subseteq A$, which implies the statement. □

Theorem 2.0.8. Let T be a minimal homeomorphism and let Γ be either $[[T]]$ or $[[T]]'$. Then for every normal subgroup H of Γ, we have $\Gamma' \leq H$.

Proof. We will show that for all elements g, h in Γ their commutator $[g, h]$ is in H. Let $f \in H$. Let E be a clopen non-empty set such that such that $f(E) \cap E = \emptyset$. By the compactness of $\mathcal{M}(T)$ we have

$$3\delta = \inf(\mu(E) : \mu \in \mathcal{M}(T)) > 0$$

By Lemma 2.0.5 and Lemma 2.0.6 we can find $g_i, h_j \in \Gamma$ such that $g = g_1 \dots g_n$ and $h = h_1 \dots h_n$ and

$$\mu(supp(g_i)) < \delta/2, \quad \mu(supp(h_i)) < \delta/2$$

for all $\mu \in \mathcal{M}(T)$. We claim that for all g and h with $\mu(supp(g) \cup supp(h)) < \delta$ we have $[g, h]$ are in H. Since the commutator $[g_1 \dots g_n, h_1 \dots h_n]$ belongs to the group generated by $[g_i, h_j]$, the claim implies the statement.

To prove the claim put $F = supp(g) \cup supp(h)$, then $3\mu(F) < \mu(E)$. Thus we can apply Lemma 2.0.7 to find an element α in $[[T]]'$ such that $\alpha(F) \subseteq E$. Since H is normal, we have

$$q = \alpha^{-1} f \alpha \in H.$$

Thus

$$\overline{h} = [h, q] = (h\alpha^{-1} f \alpha h^{-1})\alpha^{-1} f^{-1} \alpha$$

and $[g, \overline{h}]$ are in H.

Since $q(F) \cap F = \emptyset$, the elements g^{-1} and $qh^{-1}q^{-1}$ commute. Hence, we have

$$[g, \overline{h}] = g(hqh^{-1}q^{-1})g^{-1}(qhq^{-1}h^{-1}) = [g, h] \in H,$$

which proves the claim. □

Corollary 2.0.9 (Matui, '06)**.** Let $T \in Homeo(\mathbf{C})$ be a minimal, then $[[T]]'$ is simple.

Proof. Since $[[T]]''$ is a normal subgroup of $[[T]]$, we can apply the theorem to obtain that $[[T]]' \leq [[T]]''$. Thus, $[[T]]'' = [[T]]'$. Let now H be a normal subgroup of $[[T]]'$. Then $[[T]]'' \leq H$, therefore $[[T]]' = H$. □

Chapter 3

Finite generation of the commutator subgroup of a minimal subshift

The aim of this section is to prove that every commutator subgroup of a Cantor minimal subshift is finitely generated, Theorem 3.0.12. This result is due to Hiroki Matui, [69]. Through this section we assume that T is a minimal homeomorphism of the Cantor set.

Let U be a clopen set in $\mathbf{C}$, such that U, $T(U)$ and $T^{-1}(U)$ are pairwise disjoint. Define

$$f_U(x) = \begin{cases} T(x), & x \in T^{-1}(U) \cap U \\ T^2(x), & x \in T(U) \\ x, & \text{otherwise} \end{cases}$$

Obviously, f_U is a homeomorphism of $\mathbf{C}$ and $f_U \in [[T]]$. Moreover, we claim that f_U is in the commutator subgroup $[[T]]'$. To verify the claim, define a symmetry in $[[T]]$:

$$g(x) = \begin{cases} T(x), & x \in T^{-1}(U) \\ T^{-1}(x), & x \in U. \end{cases}$$

One verifies that $f_U = [g, f_U]$ by identifying f_U with cycle (123) and g with cycle (12).

Consider the following set of elements of $[[T]]'$

$$\mathcal{U} = \{f_U : U \text{ is clopen set and } U, T(U), T^{-1}(U) \text{ are pairwise disjoint}\}$$

Lemma 3.0.10. The commutator subgroup of the full topological group $[[T]]'$ is generated by $\mathcal{U}$.

Proof. Let H be a subgroup of $[[T]]$ generated by $\mathcal{U}$. We start by showing that if $g \in [[T]]$ and $g^3 = e$, then g is in H. Since each f_U is of order 3, this will imply that H is normal. Therefore because of simplicity of $[[T]]'$ we would be able to conclude that $H = [[T]]'$.

By Lemma 1.1.1 and Lemma 1.1.2 we can find a clopen set A such that A, $g(A)$ and $g^2(A)$ are pairwise disjoint and $supp(g) = A \sqcup g(A) \sqcup g^2(A)$. Let now B_i be a clopen partition of $\mathbf{C}$ such that the restriction of g to each B_i coincides with a certain power of T. Consider the following partitions of A:

$$\begin{aligned}
\mathcal{P}_0 &= \{B_i \cap A\}_{1 \le i \le n}, \\
\mathcal{P}_1 &= g^{-1}\{B_i \cap g(A)\}_{1 \le i \le n}, \\
\mathcal{P}_2 &= g^{-2}\{B_i \cap g^2(A)\}_{1 \le i \le n}.
\end{aligned}$$

Denote the common refinement of $\mathcal{P}_0$, $\mathcal{P}_1$ and $\mathcal{P}_2$ by $\{A_j\}_{1 \le j \le m}$. It has the property that for every $1 \le j \le m$ there are integers k_j, l_j such that

$$g|_{A_j} = T^{k_j}|_{A_j}, \quad g|_{g(A_j)} = T^{l_j}|_{g(A_j)}, \quad g|_{g^2(A_j)} = T^{-k_j - l_j}|_{g^2(A_j)}.$$

Now we can decompose $g = g_1 \dots g_m$ as a product of commuting elements of $[[T]]$ defined by the restriction of g onto $A_j \cup g(A_j) \cup g^2(A_j)$. This implies that it is sufficient to consider the case when g is in $[[T]]$ of the order 3 and there exists a clopen set $A \subset \mathbf{C}$ such that there are k and l with

$$g|_A = T^k|_A, \quad g|_{g(A)} = T^l|_{g(A)}, \quad g|_{g^2(A)} = T^{-k-l}|_{g^2(A)}.$$

Since for any $x \in A$ there exists a clopen neighborhood $U_x \subseteq A$ such that $\{T^i(U_x)\}_{1 \le i \le k+l}$ are pairwise disjoint and A is compact, we can select a finite family U_x that covers A. Let $C_1, \dots, C_n$ be the partition of A generated by these finite family. Let $g = g_1 \dots g_n$ be the decomposition of g into a product of commuting elements of $[[T]]$ defined by taking g_i to be the restriction of g to $C_i \cup g(C_i) \cup g^2(C_i)$.

Thus this reduces the argument to the case when $g \in [[T]]$ has the following property: $g^3 = id$ and there exists a clopen set $A \subset \mathbf{C}$ such that there are k and l with

$$g|_A = T^k|_A, \quad g|_{g(A)} = T^l|_{g(A)}, \quad g|_{g^2(A)} = T^{-k-l}|_{g^2(A)},$$

and $T^i(A) \cap T^j(A) = \emptyset$ for all $1 \leq i, j \leq k + l$. This element can be considered as a cycle $(k\ l\ k + l)$ of the permutation group S_{k+l+1} and each cycle $(i-1\ i\ i+1)$ is given by $f_{T^i(A)}$. Moreover, g is in the alternating group A_{k+l+1}, which contains all 3-cycles. Thus g is a product of elements of $\mathcal{U}$, which finishes the lemma. $\square$

Note that the proof of lemma shows slightly more. Namely, for every prime number p and an element of order p in $[[T]]$, this element belongs to the commutator subgroup.

Lemma 3.0.11. Let U and V be clopen subsets of $\mathbf{C}$, then the following holds

(i) If $T^2(V)$, $T(V)$, V, $T^{-1}(V)$, $T^{-1}(V)$ are pairwise disjoint and $U \subseteq V$, then for $\tau_U = f_{T^{-1}(U)} f_{T(U)}$ we have

$$\tau_V f_U \tau_V^{-1} = f_{T(U)}$$
$$\tau_V^{-1} f_U \tau_V = f_{T^{-1}(U)}$$

(ii) If V, U, $T^{-1}(U)$, $T(U) \cup T^{-1}(V)$, $T(V)$ are pairwise disjoint then

$$[f_V, f_U^{-1}] = f_{T(U) \cap T^{-1}(V)}.$$

Proof. The proof of the lemma boils down to the identification of the elements involved in the statement with permutations.

(i). The support of $\tau_{V \setminus U}$ is disjoint from supports of other homomorphism, thus

$$\tau_V f_U \tau_V^{-1} = \tau_U f_U \tau_U^{-1} = f_{T(U)},$$

where the last identity is the consequence of the identity in the permutation group $(01234)(123)(04321) = (012)$.

(ii). Let $C = T(U) \cap T^{-1}(V)$. We can decompose $f_U = f_{T^{-1}(C)} f_{U \setminus T^{-1}(C)}$ and $f_V = f_{T(C)} f_{V \setminus T(C)}$. Thus

$$[f_V, f_U^{-1}] = [f_{T(C)}, f_{T^{-1}(C)}^{-1}] = f_{T(C)} f_{T^{-1}(C)}^{-1} f_{T(C)}^{-1} f_{T^{-1}(C)} = f_C,$$

where the last identity is equivalent to the identity in the permutation group:

$$(234)(021)(243)(012) = (123).$$

$\square$

Theorem 3.0.12. Let $T \in Homeo(\mathbf{C})$ is minimal homeomorphism. The commutator subgroup $[[T]]'$ is simple if and only if T is conjugate to a minimal subshift.

Proof. Assume that $T \in Homeo(\mathbf{C})$ is a minimal subshift, i.e., T acts as a shift on the Cantor set $A^{\mathbb{Z}}$ for some finite alphabet A and there exists a clopen T-invariant subset $X \subset A^{\mathbb{Z}}$ such that the action of T on X is minimal. Moreover, enlarging the alphabet and using the characterization of the minimal subshifts in terms of homogeneous sequences, we can assume that $x(i) \neq x(j)$ for every $|i - j| < 4$ and $x \in X$.

For every $n, m \in \mathbb{N}$ and $a_i \in A$, $-m \leq i \leq n$, define the cylinder sets in X by

$$\langle\langle a_{-m} \dots a_{-1} \underline{a_0} a_1 \dots a_n \rangle\rangle = \{x \in X : x(i) = a_i, -m \leq i \leq n\},$$

here the underlining of a_0 means that a_0 is in the 0's coordinate of $\mathbb{Z}$-enumeration. Since $x(i) \neq x(j)$ for every $|i - j| < 4$ we have that for every cylinder set U the sets $T^{-2}(U), T^{-1}(U), U, T(U), T^2(U)$ are pairwise disjoint. Let H be a subgroup of $[[T]]'$ generated by the finite set of cylinders:

$$\{f_U : U = \langle\langle a\underline{b}c \rangle\rangle, a, b, c \in A\}.$$

We will show that $H = [[T]]'$. By Lemma 3.0.10 it is sufficient to show that for every cylinder sets $U \in X$, we have $f_U \in H$.

Since

$$f_{T(\langle\langle \underline{a} \rangle\rangle)} = \prod_{b \in A} f_{\langle\langle a\underline{b} \rangle\rangle}, \quad f_{T^{-1}(\langle\langle \underline{a} \rangle\rangle)} = \prod_{b \in A} f(\underline{b}a),$$

we immediately have $f_{T(\langle\langle \underline{a} \rangle\rangle)}, f_{T^{-1}(\langle\langle \underline{a} \rangle\rangle)}$, thus $\tau_{\langle\langle \underline{a} \rangle\rangle}$ is in H. Applying Lemma 3.0.11 to the sets

$$U = \langle\langle a_{-m} \dots a_{-1} \underline{a} a_1 \dots a_n \rangle\rangle \subseteq \langle\langle \underline{a_0} \rangle\rangle = V$$

we obtain:

$$\tau_{\langle\langle \underline{a_0}\rangle\rangle} f_U \tau^{-1}_{\langle\langle \underline{a_0}\rangle\rangle} = f_{T(U)}, \quad \tau^{-1}_{\langle\langle \underline{a_0}\rangle\rangle} f_U \tau_{\langle\langle \underline{a_0}\rangle\rangle} = f_{T^{-1}(U)}.$$

Hence we conclude the statement of the lemma by induction on $m+n$ and applying Lemma 3.0.11 (ii) to the sets $V = \langle\langle a_{-m} \dots a_{-1}\underline{a_0}a_1\rangle\rangle$ and $U = \langle\langle a_1\underline{a_2}\rangle\rangle$.

To prove the converse, assume that $[[T]]'$ is finitely generated for a minimal homeomorphism of the Cantor set $T \in Homeo(\mathbf{C})$. Let $g_1, \dots, g_n$ be the generating set of $[[T]]'$ and $n_i : \mathbf{C} \to \mathbb{Z}$ be continuous maps that satisfy:

$$g_i(x) = T^{n_i(x)}x, \quad x \in \mathbf{C}.$$

Let $\mathcal{P} = \{\mathcal{P}_1, \dots, \mathcal{P}_n\}$ be the common refinement of the partition $\{n_i^{-1}(k)\}_{k\in\mathbb{Z}}$. We will consider $\mathcal{P}$ as a finite alphabet together with shift map $s : \mathcal{P}^{\mathbb{Z}} \to \mathcal{P}^{\mathbb{Z}}$. Define a continuous map $S : X \to \mathcal{P}^{\mathbb{Z}}$ by the property that $S(x)(k) = \mathcal{P}_s$, if $T^k(x) \in \mathcal{P}_s$. It is easy to verify that S is a factor map. Define a homeomorphism $f_i \in Homeo(\mathbf{C})$ by $f_i(z) = S^k(z)$ when $z(0) \subseteq n_i^{-1}(k)$. It is easy to see that $f_i \in [[s]]$ and $Sg_i = f_iS$. It remains to show that S is injective.

Suppose $x, y \in \mathbf{C}$ are distinct and $S(x) = S(y)$. Let $g \in [[T]]'$ such that $g(x) \neq x$ and $g(y) = y$. By assumptions $[[T]]'$ is finitely generated, thus we can write g as a word on the generators $w(g_1, \dots, g_n)$.

$$\begin{aligned} Sg(x) &= Sw(g_1, \dots, g_n)(x) \\ &= w(f_1, \dots, f_n)S(x) \\ &= w(f_1, \dots, f_n)S(y) \\ &= Sw(g_1, \dots, g_n)(y) \\ &= Sg(y) = S(x). \end{aligned}$$

Hence, for some k we have $s^kS(x) = S(T(x)) = S(x)$, which contradicts to minimality of s and thus of T. $\square$

Chapter 4

Amenable groups

4.1 Means and measures.

In this section we specify a connection between means and finitely additive probability measures. This connection will be important for proving equivalences of several definitions of amenability. More on means and measures can be found in classical books on functional analysis and measure theory, for example in [33].

Let X be a set and $\mathcal{P}(X)$ be the set of all subsets of X. For $E \in \mathcal{P}(X)$ denote by χ_E the characteristic function of the set E, i.e., $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ otherwise.

Definition 4.1.1. A map $\mu : \mathcal{P}(X) \to [0, 1]$ is called finitely additive measure if it satisfies:

(i) $\mu(X) = 1$

(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$ and $A, B \in \mathcal{P}(X)$.

Denote the set of all finitely additive probability measures on X by $PM(X)$. One of the examples of finitely additive measures is the counting measure on supported on a finite set, i.e., $\mu(F) = |E \cap F|/|F|$ for some finite set F in X.

Definition 4.1.2. A mean on a set X is a functional $m \in l_\infty(X)^*$ which satisfies

(i) $m(\chi_X) = 1$,

(ii) $m(f) \geq 0$ for all $f \geq 0$, $f \in l_\infty(X)$.

Define $M(X)$ to be the set of all means on X,

$$Proba(X) = \{f : X \to \mathbb{R}_+ : \sum_{x \in X} f(x) = 1\}$$

$$Proba_{fin}(X) = \{f : X \to \mathbb{R}_+ : \sum_{x \in X} f(x) = 1, f \text{ is finitely supported}\}$$

Then $Proba_{fin}(X) \subseteq Proba(X) \subseteq M(X)$. Indeed, for each $h \in Proba(X)$ we can define a mean $m_h \in M(X)$ by

$$m_h(f) = \sum_{x \in X} h(x) f(x)$$

Fact 4.1.3. We list the following classical properties of means:

(i) For each $m \in M(X)$ we have $\|m\| = 1$.

(ii) The set of all means $M(X) \subset l_\infty(X)^*$ is convex and closed in the weak*-topology.

(iii) $Proba_{fin}(X)$ and $Proba(X)$ are convex in $M(X)$.

(iv) The set $Proba_{fin}(X)$, and thus $Proba(X)$, is weak*-dense in $M(X)$.

Proof. To prove (i), observe that $\|m\| \geq 1$, since $m(\chi_X) = 1$. Now, if $f, h \in l_\infty(X)$ and $f \leq h$ then $m(h - f) \geq 0$ and thus $m(f) \leq m(h)$. Applying this to $f \leq \|f\|_\infty \cdot \chi_X$, we obtain $m(f) \leq \|f\|_\infty \cdot m(\chi_X) = \|f\|_\infty$, therefore $\|m\| = 1$.

The steps (ii) and (iii) are trivial.

To see (iv), assume that $m \in M(X)$ is not a weak*-limit of means in $Proba_{fin}(X)$. By Hahn-Banach theorem, we can find a function f in $l_\infty(X)$ and $\delta > 0$, such that $m(f) \geq \delta + \overline{m}(f)$ for all $\overline{m} \in Proba_{fin}(X)$. In particular, this holds for Dirac measures $\delta_x \in Proba_{fin}(X)$, for which we have $\delta_x(f) = f(x)$. Therefore, $m(f) > \sup_{x \in X} f(x)$ for all $f \in l_\infty(X)$, this is impossible. $\square$

For each mean $m \in M(X)$ we can associate a finitely additive measure on X:

$$\hat{m}(A) = m(\chi_A), \text{ for all } A \in \mathcal{P}(X).$$

Indeed, since $\chi_A \leq \chi_X$ we have that $\hat{m}$ takes its values in $[0, 1]$. Moreover, the linearity of m implies $\hat{m}(A \cup B) = \hat{m}(A) + \hat{m}(B)$ for all $A, B \in \mathcal{P}(X)$ with $A \cap B = \emptyset$.

Theorem 4.1.4. The map between means and finitely additive probability measures given by $m \mapsto \hat{m}$ is bijective.

Proof. Let $\mathcal{E}(X)$ be the set of all $\mathbb{R}$-valued functions on X which take only finitely many values. Then $\mathcal{E}(X)$ is dense in $l_\infty(X)$. Indeed, for each positive function in $h \in l_\infty(X)$ define $\lambda_i = \frac{i}{n}\|h\|_\infty$ and

$$f_n(x) = \min\{\lambda_i : h(x) \leq \lambda_i\}.$$

Then $f_n \in \mathcal{E}(X)$ and $\|h - f_n\|_\infty \leq \|h\|_\infty / n$. Since every function $h \in l_\infty(X)$ can be decomposed as a difference of two positive function, we have that $\mathcal{E}(X)$ is dense in $l_\infty(X)$.

For $\mu \in PM(X)$ and $h \in \mathcal{E}(X)$ define

$$\overline{\mu}(h) = \sum_{t \in \mathbb{R}} \mu(h^{-1}(t))t.$$

Since $h = \sum_{i \in I} \lambda_i \chi_{A_i}$ for some finite set I and μ is finitely additive, we have

$$\overline{\mu}(h) = \sum_{i \in I} \lambda_i \mu(A_i).$$

Moreover the map $\overline{\mu}$ is linear. Indeed, for all $\lambda \in \mathbb{R}$ we have

$$\overline{\mu}(\lambda h) = \sum_{i \in I} \lambda \mu(A_i) = \lambda \overline{\mu}(h).$$

For $h, g \in \mathcal{E}(X)$ we have

$$\begin{aligned}
\overline{\mu}(h + g) &= \sum_{(x,y) \in Image(h) \times Image(g)} \mu(h^{-1}(x) \cap g^{-1}(y))(x + y) \\
&= \sum_{x \in Image(h)} \mu(h^{-1}(x)) + \sum_{y \in Image(g)} \mu(h^{-1}(y)) \\
&= \mu(h) + \mu(h).
\end{aligned}$$

Since $|\overline{\mu}(h)| \leq \sum_{i\in I} |\lambda_i|\mu(A_i) \leq \|h\|_\infty \cdot \mu(X) = \|h\|_\infty$, we can extend $\overline{\mu}$ to a linear functional m on $l_\infty(X)$ with $\|m\| \leq 1$. Moreover, $m(\chi_X) = \mu(X) = 1$. By the construction, if $f \in \mathcal{E}(X)$ is a positive function then $m(f) \geq 0$. As we showed above, each positive function in $l_\infty(X)$ can be approximated by positive functions from $\mathcal{E}(X)$, thus we have $m(f) \geq 0$ for all positive functions $f \in l_\infty(X)$.

Since for every $E \subset X$

$$m(\chi_E) = \mu(E)$$

we have that $\hat{m} = \mu$ and thus the statement of the theorem follows. $\square$

Consider now an action of a group G on a set X. It is straight forward to check that a mean m is G-invariant if and only if the finitely additive probability measure $\hat{m}$ is G-invariant.

4.2 First definitions: invariant mean, Følner condition.

The classical definition of amenable group which goes back to von Neumann is the following.

Definition 4.2.1. A group Γ is amenable if there exists a finitely additive measure μ on all subsets of Γ into $[0,1]$ with $\mu(\Gamma) = 1$ and satisfying

$$\mu(gE) = \mu(E)$$

for every $E \subseteq \Gamma$ and $g \in \Gamma$.

The left regular representation of Γ, $\lambda : \Gamma \to U(l^2(\Gamma))$, is the representation acting on a Hilbert space $l^2(\Gamma)$ by unitary operators as follows:

$$\lambda_g(\delta_t) = \delta_{st} \text{ for all } g, t \in \Gamma,$$

where $\{\delta_t\}_{t\in\Gamma}$ is the canonical orthonormal basis of $l^2(\Gamma)$. Analogously, *the right regular representation* is defined by $\rho_s(\delta_t) = \delta_{ts^{-1}}$.

Von Neumann's main novelty for studying amenability is to consider the space of functions and means on a group. Denote by $l^\infty(\Gamma)$ the space of bounded functions on Γ. The space $l^\infty(\Gamma)$ can be considered as multiplication operators on $l^2(\Gamma)$: $f\delta_t = f(t)\delta_t$ for $t \in \Gamma$ and $f \in l^\infty(\Gamma)$. This gives an embedding $l^\infty(\Gamma) \subseteq B(l^2(\Gamma))$. The action of Γ on $l^\infty(\Gamma)$ is defined by $g.f(t) = f(g^{-1}t)$. In terms of bounded on $l^2(\Gamma)$ operators this is nothing but the action by conjugation, i.e., $g.f = \lambda(g)f\lambda(g)^{-1}$.

A mean on the group Γ is a linear functional μ on $l^\infty(\Gamma)$ such that $\mu(\chi_\Gamma) = 1$, $\mu(f) \geq 0$ for all $f \geq 0$, $f \in l^\infty(\Gamma)$. It is called Γ-invariant if $\mu(t.f) = \mu(f)$ for all $f \in l^\infty(\Gamma)$ and $t \in \Gamma$.

We denote **the space of probability measures on** Γ by

$$Prob(\Gamma) = \{\mu \in l^1(\Gamma) : \|\mu\|_1 = 1 \text{ and } \mu \geq 0\}.$$

A subset $S \subset \Gamma$ is called symmetric if $S = S^{-1} = \{s^{-1} : s \in S\}$.

From the previous section we know that there is one-to-one correspondence between means and finitely additive measures. Moreover, it is straightforward to check that to Γ-invariant finitely additive measure this correspondence associates Γ-invariant mean. Thus we immediately obtain the following definition which is equivalent to the Definition 4.2.1.

Definition 4.2.2. A group Γ is amenable if it admits an invariant mean.

As our tools will develop we will present more and more sophisticated definitions of amenability. The main purpose of this is to cover all known examples of amenable groups. The following is the first set of equivalent definitions.

To begin with here is the first list of the equivalent definitions of amenability.

Theorem 4.2.3. For a discrete group Γ the following are equivalent:

1. Γ is amenable.

2. *Approximately invariant mean.* For any finite subset $E \subset \Gamma$ there is $\mu \in Prob(\Gamma)$ such that $\|s.\mu - \mu\|_1 \leq \varepsilon$ for all $s \in E$.

3. *Følner condition.* For any finite subset $E \subset \Gamma$ and $\varepsilon > 0$, there exists a finite subset $F \subset \Gamma$ such that

$$|gF\Delta F| \leq \varepsilon|F| \text{ for all } g \in E.$$

Proof. (1) $\implies$ (2). Let E be a finite set and $\mu \in l^\infty(\Gamma)^*$ be an invariant mean. Since $l^1(\Gamma)$ is dense in $l^\infty(\Gamma)^*$ in weak*-topology, let $\mu_i \in Prob(\Gamma)$ weak*-converges to μ. This implies that $s.\mu_i - \mu_i$ converges in weak*-topology to zero. Since weak*-convergence for functions in $l^\infty(\Gamma)$ and thus weakly in $l^1(\Gamma)$ for all $s \in \Gamma$. Consider the weak closure of the convex set

$$\{\bigoplus_{s\in E} s.\mu - \mu : \mu \in Prob(\Gamma)\}.$$

We have that this closure contains zero. By Hahn-Banach theorem, it is also norm closed. Thus (2) follows.

(2) $\implies$ (3). Given $E \subset \Gamma$ and $\varepsilon > 0$, let μ satisfy (2). Let $f \in l^1(\Gamma)$ and $r \geq 0$. Define $F(f,r) = \{t \in \Gamma : f(t) > r\}$.

For positive functions f, h in $l^1(\Gamma)$ with $\|f\|_1 = \|h\|_1 = 1$, we have $|\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)| = 1$ if and only if $f(t) \leq r \leq h(t)$ or $h(t) \leq r \leq f(t)$. Hence for two functions bounded above by 1 we have

$$|f(t) - h(t)| = \int_0^1 |\chi_{F(f,r)}(t) - \chi_{F(h,r)}(t)| dr.$$

Thus we can apply this to $\mu(t)$ and $s.\mu(t)$:

$$\begin{aligned}
\|s.\mu - \mu\|_1 &= \sum_{t\in\Gamma} |s.\mu(t) - \mu(t)| \\
&= \sum_{t\in\Gamma} \int_0^1 |\chi_{F(s.\mu,r)}(t) - \chi_{F(\mu,r)}(t)| dr \\
&= \int_0^1 \left(\sum_{t\in\Gamma} |\chi_{s.F(\mu,r)}(t) - \chi_{F(\mu,r)}(t)| \right) dr \\
&= \int_0^1 |s.F(\mu,r) \Delta F(\mu,r)| dr
\end{aligned}$$

Since $\|f\|_1 = 1$ and μ satisfies (2), it follows

$$\int_0^1 \sum_{s\in E} |s.F(\mu,r)\Delta F(\mu,r)|dr \leq \varepsilon|E|$$

$$= \varepsilon|E| \sum_{t\in\Gamma} \mu(t)$$

$$= \varepsilon|E| \sum_{t\in\Gamma} \int_0^{\mu(t)} dr$$

$$= \varepsilon|E| \int_0^1 |\{t\in\Gamma : \mu(t) > r\}|dr$$

$$= \varepsilon|E| \int_0^1 |F(\mu,r)|dr.$$

Thus there exists r such that

$$\sum_{s\in E} |s.F(\mu,r)\Delta F(\mu,r)| \leq \varepsilon|E||F(\mu,r)|.$$

(3) $\implies$ (1). Let E_i be an increasing to Γ sequence of finite subsets and $\{\varepsilon_i\}$ be a converging to zero sequence. By (3) we can find F_i that satisfy

$$|gF_i\Delta F_i| \leq \varepsilon|F_i| \text{ for all } g\in E_i.$$

Denote by $\mu_i = \frac{1}{|F_i|}\chi_{F_i} \in Prob(\Gamma)$, then

$$\|s.\mu_i - \mu_i\|_1 = \frac{1}{|F_i|}|gF_i\Delta F_i|.$$

Let $\mu \in l^\infty(\Gamma)^*$ be a cluster point in the weak*-topology of the sequence μ_i, then μ is an invariant mean. $\square$

Let S be a generating set of the group Γ. A sequence $\{F_i\}_{i\in\mathbb{N}}$ is called a *Følner sequence* for S if there exists a sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ that converges to zero that

$$|gF_i\Delta F_i| \leq \varepsilon_i|F_i| \text{ for all } g\in S \text{ and all } i\in\mathbb{N}.$$

Let $s,h\in\Gamma$ and F be a finite subset of Γ, then

$$|shF\Delta F| = |(shF\Delta sF)\cup(sF\Delta F)| \leq |hF\Delta F| + |sF\Delta F|.$$

Since any finite set is contained in a ball of a large enough radius of the Cayley graph, by inequality above we have that a finitely generated group Γ is amenable if and only if it admits a Følner sequence for some generating set.

4.3 First examples

Below we present first examples of amenable and non-amenable groups that can be constructed using the basic definitions.

Finite groups. Finite groups are amenable because they are Følner sets themselves.

Groups of subexponential growth. A group Γ has *subexponential growth* if $\limsup_n |S^n|^{1/n} = 1$ for any finite subset $S \subset \Gamma$, where $S^n = \{s_1 \dots s_n : s_1, \dots, s_n \in S\}$. Obviously, for a finitely generated group it is sufficient to verify the above condition on a symmetric generating set.

The fact that all non-amenable groups have exponential growth can be deduced from the Følner condition.

Let Γ be a group of subexponential growth. Let E be a finite symmetric subset of Γ and denote $B_n = E^n$. By the definition of subexponential growth we have that for every $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$|B_{k+1}|/|B_k| \leq (1 + \varepsilon).$$

Indeed, to reach a contradiction assume that for there exists $\varepsilon > 0$ such that for all k

$$|B_{k+1}| > (1 + \varepsilon)|B_k|.$$

Hence $|B_{k+1}| > (1 + \varepsilon)^k |B_1|$ and therefore $\limsup_n |B_k|^{1/k} >= 1 + \varepsilon$, which is a contradiction.

Now for any $g \in E$ we have

$$\frac{|gB_k \Delta B_k|}{|B_k|} \leq \frac{2(|B_{k+1}| - |B_k|)}{|B_k|} \leq 2\varepsilon.$$

Thus Γ is amenable.

Abelian groups. There are many ways to see that abelian groups are amenable. One of them is to notice that finitely generated abelian groups are of subexponential growth.

Free groups. The free group $\mathbb{F}_2$ of rank 2 is the typical example of non-amenable group. Let $a, b \in \mathbb{F}_2$ be the free generators of $\mathbb{F}_2$. Denote by $\omega(x)$ the set of all reduced words in $\mathbb{F}_2$ that start with x. Thus the group can be decomposed as follows

$$\mathbb{F}_2 = \{e\} \cup \omega(a) \cup \omega(a^{-1}) \cup \omega(b) \cup \omega(b^{-1}).$$

To reach a contradiction assume that $\mathbb{F}_2$ has an invariant mean μ. Since the group is infinite and μ is invariant we have $\mu(\chi_{\{t\}}) = 0$ for all $t \in \mathbb{F}_2$. Moreover, applying the fact that $\omega(x) = x(\mathbb{F}_2 \backslash \omega(x^{-1}))$ for $x \in \{a, b, a^{-1}, b^1\}$ we obtain:

$$\begin{aligned}
1 = \mu(\chi_{\mathbb{F}_2}) &= \mu(\chi_{\{e\}}) + \mu(\chi_{\omega(a)}) \\
&+ \mu(\chi_{\omega(a^{-1})}) + \mu(\chi_{\omega(b)}) + \mu(\chi_{\omega(b^{-1})}) \\
&= \mu(\chi_{\{e\}}) + \left[1 - \mu(\chi_{\omega(a^{-1})})\right] + \left[1 - \mu(\chi_{\omega(a)})\right] \\
&+ \left[1 - \mu(\chi_{\omega(b^{-1})})\right] + \left[1 - \mu(\chi_{\omega(b)})\right] = 3.
\end{aligned}$$

which is a contradiction.

4.4 Operations that preserve amenability

In this section we list basic operations that preserve amenability.

Subgroups. Amenability passes to subgroups. Indeed, let H be a subgroup of an amenable group Γ and $\mathcal{R}$ be a complete set of representatives of the right cosets of H.

Given $\varepsilon > 0$ and a finite set $F \subset H$, let $\mu \in Prob(\Gamma)$ be an approximately invariant mean that satisfies $\|s.\mu - \mu\|_1 \leq \varepsilon$ for all $s \in F$. Define $\tilde{\mu} \in Prob(H)$ by

$$\tilde{\mu}(h) = \sum_{r \in \mathcal{R}} \mu(hr).$$

Since for all $s \in F$ we have

$$
\begin{aligned}
\|s.\tilde{\mu} - \tilde{\mu}\|_1 &= \sum_{h \in H} |s.\tilde{\mu}(h) - \tilde{\mu}(h)| \\
&= \sum_{h \in H} |\sum_{r \in \mathcal{R}} s.\mu(hr) - \mu(hr)| \\
&\leq \|s.\mu - \mu\|_1 \leq \varepsilon,
\end{aligned}
$$

it follows that $\tilde{\mu}$ is an approximate mean for F, thus H is amenable.

Quotients. Let Γ be an amenable group with normal subgroup H, then G/H is amenable. Define a map $\phi : l^\infty(\Gamma/H) \to l^\infty(\Gamma)$ by $\phi(f)(t) = f(tH)$. Then if μ is an invariant mean of Γ, we have that $\mu \circ \phi$ is an invariant mean of Γ/H. Indeed, for every $f \in l^\infty(\Gamma/H)$ we have

$$
\mu(\phi(g.f)) = \mu(t \mapsto f(g^{-1}tH)) = \mu(t \mapsto f(tH))
$$

Extensions. Let Γ be a group with normal subgroup H such that both H and Γ/H are amenable, then Γ is amenable.

Let μ_H and $\mu_{\Gamma/H}$ be invariant means of H and Γ/H correspondingly. For $\phi \in l^\infty(\Gamma)$, let $\phi_\mu \in l^\infty(\Gamma/H)$ be defined by $\phi_\mu(gH) = \mu_H((g.\phi)|_H)$, since μ_H is H-invariant we have that ϕ_μ is well-defined. Then the functional $\phi \mapsto \mu_{\Gamma/H}(\phi_\mu)$ is an invariant mean of Γ.

Direct limits. The direct limit of groups $\Gamma = \lim\limits_{\to} \Gamma_i$ have the property that for each finite set in the limit group Γ, the group generated by this set belongs to one of Γ_i. Thus if all Γ_i are amenable we can apply Følner's definition of amenability to conclude that Γ is also amenable.

In particular, a group is amenable if and only if all its finitely generated subgroups are amenable.

Chapter 5

Amenable actions

5.1 Several equivalent definitions of amenable actions

In this section we study amenable actions. Assume a discrete group Γ acts on a set X, that is there is a map $(g, x) \mapsto gx$ from $\Gamma \times X$ to X such that $(gh)x = g(h(x))$ for all g, h in Γ. Then the group acts on the space of functions on X by $g.f(x) = f(g^{-1}x)$. A mean on X is a linear functional $m \in l^\infty(X)^*$ which satisfies $m(1_X) = 1$, $m(f) \geq 0$ for all $f \in l^\infty(X)$. This automatically implies $\|m\| \leq 1$. Denote the set of all means on X by $M(X)$. The group Γ acts on $M(X)$, a fixed point of this action (if such exists) is an invariant mean.

There is one-to-one correspondence between means on X and finitely additive probability measures on X, see Appendix 4.1 for detailed overview.

Definition 5.1.1. An action of a discrete group Γ on a set X is amenable is X admits an invariant mean.

We will identity the set $\{0,1\}^X$ of all sequences indexed by X with values in $\{0,1\}$ with set of all subsets of X.

Theorem 5.1.2. Let a discrete group Γ act on a set X. Then the following are equivalent:

(i) An action of a discrete group Γ on a set X is amenable;

(ii) There exists a map $\mu : \{0,1\}^X \to [0,1]$, which satisfies

- μ is finitely additive, $\mu(X) = 1$,
- $\mu(gE) = \mu(E)$ for all $E \subseteq X$ and $g \in \Gamma$.

(iii) *Følner condition.* For every finite set $E \subset \Gamma$ and for every $\varepsilon > 0$ there exists $F \subseteq X$ such that for every $g \in E$ we have:

$$|gF \Delta F| \leq \varepsilon \cdot |F|$$

(iv) *Reiters condition (or approximate mean condition).* For every finite set $S \subset \Gamma$ and for every $\varepsilon > 0$ there exists a non-negative function $\phi \in l^1(X)$ such that $\|\phi\| = 1$ and $\|g.\phi - \phi\|_1 \leq \varepsilon$.

Proof. The proof of the theorem is exactly the same as in Lemma 4.2.3. $\square$

Remark 5.1.3. Note that if G-invariant finitely additive probability measure μ on X gives a full weight to a subset X', i.e., $\mu(X') = 1$, then we can define a finitely additive probability measure on X by $\overline{\mu}(A) = \mu(A \cap X')$. It is easy to check that it remains G-invariant. Moreover, going through the equivalences in the last theorem, it is immediate that the Følner sets can be chosen as subsets of X' as well as the approximate means can be chosen to be supported on X'. This simple observation will be important for the applications in the next sections.

Let $\Phi : X \to Y$ be a map. Then we have a canonical map $\overline{\Phi} : l^\infty(Y) \to l^\infty(X)$ defined by $\overline{\Phi}(f) = f \circ \Phi$ for all $f \in l^\infty(Y)$. Consider the dual $\overline{\Phi}^* : l^\infty(X)^* \to l^\infty(Y)^*$ of $\overline{\Phi}$. **A push-forward** of a mean $\mu \in l^\infty(X)^*$ with respect to Φ is the mean $\overline{\Phi}^*(\mu)$, we will denote it by $\Phi_*\mu$. It is straightforward that if Φ is a Γ-map then the push-forward of a Γ-invariant mean is Γ-invariant. Let $m \in M(M(X))$ be a mean on the space of means.

The **barycenter** of m is $\overline{m} \in M(X)$ defined by $\overline{m}(f) = m(\mu \mapsto \mu(f))$ for all $f \in l^\infty(X)$. It is easy to check that if m is Γ-invariant then $\overline{m}$ is also Γ-invariant.

Theorem 5.1.4. If Γ acts amenably on a set X and the stabilizer of each point of X is amenable, then Γ is amenable.

Proof. Define a Γ-map $\Phi : X \to M(\Gamma)$ as follows. For each point $x \in X$, the stabilizer of it acts amenably on $M(\Gamma)$, thus it has a fixed point μ_x. Let Y be a set of orbit representatives and $X = \bigcup_{x \in Y} Orb(x, \Gamma)$ be a decomposition of X into the (disjoint) orbits of Γ. Then if $x \in Y$ we define $\Phi(x) = \mu_x$ and if $y = gx$ for $x \in Y$ we define $\Phi(y) = g.\mu_x$. In other words, Φ is the orbital map. It is straightforward to check that Φ is a Γ-map.

Let $\Phi_* \mu \in M(M(\Gamma))$ be the push-forward of μ. Then its barycenter is an invariant mean on Γ. $\square$

Lemma 5.1.5. If X has subexponential growth then an action of any finitely generated group on it is amenable.

Chapter 6

Faithful amenable actions of non-amenable groups

It is natural to ask if there is a non-amenable group which can act faithfully and amenably on a set. In this section we give examples of such actions and elaborate more on this question.

A basic example of this sections is the class of *wobbling groups* constructed by a metric space. This class of groups is relatively new. It is expected that there is a strong connection between the group structure of $W(X)$ and the metric space X.They were also used to prove non-amenability results in [49, Remark 0.5 C_1''], [29], [34].

6.1 Wobbling groups of metric spaces

Let (X, d) be a metric space. We define *the wobbling group* $W(X)$ as the group of all bijections g of X satisfying

$$|g|_w = \sup\{d(g(x), x) : x \in X\} < \infty.$$

A more general definition of this group appears in [23] and [29]. Wobbling groups will be important in one of the proofs of amenability of *topological full groups* in later sections.

Usually we will be interested in metric spaces coming from graphs. In the case when all connected components of X are of finite cardinality, the group

$W(X)$ is locally finite. However, this is the only case when a wobbling group can be amenable.

An infinite path in a metric space X is a sequence $x_0, x_1 \ldots \in X$ such that $x_i \neq x_j$ for all i, j and $d(x_i, x_{i+1}) < C$ for some constant C.

Proposition 6.1.1. If X contains an infinite path then $W(X)$ contains $\mathbb{F}_2$.

Proof. We will show that the group

$$\Gamma = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = \langle a, b, c : a^2 = b^2 = c^2 \rangle$$

is a subgroup of $W(X)$, which is supported on infinite path.

Let w be a reduced word in Γ of length n. We enumerate w from the right to the left. Define an action of Γ on $\{1, \ldots, n+1\}$ as follows. Let $g \in \{a, b, c\}$ be a generator. Then $g(i) = i+1$ and $g(i+1) = i$ if g is the i-th element of w and $g(i) = i$ on the rest of the points. Consecutively, we have $w(1) = n+1$. Since the set of reduced words is countable we can arrange that Γ acts as bijections of the path and extend it to the whole X by trivial action. Since $\mathbb{F}_2 < \Gamma$ the statement follows. □

The proof originates from Schreier, [94], who used it to show that $\mathbb{F}_2$ is residually finite.

6.2 Properties of wobbling groups

It is an interesting question to extract properties of the group $W(X)$ using the properties of the underlying metric space. Here we deduce some properties of the wobbling group $W(X)$ from the property of metric space X. The first one is amenability of the action of $W(X)$ provided that X is an amenable graph and the second is the absence of property (T) subgroups in $W(X)$ for X of subexponential growth.

A graph (V, E) is amenable if for every $\varepsilon > 0$ there exists a finite set of vertices V_ε such that

$$|\partial_E(V_\varepsilon)| \leq \varepsilon |V_\epsilon|,$$

here $\partial_E(V_\varepsilon)$ stands for the number of edges connecting V_ε and its complement.

It is straightforward that the group acts on a set amenably if and only if the Schreier graph of this action is amenable. Assuming that (V, E) is amenable it is easy to check that the Schreier graph of any finitely generated subgroup of $W(V)$ is amenable. Thus we have the following lemma.

Lemma 6.2.1. If X is an amenable graph, then the action of $W(X)$ on X is amenable.

Below we prove that $W(X)$ cannot contain property (T) groups when X is of subexponential growth. A very similar observation, attributed to Kazhdan, was made by Gromov ([49] Remark 0.5.F): a discrete property (T) group G cannot contain a subgroup G' such that G/G' has subexponential growth unless G/G' is finite.

Theorem 6.2.2. Let X be a metric space with uniform subexponential growth :

$$\lim_n \log \sup_{x \in X} |B(x,n)|/n = 0.$$

Then $W(X)$ does not contain an infinite countable property (T) group.

Proof. Assume $G < W(X)$ is a finitely generated property (T) group, with finite symmetric generating set S. We will prove that G is finite. To do so we prove that the G-orbits on X are finite, with a uniform bound. Assume that $1 \in S$. If $m = max\{|g|_w : g \in S\}$, then $S^n x \subset B(x, mn)$ for every $x \in X$, so that by assumption, the growth of $S^n x$ is subexponential with uniform by $x \in X$ constant. We will show that the expanding properties for actions of (T) groups will imply that the orbit of x is finite with size uniformly bounded in x. The later implies that G is finite.

Let Y be a G-orbit of x. Since the Schreier graph of the action of G on Y is subexponention, by Lemma 5.1.5 we have that this action is amenable. Property (T) implies that there exists $\varepsilon > 0$, such that for every unitary action of G on a Hilbert space H without invariant vectors we have

$$\max_{g \in S} \|g \cdot \xi - \xi\|^2 \geq \varepsilon \|\xi\|^2 \tag{6.1}$$

for every $\xi \in H$.

Assume that Y is infinite and consider the Hilbert space $l^2(Y)$. Let F be a finite subset of Y. Applying the inequality above the normalized indicator function $\chi_F / \|\chi_F\|_2$ we obtain

$$\max_{g \in S} |gF\Delta F| \geq \varepsilon \cdot \|\chi_F\|^2 = \varepsilon |F|,$$

which implies that the action of G on Y is not amenable, thus Y is finite.

Assuming that Y is finite, we will estimate it's size uniformly by x. Consider a Hilbert space $l_2(Y)$ with vector

$$\xi = \chi_F - (|F|/|F^c| \cdot \chi_{F^c}),$$

where F is any finite subset of Y such that $2|F| \leq |Y|$. Then $\|\xi\|_2^2 = 2|F|$ and

$$\begin{aligned} \|g\xi - \xi\|_2^2 =& \|(1 + |F|/|F^c|)(\chi_{gF} - \chi_F)\|_2^2 \\ =& (1 + |F|/|F^c|)^2 |gF\Delta F| \\ \leq& 4|gF\Delta F|. \end{aligned}$$

Applying the inequality (6.1) we obtain

$$\max_{s \in S} |gF\Delta F| \geq \frac{1}{4}\|g\xi - \xi\|_2^2 \geq \frac{\varepsilon}{2}\|\xi\|_2^2 = \frac{1}{2}|F|.$$

Thus as soon as $2|F| \leq |Y|$ we have $\max_{s \in S} |gF\Delta F| \geq \frac{1}{2}|F|$. Now assume that

$$2|S^n x| \leq |Y|$$

for some n and x. Then there exists $g \in S$ such that

$$|gS^n x \Delta S^n x| \geq \frac{\varepsilon}{2}|S^n x|.$$

Note that $|S^n x \backslash gS^n x| = |g^{-1}S^n x \backslash S^n x|$, thys

$$\begin{aligned} |gS^n x \Delta S^n x| &= |gS^n x \backslash S^n x| + |g^{-1}S^n x \backslash S^n x| \\ &\leq 2 \max_{h \in \{g, g^{-1}\}} |hS^n x \backslash S^n x| \\ &\leq 2|S^{n+1} x \backslash S^n x|. \end{aligned}$$

Thus $2|S^{n+1}x - S^n x| \geq \frac{\varepsilon}{2}|S^n x|$ which implies

$$|S^{n+1}x| \geq \left(1 + \frac{\varepsilon}{4}\right)|S^n x|.$$

Since $2|S^n x| \leq |Y|$ implies $2|S^m x| \leq |Y|$ for all $m \leq n$ and $|S^0 x| = 1$ we have

$$|S^{n+1} \geq \left(1 + \frac{\varepsilon}{4}\right)^{n+1}$$

provided $2|S^n x| \leq |Y|$. Suppose that there exists a sequence of orbits Y_k of arbitrary large size. Hence we can find a divergent sequence n_k and $x_k \in Y_k$ such that

$$2|S^{n_k} x_k| \leq |Y_k|,$$

thus

$$\frac{\log |S^{n_k+1} x_k|}{n_k + 1} \geq 1 + \frac{\varepsilon}{4},$$

which is a contradiction. □

We will show that it is possible to construct amenable spaces X such that $W(X)$ contains property (T) groups. We first remark that the groups $W(X)$ behave well with respect to coarse embeddings. A map $q : (X, d_X) \to (Y, d_Y)$ between metric spaces is *a coarse embedding* if there exists nondecreasing functions $\varphi_+, \varphi_- : [0, \infty[\to \mathbb{R}$ such that $\lim_{t \to \infty} \varphi_-(t) = \infty$ and

$$\varphi_-(d_X(x, x')) \leq d_Y(q(x), q(x')) \leq \varphi_+(d_X(x, x'))$$

for every $x, x' \in X$.

Lemma 6.2.3. Let $q : (X, d_X) \to (Y, d_Y)$ be a map such that there is an increasing function $\varphi_+ : \mathbb{R}^+ \to \mathbb{R}^+$ such that $d_Y(qx, qy) \leq \varphi_+(d_X(x, y))$, and such that the preimage $q^{-1}(y)$ of every $y \in Y$ has cardinality less than some constant K (e.g. q is a coarse embedding and X has bounded geometry). Let F be a finite metric space of cardinality K. Then $W(X)$ is isomorphic to a subgroup of $W(Y \times F)$.

Proof. In this statement $Y \times F$ is equipped with the distance

$$d((y, f), (y', f')) = d_Y(y, y') + d_F(f, f').$$

Since F is bigger than $q^{-1}(y)$ for all y, there is a map $f : X \to F$ such that the map

$$\widetilde{q} : x \in X \mapsto (q(x), f(x)) \in Y \times F$$

is injective. We can therefore define an action of $W(X)$ on $Y \times F$ by setting $g(\widetilde{q}(x)) = \widetilde{q}(gx)$ and $g(y, f) = (y, f)$ if $(y, f) \notin \widetilde{q}(X)$. The existence of φ_+

guarantees that this action is by wobblings, i.e., that it defines an embedding of $W(X)$ into $W(Y \times F)$. $\square$

In a contrast to Theorem 6.2.2 we have the following example of R. Tessera.

Theorem 6.2.4. There is a solvable group Γ such that $W(\Gamma)$ contains the property (T) group $SL(3, \mathbb{Z})$.

Proof. The proof uses the notion of asymptotic dimension (see [9]). By [9, Corollary 94], $SL(3, \mathbb{Z})$ has finite asymptotic dimension. By [9, Theorem 44] this implies that $SL(3, \mathbb{Z})$ embeds coarsely into a finite product of binary trees. Take Γ_0 a solvable group with a free semigroup. In particular it coarsely contains a binary tree, so $SL(3, \mathbb{Z})$ embeds coarsely in Γ_0^n for some n. By Lemma 6.2.3, there is a finite group F such that $W(SL(3, \mathbb{Z}))$ embeds as a subgroup in $W(F \times \Gamma_0^n)$. But $W(SL(3, \mathbb{Z}))$ contains $SL(3, \mathbb{Z})$ with action given by translation. $\square$

Remark 6.2.5. The proof actually shows that for every group Λ with finite asymtotic dimension, there is an integer n such that Λ is isomorphic to a subgroup of $W(F \times \Gamma^n)$ whenever there is a Cayley graph of Γ that contains an infinite binary tree as a subgraph. This includes lots of groups Γ with exponential growth. In some sense this says that the assumptions of Theorem 6.2.2 are not so restrictive.

Chapter 7

Lamplighter actions and extensive amenability

Assume that a discrete group G is acting on a set X. In this chapter we will assume that the action is transitive, thus the Schreier graph of the action is connected. This assumption is natural, since in order to show that the action of G on X is amenable it is enough to show that the action of G on some orbit is amenable.

Consider the direct sum $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$, i.e., the group of all finitely supported sequences with values in $\{0,1\}$ with addition mod 2. Another interpretation of $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ is as the set of all finite subsets of X, denoted by $\mathcal{P}_f(X)$, with multiplication given by the symmetric difference. We will use all this interpretations of $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ depending on which is more appropriate to a context.

The action of an element $g \in G$ on X induces an automorphism of $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ given by the action on the indexes of a sequence $(w_x)_{x\in X} \in \bigoplus_X \mathbb{Z}/2\mathbb{Z}$ by

$$g(w_x)_{x\in X} = (w_{g^{-1}x})_{x\in X}.$$

Thus we can form a semidirect product $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$, which is also called *wreath product* of $\mathbb{Z}/2\mathbb{Z}$ and G, denoted by $\mathbb{Z}/2\mathbb{Z} \wr_X G$. The multiplication is given as follows:

$$(E, g) \cdot (F, h) = (E \Delta g(F), gh) \text{ for } g, h \in G \text{ and } E, F \in \mathcal{P}_f(X).$$

The lamplighter $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$ acts on the cosets $\bigoplus_X (\mathbb{Z}/2\mathbb{Z} \rtimes G)/G$ by multiplication on the left, which can be viewed as an action on $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ by the following rule:

$$(E, g)(F) = g(F)\Delta E \text{ for } g \in G \text{ and } E \in \mathcal{P}_f(X).$$

Definition 7.0.6. An action of G on X is **extensively amenable** if the affine action of the semidirect product $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$ on $\bigoplus_X \mathbb{Z}/2\mathbb{Z}$ is amenable.

In fact, as we will see in Theorem 7.0.8, the $\mathbb{Z}/2\mathbb{Z}$ can be replaced by any amenable group in the definition above.

Note that if the group G is amenable then so is $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$, thus any action of G is extensively amenable. However, as we will see later, the action of the wobbling group $W(\mathbb{Z})$ on $\mathbb{Z}$ is extensively amenable. Never the less $W(\mathbb{Z})$ contains the free non-abelian group on two generators.

Consider the map $\phi : \mathcal{P}_f(X)\backslash\emptyset \to Proba(X)$, which sends a finite subset to the uniform measure on it. It is a G-map. Then the extensive amenability of the action of G implies, in particular, that there exists G-invariant mean μ on $\mathcal{P}_f(X)$. The push-forward of μ along ϕ is a G-invariant mean on X. Thus we have just showed the following.

Lemma 7.0.7. If the action of G on X is extensively amenable, then this action is amenable.

Note that the converse is not true, we provide examples in the next section.

In the case of non-amenable groups it is always a difficult question to decide whether a particular amenable action is extensively amenable. A good example of this difficulty will be illustrated on the wobbling groups. Before proceeding to examples, we will provide more equivalent definitions of extensive amenability in the theorem below.

Since the definitions we are aiming to present have many different flavors, let us first discuss some notation we are going to use. We consider a measure space $(\{0,1\}^X, \mu)$, where the set of all sequences in $\{0,1\}^X$ is equipped with measure μ which is equal to the product measure of the uniform measures on $\{0,1\}$. This defines the Hilbert space $L^2(\{0,1\}^X, \mu)$ of all square integrable functions on $\{0,1\}^X$ with respect to the measure μ and the inner product

$$\langle f, g\rangle = \int_{\{0,1\}^X} f\overline{g} \quad d\mu$$

Denote by χ_A the characteristic function of a set $A \subset \{0,1\}^X$.

Theorem 7.0.8. Let G act transitively on X and fix a point p in X. The following are equivalent:

(i) There exists a net of unit vectors $f_n \in L^2(\{0,1\}^X, \mu)$ such that for every $g \in G$

$$\|gf_n - f_n\|_2 \to 0 \text{ and } \|f_n \cdot \chi_{\{(w_x)\in\{0,1\}^X : w_p = 0\}}\|_2 \to 1;$$

(ii) The action of G on X is extensively amenable;

(iii) There exists a constant $C > 0$ such that the action of G on $\mathcal{P}_f(X)$ admits an invariant mean giving weight C to the collection of sets containing p;

(iv) The action of G on $\mathcal{P}_f(X)$ admits an invariant mean giving the full weight to the collection of sets containing p;

(v) The action of G on $\mathcal{P}_f(X)$ admits an invariant mean such that for all $E \in \mathcal{P}_f(X)$ it gives the full weight to the collection of sets containing E;

Proof. (i) $\Longrightarrow$ (ii). Let $f_n \in L^2(\{0,1\}^X, \mu)$ be a net of functions which satisfy (i). Note that, replacing f_n with $f_n \cdot \chi_{\{w_p=0\}}$ and normalizing we obtain a net of unit vectors in $L^2(\{0,1\}^X, \mu)$ such that $f_n \cdot \chi_{\{w_p=0\}} = f_n$ and

$$\|gf_n - f_n\|_2 \to 0 \text{ for all } g \in G.$$

The group $\{0,1\}^X$ is the Pontriagin dual of $\mathcal{P}_f(X)$, with the pairing function $\{0,1\}^X \times \mathcal{P}_f(X)$ given by

$$\langle w, E\rangle = exp(i\pi \sum_{x\in E} w_x)$$

Thus we can define the Fourier transform $\hat{f} \in l^2(\mathcal{P}_f(X))$ of $f \in L^2(\{0,1\}^X, \mu)$ by

$$\hat{f}(E) = \int_{\{0,1\}^X} f(w)\langle w, E\rangle d\mu(w).$$

We will show that the net $\hat{f}_n$ is approximately invariant under the action of $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$.

From the properties of Fourier transform and an easy fact that it is G-equivariant, we have

$$\|g\hat{f}_n - \hat{f}_n\|_2 = \|(gf_n - f_n)\hat{}\,\|_2 = \|gf_n - f_n\|_2 \to 0.$$

Thus $\hat{f}_n$ remains G-invariant.

Since the action of G on X is transitive it is sufficient to show that $\hat{f}_n$ is $\{p\}$-almost invariant, where $\{p\} \in \mathcal{P}_f(X)$:

$$\begin{aligned}
\{p\}\hat{f}_n(E) =& \hat{f}_n(E\Delta\{p\}) \\
=& \int_{\{0,1\}^X} f_n\langle w, E\Delta\{p\}\rangle d\mu(w) \\
=& \int_{\{0,1\}^X} f_n\chi_{\{w_p=0\}}(w)\langle w, E\Delta\{p\}\rangle d\mu(w) \\
=& \int_{\{0,1\}^X} f_n\chi_{\{w_p=0\}}(w)exp(i\pi \sum_{x\in E\Delta\{p\}} w(x))d\mu(w) \\
=& \int_{\{0,1\}^X} f_n\chi_{\{w_p=0\}}(w)exp(i\pi \sum_{x\in E} w(x))d\mu(w) \\
=& \hat{f}_n(E)
\end{aligned}$$

Thus $\hat{f}_n$ is $\{p\}$-invariant. Since $\hat{f}_n \in l^2(\mathcal{P}_f(X))$ are almost invariant, we have that $\hat{f}_n^2 \in l_1(\mathcal{P}_f(X))$ are also almost invariant. Indeed, this follows from Cauch-Schwarz inequality:

$$\begin{aligned}
\|gf^2 - f^2\|_1 &= \|(gf-f)(gf+f)\|_1 \\
&\le \|(gf-f)\|_2\|(gf+f)\|_2 \\
&\le 2\|(gf-f)\|_2,
\end{aligned}$$

for every unit vector $f \in l^2(\mathcal{P}_f(X))$ and $g \in G$.

Taking a cluster point of the net $\hat{f}_n^2$ in weak$*$-topology we obtain an $\bigoplus_X \mathbb{Z}/2\mathbb{Z} \rtimes G$-invariant mean on $\mathcal{P}_f(X)$, thus the action is extensively amenable.

To show that (ii) implies (iii), assume that m is $\mathcal{P}_f(X) \rtimes G$-invariant mean on $\mathcal{P}_f(X)$. Let $\mathcal{F}_p$ be the set of all finite subsets of X containing p. Then $\{p\}.\mathcal{F}_p = \mathcal{F}_p^c$, therefore we have $m(\mathcal{F}_p) = m(\{p\}.\mathcal{F}_p) = m(\mathcal{F}_p^c) = 1/2$.

(iii) $\Longrightarrow$ (iv). Let now m be a G-invariant mean on $\mathcal{P}_f(X)$ with $m(\mathcal{F}_p) = C$ for some $C > 0$.

Fix $k \in \mathbb{N}$ and define G-equivariant map

$$U_k : \mathcal{P}_f(X)^k \to \mathcal{P}_f(X)$$

by

$$U_k(F_1, \ldots, F_k) = F_1 \cup \ldots \cup F_k$$

Let $m^{\times k}$ be a product measure on $\mathcal{P}_f(X)^k$, i.e., for all $F_1, \ldots, F_k \in \mathcal{P}_f(X)$ we have

$$m^{\times k}(F_1, \ldots, F_k) = m(F_1) \cdot \ldots \cdot m(F_k).$$

Note that $m^{\times k}$ is invariant under diagonal action of G.

Define m_k to be the push-forward of $m^{\times k}$ with respect to U_k. Since U_k and $m^{\times k}$ are G-invariant, m_k is also G-invariant. Moreover, for all $E \in \mathcal{X}$ we have

$$U_k^{-1}(\mathcal{F}_p^c) \subseteq \mathcal{F}_p^c \times \ldots \times \mathcal{F}_p^c.$$

Thus $1 - m_k(\mathcal{F}_p) \leq (1 - m(\mathcal{F}_p))^k = (1 - C)^k$. Taking a cluster point in the weak*-topology we obtain a mean that satisfies (v).

(v) $\Longrightarrow$ (iv). It is obvious.

(v) $\Longrightarrow$ (v). For a finite set $E \subset X$ the set of finite subsets that contain E is $\bigcap_{x \in E} \mathcal{F}_x$ Since the action of G on X is transitive we have $\mu(\mathcal{F}_x) = 1$ for every $x \in X$, thus $\mu(E) = 1$ and (v) follows.

(iv) $\Longrightarrow$ (i). Let m be G-invariant mean giving a full weight to $\mathcal{F}_p$. Then there exists a net $m_n \in l_1(\mathcal{P}_f(X))$ such that $\|g.m_n - m_n\|_1 \to 0$ for all $g \in G$ and $m_n(\mathcal{F}_p) = 1$. For a sequence $w \in \{0,1\}^X$, define

$$f_n(w) = \sum_{E \in \mathcal{P}_f(X)} m_n(F) 2^{|F|} \chi_F(w),$$

where $\chi_F(w)$ is 1 is $F \cap w \neq \emptyset$ and 0 otherwise. Since m_n is supported on $\mathcal{F}_p$ we have $f_n \cdot \chi_{\{(w_x) \in \{0,1\}^X : w_p = 0\}} = f_n$. Since $\|\chi_F\|_1 = 2^{-|F|}$ we have that $\|f_n\| = 1$.

Moreover f_n is G-invariant. Indeed,

$$\begin{aligned} \|g.f_n - f_n\| &\leq \sum_{E \in \mathcal{P}_f(X)} \|m_n(E) 2^{|F|} g.\chi_F - m_n(E) 2^{|F|} \chi_F\|_1 \\ &= \sum_{E \in \mathcal{P}_f(X)} \|m_n(gE) 2^{|F|} \chi_F - m_n(E) 2^{|F|} \chi_F\|_1 \\ &= \sum_{E \in \mathcal{P}_f(X)} \|m_n(gE) - m_n(E)\|_1 \\ &= \|g.m_n - m_n\|. \end{aligned}$$

Since $\|g.f_n^{1/2} - f_n^{1/2}\|_2^2 \leq \|g.f_n - f_n\|_1$ we obtain the desired.

□

The following lemma is an interesting observation of the fact that the group $\mathbb{Z}/2\mathbb{Z}$ is not very essential for extensively amenable actions and can be replaced by any other amenable group.

Lemma 7.0.9. Assume that the action of G on X is transitive and extensively amenable. Then for any amenable group H the action of $\bigoplus_X H \rtimes G$ on $\bigoplus_X H$ is amenable;

Proof. Fix a point p in X. We may assume that G is generated by a finite set S. Let A be a finite subset of H. For every $\varepsilon > 0$ we need to find a finite set E in $\bigoplus_X H$ such that it is ε-invariant for both S and $\{(e, h\delta_p) : h \in A\}$.

Fix $\varepsilon > 0$. By assumptions we can find a finite set $F \subset \mathcal{P}_f(X)$, which is (S, ε)-invariant and such that all elements of F contain the point p. Without loss of generality we may assume that all elements of F are of the same size which is equal to k. Indeed, since the action of G preserves cardinality, we can decompose F as disjoint union $F = F_1 \cup \ldots \cup F_n$ each element of F_i has the same cardinality. Assume that none of these components is $(S, |S| \cdot \varepsilon)$-invariant, i.e., for every F_i there exists $s \in S$ such that

$$|sF_i \backslash F_i| > \varepsilon \cdot |S| \cdot |F_i|.$$

Note that $|sF\backslash F| = \sum\limits_{i=1}^{n} |sF_i\backslash F_i|$. Summing the last equation by s we obtain:

$$\sum_{s\in S} |sF_i\backslash F_i| = \sum_{s\in S}\sum_{i=1}^{n} |sF_i\backslash F_i| > \varepsilon \cdot |S| \cdot \sum_{i=1}^{n} |F_i| = \varepsilon \cdot |S| \cdot |F|,$$

which is impossible.

Let F_A be a (A,ε)-Følner set. Consider the set taken with multiplicities:

$$E = \{\phi \in \bigoplus_X H : \mathrm{supp}(\phi) \in F, \phi(X) \subset F_A\}.$$

Here each ϕ comes with multiplicity $\{|\{C \in F : supp(\phi) \subseteq C\}|\}$. Then $|E| = |F_A|^k \cdot |F|$ and

$$|sE\backslash E| \leq \varepsilon |F_A| \cdot |F| = \varepsilon |E|.$$

Moreover, for all $h \in A$ we have

$$|(e, h\delta_p)E\backslash E| \leq |hF_A\backslash F_A| \cdot |F_A|^{k-1} \cdot |F| = \varepsilon/|F_A| \cdot |E|.$$

Define a function $f : \bigoplus\limits_X H \to \mathbb{R}_+$:

$$f(\nu) = \sum_{\phi\in E} \delta_\phi(\nu) \text{ for all } \nu \in \bigoplus_X H,$$

Here we write sum instead of χ_E in order to specify that the values of f can depend on the multiplicities that appear in the set E. It is immediate that $\|f\|_1 = |E|$ and $\|g.f - f\|_1 = |sE\Delta E|$, therefore we have the statement of the lemma. $\square$

We will use the following theorem for applications in the later sections.

Theorem 7.0.10. Assume that the action of G on X is extensively amenable and $H < G$ is a subgroup. Assume, in addition, that a set $Y \subset X$ is invariant under the action of H. Then the action of H on Y is extensively amenable, in particular, it is amenable.

Proof. Define a $\mathcal{P}_f(Y) \rtimes H$-equivariant map on $\mathcal{P}_f(X)$ into $\mathcal{P}_f(Y)$ by intersecting a finite subset of X with Y. The push-forward of $\bigoplus\limits_X \mathbb{Z}/2\mathbb{Z} \rtimes G$-invariant mean on $\bigoplus\limits_X \mathbb{Z}/2\mathbb{Z}$ along our H-map is $\bigoplus\limits_Y \mathbb{Z}/2\mathbb{Z} \rtimes H$-invariant mean on $\bigoplus\limits_Y \mathbb{Z}/2\mathbb{Z}$. $\square$

7.1 Recurrent actions: definition and basic properties

Let G be a finitely generated group with finite symmetric generating set S.

If G acts transitively on X the **Schreier graph** $\Gamma(X, G, S)$ is the graph with the set of vertices identified with X, the set of edges is $S \times X$, where an edge (s, x) connects x to $s(x)$.

Choose a measure μ on G such that support of μ is a finite generating set of G and $\mu(g) = \mu(g^{-1})$ for all $g \in G$. Consider the Markov chain on X with transition probability from x to y equal to $p(x, y) = \sum_{g \in G, g(x)=y} \mu(g)$.

Definition 7.1.1. The action is called **recurrent** if the probability of returning to x_0 after starting at x_0 is equal to 1 for some (hence for any) $x_0 \in X$. An action is transient if it is not recurrent.

It is well known (see [101, Theorems 3.1, 3.2]) that recurrence of the described Markov chain does not depend on the choice of the measure μ, if the measure is symmetric, and has finite support generating the group.

Definition 7.1.2. An action of G on X is **recurrent** if a Markov chain that corresponds to a measure that supported on a symmetric finite generating set of G is recurrent.

Note that, the action of G on itself is recurrent if and only if G is virtually $\{0\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$. Moreover, all recurrent actions are amenable.

The following theorem is a part of more general Nash-Williams criteria for recurrency. It will be very useful in the applications.

Theorem 7.1.3. Let Γ be a connected graph of uniformly bounded degree with set of vertices V. Suppose that there exists an increasing sequence of finite subsets $F_n \subset V$ such that $\bigcup_{n \geq 1} F_n = V$, the subsets ∂F_n are pairwise disjoint, and

$$\sum_{n \geq 1} \frac{1}{|\partial F_n|} = \infty,$$

where ∂F_n is the set of vertices of F_n adjacent to the vertices of $V \setminus F_n$. Then the simple random walk on Γ is recurrent.

We will also use a characterization of transience of a random walk on a locally finite connected graph (V, E) in terms of electrical network. The **capacity** of a point $x_0 \in V$ is the quantity defined by

$$cap(x_0) = \inf \left\{ \left(\sum_{(x,x') \in E} |a(x) - a(x')|^2 \right)^{1/2} \right\}$$

where the infimum is taken over all finitely supported functions $a : V \to \mathbb{C}$ with $a(x_0) = 1$. We will use the following

Theorem 7.1.4 ([101], Theorem 2.12)**.** The simple random walk on a locally finite connected graph (V, E) is transient if and only if $cap(x_0) > 0$ for some (and hence for all) $x_0 \in V$.

7.2 Recurrent actions are extensively amenable

In this section we discuss the relation of recurrent actions to amenability of lamps. The connecting point is the definition of amenability of lamps given in Theorem 7.0.8 (i).

Let $\mathcal{H}_i$ be a collection of Hilbert spaces indexed by a set I. Fix a sequence of normal vectors $\xi_i \in \mathcal{H}_i$. Then the algebraic (incomplete) tensor product of $\mathcal{H}_i$ is the set of all linear combinations of $\bigotimes_{i \in I} \phi_i$, where all but finitely many ϕ_i are equal to ξ_i. It carries an inner product, which is defined by

$$\langle \bigotimes_{i \in I} \phi_i, \bigotimes_{i \in I} \nu_i \rangle = \prod_{i \in I} \langle \phi_i, \nu_i \rangle_{\mathcal{H}_i}$$

An infinite tensor product of Hilbert spaces is the Hilbert space is defined to be the completion of the algebraic tensor product by the norm defined by the above inner product.

Consider a Hilbert space of square integrable functions $L_2(\{0,1\}^X, \mu)$ with respect to the measure μ given by the product of measure m on $\{0,1\}$, where $m(0) = m(1) = \frac{1}{2}$.

It is natural to consider the Hilbert space $L_2(\{0,1\}^X, \mu)$ as an infinite tensor power of the Hilbert space $L_2(\{0,1\}^X, m)$.

A function $f \in L_2(\{0,1\}^X, \mu)$ is called *a product of independent random variables* if there are functions $f_x : \{0,1\} \to \mathbb{C}$ such that $f(w) = \prod_{x \in X} f_x(w_x)$.

Equivalently, if we consider $L_2(\{0,1\}^X, \mu)$ as the infinite tensor power, then the condition that f is a product of random independent variables means that f is an elementary tensor in $L_2(\{0,1\}^X, \mu)$.

Theorem 7.2.1. Let G be a finitely generated group acting transitively on a set X and fix a point p in X. There exists a sequence of functions $\{f_n\}$ in $L_2(\{0,1\}^X, \mu)$ with $\|f_n\|_2 = 1$ given by a product of random independent variables that satisfy

(i) $\|gf_n - f_n\|_2 \to 0$ for all $g \in G$,

(ii) $\|f_n \cdot \chi_{\{(w_x)\in\{0,1\}^X : w_p = 0\}}\|_2 \to 1$,

if and only if the action of G on X is recurrent.

Proof 1. Denote by (X, E) the Schreier graph of the action of G on X with respect to S. Suppose that the simple random walk on (X, E) is recurrent. By Theorem 7.1.4, there exists $a_n = (a_{x,n})_x$ a sequence of finitely supported functions such that $a_{p,n} = 1$ and

$$\sum_{x\sim x'} |a_{x,n} - a_{x',n}|^2 \to 0.$$

Without loss of generality we may assume that $0 \le a_{x,n} \le 1$. Indeed, we can replace all values $a_{x,n}$ that are greater than 1 by 1 and those that are smaller than 0 by 0, this would not increase the differences $|a_{x,n} - a_{x',n}|$.

For $0 \le t \le 1$ consider the unit vector $\xi_t \in L_2(\{0,1\}, m)$ defined by

$$(\xi_t(0), \xi_t(1)) = (\sqrt{2}\cos(t\pi/4), \sqrt{2}\sin(t\pi/4)).$$

Define $f_{x,n} = \xi_{1-a_{x,n}}$ and $f_n = \bigotimes_{x\in X} f_{x,n}$.

To show that $\|gf_n - f_n\|_2 \to 0$ for all $g \in G$, it the same as to show that $\langle gf_n, f_n\rangle \to 1$ for all $g \in \Gamma$. It is sufficient to show this for $g \in S$. Since $\cos(x) \ge e^{-x^2}$, whenever $|x| \le \pi/4$, we have

$$\begin{aligned}\langle gf_n, f_n\rangle &= \prod_x \langle f_{x,n}, f_{gx,n}\rangle \\ &= \prod_x \cos\frac{\pi}{4}(a_{x,n} - a_{gx,n}) \\ &\geq \prod_x \exp\left(-\frac{\pi^2}{16}(a_{x,n} - a_{gx,n})^2\right) \\ &\geq \exp\left(-\frac{\pi^2}{16}\sum_{x\sim x'} |a_{x,n} - a_{x',n}|^2\right)\end{aligned}$$

By the selection of $a_{x,n}$, the last value converges to 1.

Since $f_{p,n} = \xi_0 = (1,0)$ we have

$$f_n \chi_{\{(w_x)\in\{0,1\}^X : w_p = 0\}} = f_n.$$

Let us prove the other direction of the theorem. Define the following pseudometric on the unit sphere of $L_2(\{0,1\}, m)$ by

$$d(\xi, \eta) = \inf_{w\in\mathbb{C}, |w|=1} \|w\xi - \eta\| = \sqrt{2 - 2|\langle \xi, \eta\rangle|}.$$

Assume that there is a sequence of products of random independent variables $\{f_n\}$ in $L_2(\{0,1\}^X, \mu)$ that satisfy the conditions of the theorem, i.e.,

$$f_n(w) = \prod_{x\in X} f_{n,x}(w_x).$$

We can assume that the product is finite. Replacing $f_{n,x}$ by $f_{n,x}/\|f_{n,x}\|$ we can assume that $\|f_{n,x}\|_{l_2(\{0,1\},m)} = 1$. Define $a_{x,n} = d(f_{x,n}, 1)$. It is straightforward that $(a_{x,n})_{x\in X}$ has finite support and

$$\lim_n a_{p,n} = \sqrt{2 - \sqrt{2}} > 0.$$

Moreover for every $g \in G$

$$\begin{aligned}|\langle gf_n, f_n\rangle| &= \prod_x |\langle f_{n,x}, f_{n,gx}\rangle| \\ &= \prod_x (1 - d(f_{n,x}, f_{n,gx})^2/2) \\ &\leq \exp\left(-\sum_x d(f_{n,x}, f_{n,gx})^2/2\right).\end{aligned}$$

Since by assumption $|\langle gf_n, f_n\rangle| \to 1$ and $\sum_x d(f_{n,x}, f_{n,gx})^2 \geq 0$ we have

$$\sum_x d(f_{n,x}, f_{n,gx})^2 \to 0.$$

By definition of the Schreier graph and the triangle inequality for d,

$$\begin{aligned}\sum_{(x,x')\in E} |a_{x,n} - a_{x',n}|^2 &= \sum_{g\in S}\sum_x |a_{x,n} - a_{gx,n}|^2 \\ &\leq \sum_{g\in S}\sum_x d(f_{n,x}, f_{n,gx})^2 \to 0.\end{aligned}$$

This proves that $cap(p) = 0$ in (X, E), and hence by Theorem 7.1.4 that the simple random walk on (X, E) is recurrent. $\square$

A more direct proof of the amenability of lamps from recurrency of the action is the following.

Direct proof of recurrency implies extensive amenability. We again use the characterization of recurrency in terms of capacity, which implies that there exists $a_n : X \to \mathbb{R}_+$ be a sequence of finitely supported functions such that for a fixed point $p \in X$ we have $a_n(p) = 1$ for all n and

$$\|ga_n - a_n\|_2 \to 0 \text{ for all } g \in G.$$

Moreover, we can assume $0 \leq a_n(x) \leq 1$ for all $x \in X$ and n.

Define $\xi_n : \mathcal{P}_f(X) \to \mathbb{R}_+$ by $\xi_n(\emptyset) = 1$ and

$$\xi_n(F) = \prod_{x\in F} a_n(x).$$

We claim that $\nu_n := \xi_n/\|\xi_n\|_2 \in l_2(\mathcal{P}_f(X)$ is almost invariant under the action of $\mathcal{P}_f(X) \rtimes G$. Thus taking a cluster point in the weak*-topology of $\nu_n^2 \in l_1(\mathcal{P}_f(X))$ we obtain a $\mathcal{P}_f(X) \rtimes G$-invariant mean on $\mathcal{P}_f(X)$.

To prove the claim, note that since $a_n(p) = 1$ for all n the functions ν_n are automatically invariant under the action of $\{p\} \in \mathcal{P}_f(X)$. From the transitivity of action of G on X we have that it is sufficient to show that ν_n are almost invariant under the action of G. Since $\|g\nu_n - \nu_n\| = 2 - 2\langle g\nu_n, \nu_n\rangle$, it is sufficient to show that $\langle g\nu_n, \nu_n\rangle \to 1$. The direct verification shows that

$$
\begin{aligned}
\|\xi_n\|^2 &= \langle \xi_n, \xi_n \rangle \\
&= \prod_{x \in X} \left(1 + a_n(x)^2\right) \\
&= \prod_{x \in X} \left(1 + a_n(g^{-1}x)^2\right)
\end{aligned}
$$

and

$$
\langle g\xi_n, \xi_n \rangle = \prod_{x \in X} \left(1 + a_n(g^{-1}x)a_n(x)\right).
$$

Thus we have

$$
\left(\frac{\langle \xi_n, \xi_n \rangle}{\langle g\xi_n, \xi_n \rangle}\right)^2 = \prod_{x \in X} \frac{(1 + a_n(x)^2)\,(1 + a_n(g^{-1}x)^2)}{\left(1 + a_n(g^{-1}x)a_n(x)\right)^2}
$$

Since $\log(t) \le t - 1$ for all $t > 0$ and $0 \le a_n(x) \le 1$ we have

$$
\begin{aligned}
0 &\le 2\log \frac{\langle \xi_n, \xi_n \rangle}{\langle g\xi_n, \xi_n \rangle} \\
&= \sum_{x \in X} \log \frac{(1 + a_n(x)^2)\,(1 + a_n(g^{-1}x)^2)}{\left(1 + a_n(g^{-1}x)a_n(x)\right)^2} \\
&= \sum_{x \in X} \frac{\left(a_n(x) - a_n(g^{-1}x)\right)^2}{\left(1 + a_n(g^{-1}x)a_n(x)\right)^2} \\
&= \|ga_n - a_n\| \to 0.
\end{aligned}
$$

$\square$

Chapter 8

Amenability of the full topological group of Cantor minimal systems

By minimality, the full topological group $[[T]]$ acts on any T-orbit faithfully. Identifying an orbit of T with $\mathbb{Z}$ we have that $[[T]]$ is a subgroup of $W(\mathbb{Z})$. More precisely, fix a point $p \in C$ and define a homomorphism

$$\pi_p \colon [[T]] \longrightarrow W(\mathbb{Z})$$

by the formula

$$g(T^j p) = T^{\pi_p(g)(j)} p, \text{ for all } g \in [[T]], j \in \mathbb{Z}.$$

The homomorphism π_p is injective if the orbit of p is dense.

A subgroup G of $W(\mathbb{Z})$ has the *ubiquitous pattern property* if for every finite set $F \subseteq G$ and every $n \in \mathbb{N}$ there exists a constant $k = k(n, F)$ such that for every $j \in \mathbb{Z}$ there exists $t \in \mathbb{Z}$ such that $[t-n, t+n] \subseteq [j-k, j+k]$ and such that for every $i \in [-n, n]$ and every $g \in F$ we have $g(i+t) = g(i)+t$.

Lemma 8.0.2. Let (T, C) be a Cantor minimal system, then $\pi_p([[T]])$ has the ubiquitous pattern property.

Proof. Fix a finite subset F in $[[T]]$ and let $n \in \mathbb{N}$. By definition there is a finite clopen partition $\mathcal{P}$ of C such that each $g \in F$ is a power of T when restricted to any element of $\mathcal{P}$. Thus there is an open neighborhood V of p

such that for all $i \in [-n, n]$ the set $T^i V$ is contained in some $D \in \mathcal{P}$. We have that the set

$$\bigcup_{q \geq 1} \bigcup_{|r| \leq q} T^r V$$

is non-empty open and T-invariant. By minimality of T, this set coinsides with C. By compactness, there is $q \in \mathbb{N}$ such that

$$C = \bigcup_{|r| \leq q} T^r V.$$

Set $k = k(n, F) = q + n$. For all $j \in \mathbb{Z}$

$$C = T^{-j} C = T^{-(j+q)} V \cup \ldots \cup T^{-(j-q)} V$$

Therefore there is an integer $t \in [j - q, j + q]$ such that $p \in T^{-t} V$. In particular, we have

$$[t - n, t + n] \subseteq [j - k, j + k].$$

Now $T^t p \in V$ and thus both $T^i p$ and $T^{i+t} p$ are in $T^i V$ for all i. Therefore, when $i \in [-n, n]$, every $g \in F$ acts on $T^i p$ and on $T^{i+t} p$ as the same power of T. By identification of the T-orbit of p with $\mathbb{Z}$, we have that the last property is exactly the ubiquitous pattern property. □

It is known that the stabiliser of the orbit of positive powers of T, i.e., the set $\{T^j p : j \in \mathbb{N}\}$, in the topological full group of a minimal Cantor system is locally finite [37]. Here is an alternative proof of this fact.

Lemma 8.0.3. The stabilizer of natural numbers, $Stab_{[[T]]}(\mathbb{N})$, under the action of $[[T]]$ on $\mathbb{Z}$ is locally finite.

Proof. Let F be a finite subset in $Stab_{[[T]]}(\mathbb{N})$. We will show that $\mathbb{Z}$ can be decomposed as a disjoint union of F-invariant finite intervals with uniformly bounded length. This implies that the group generated by F is finite, since it is a subgroup of a power of a finite group and therefore we can conclude that $Stab_{[[T]]}(\mathbb{N})$ is locally finite.

Let $c = \max(|g(j) - j| : j \in \mathbb{Z}, g \in F)$. Let $k = k(c, F)$ be the constant from the definition of the ubiquitous pattern property. Decompose $\mathbb{Z}$ as a disjoint union of consecutive intervals I_i of the length $2k + 1$ and such one of the intervals is $[-k, k]$. Let $E_0 = [0, 2N]$ and $t_0 = 0$. Then by ubiquitous pattern property we can find $E_i \subseteq I_i$ and t_i such that $E_i = E_0 + t_i$ and $g(s) + t_i = g(s + t_i)$ for every $f \in F$ and $s \in E_0$.

Since none of the positive integers are mapped by F to negative integers and by the choice of E_n we have that the intervals $[t_i, t_{i+1}]$ are F-invariant. Moreover, the length of each $[t_i, t_{i+1}]$ is bounded by $2|I_i| = 4k + 2$. Thus the group generated by F is finite and therefore $Stab_{[[T]]}(\mathbb{N})$ is locally finite. $\square$

Now we have all ingredients to prove the amenability of $[[T]]$.

Theorem 8.0.4. Let (T, C) be a Cantor minimal system. Then the full topological group $[[T]]$ is amenable.

Proof. Let G be a finitely generated subgroup of $[[T]] < W(\mathbb{Z})$. The following map $\pi : G \to \mathcal{P}(\mathbb{Z}) \rtimes G$ is an injective homomorphism

$$\pi(g) = (g(\mathbb{N})\Delta\mathbb{N}, g).$$

Indeed,

$$\begin{aligned}
\pi(g)\pi(h) =& (g(\mathbb{N})\Delta\mathbb{N}, g) \cdot (h(\mathbb{N})\Delta\mathbb{N}, h) \\
=& (g(\mathbb{N})\Delta\mathbb{N}\Delta gh(\mathbb{N})\Delta g(\mathbb{N}), gh) \\
=& (gh(\mathbb{N})\Delta\mathbb{N}, gh) \\
=& \pi(gh).
\end{aligned}$$

By the property of $W(\mathbb{Z})$ we can select a strictly increasing to $\mathbb{Z}$ sequence of intervals $[-n_i, n_i]$ such that the boarder of $[-n_i, n_i]$ under the generating set of G is contained in $[-n_{i+1}, n_{i+1}]$ and is of uniformly bounded size. By Theorem **??** we have that the action of G on $\mathbb{Z}$ is recurrent. Therefore, by Theorem 7.2.1, this action has amenable lamps. Thus it is left to show that the action of $\pi([[T]])$ on $\mathcal{P}_f(\mathbb{Z})$ has amenable stabilizers.

Consider firstly the stabilizer of the empty set, $Stab_{\pi(G)}(\emptyset)$. Let $g \in Stab_{\pi(G)}(\emptyset)$, then

$$(g(\mathbb{N})\Delta\mathbb{N}, g)(\emptyset) = g(\mathbb{N})\Delta\mathbb{N} = \emptyset.$$

Therefore $Stab_{\pi(G)}(\emptyset) = Stab_G(\mathbb{N})$, which is amenable by Lemma 8.0.3.

To show that stabilizers of all other points are amenable, note that the action of G on $\mathbb{Z}$ is amenable. This follows either from amenability of lamps or more straightforward by taking $F_n = [-n, n]$ as a Følner sequence for the action of G on $\mathbb{Z}$.

To reach a contradiction assume that there exists $E = \{x_0, \ldots, x_n\} \in \mathcal{P}_f(\mathbb{Z})$ such that $Stab_{\pi(G)}(E)$ is not amenable. We have that the group $Stab_{\pi(G)}(E) \cap Stab_G(x_0)$ is also not amenable. Indeed, the action of $Stab_{\pi(G)}(E)$ on the orbit of x_0 is amenable and the stabilizers of points of this action are conjugate to $Stab_{\pi(G)}(E) \cap Stab_G(x_0)$, thus, Theorem 5.1.4 applies. Repeating this argument we obtain that the group $Stab_{\pi(G)}(E) \cap Stab_G(x_0) \cap \ldots \cap Stab_G(x_n)$ is not amenable. However, for $g \in Stab_{\pi(G)}(E) \cap Stab_G(x_0) \cap \ldots \cap Stab_G(x_n)$, we have

$$(g(\mathbb{N})\Delta\mathbb{N}, g)(E) = g(E)\Delta g(\mathbb{N})\Delta\mathbb{N} = E\Delta g(\mathbb{N})\Delta\mathbb{N} = E$$

that implies $g(\mathbb{N})\Delta\mathbb{N} = \emptyset$ and therefore $Stab_{\pi(G)}(E) \cap Stab_G(x_0) \cap \ldots \cap Stab_G(x_n)$ is a subgroup of an amenable group $Stab_{\pi(G)}(\emptyset)$, which contradicts to our assumption. Thus, G is amenable. Since every finitely generated subgroup of $[[T]]$ is amenable, $[[T]]$ is also amenable. $\square$

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